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DECOMPOSING A NEW NONLINEAR DIFFERENTIAL-DIFFERENCE SYSTEM UNDER A BARGMANN IMPLICIT SYMMETRY CONSTRAINT*

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Abstract Firstly, a hierarchy of integrable lattice equations and its bi-Hamiltonian structures are established by applying the discrete trace identity. Secondly, under an implicit Bargmann symmetry constraint, every lattice equation in the nonlinear differential-difference system is decomposed by an completely integrable symplectic map and a finite-dimensional Hamiltonian system. Finally, the spatial part and the temporal part of the Lax pairs and adjoint Lax pairs are all constrained as finite dimensional Liouville integrable Hamiltonian systems.

Keywords Integrable lattice equations, symplectic map, implicit symmetry constraint, finite-dimensional Hamiltonian system.

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1. Introduction

There is a close relationship between symmetries and integrability [9, 10, 40]. Almost all classical integrable equations were demonstrated as equations with infinitely many K-symmetries [1, 27]. For soliton theory, to research symmetries of equations and symmetry properties of solutions made great improvements [7, 13, 38]. The symmetry constraint method for soliton hierarchies is proposed through the nonlinearization technique (including mono-nonlinearization [3] and binary nonlinearization [34]. Mono-nonlinearization involves only the Lax pairs for soliton equations and binary nonlinearization involves both the Lax pairs and the adjoint Lax pairs for soliton equations. The nonlinearization method to get the quasi-periodic solutions of soliton equations is a systematic approach to get finite-dimensional Liouville integrable systems under certain symmetry constraints between the potentials and the eigenfunctions. Up to now, the binary nonlinearization of integrable systems has obtained great achievements [8, 17, 24, 28, 36, 45].

In recent years, we focus our attention on the binary nonlinearization method because it has helped in a significant way to search for solutions of soliton equations [25, 26, 49]. Above all, integrable lattice equations associated with 2-order

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spectral problem play a fundamental role in integrable system. This paper begins with an interesting 2-order spectral problem, a hierarchy of integrable lattice equations and its bi-Hamiltonian structures are discussed. Considering the symmetry constraints provide a direct method to construct solutions of higher-dimensional integrable systems by solving lower-dimensional integrable systems, we try to get a family of finite-dimensional integrable systems from the obtained hierarchy by binary nonlinearization. In this paper, we will deduce the following new integrable equation

$$\begin{cases} u_{n_t} = u_n(u_n v_{n-1} - u_{n+1} v_n), \\ v_{n_t} = v_n(u_n - u_{n+1}) + v_n(u_n v_{n-1} - u_{n+1} v_n). \end{cases} \tag{1.1}$$

and derive an implicit symmetry constraint in detail. Through the implicit symmetry constraint, we would like to factor each lattice soliton equation in the resulting hierarchy by an integrable symplectic map and a finite-dimensional Liouville integrable Hamiltonian system.

This paper is structured as follows. The next section, a new hierarchy of lattice equations is deduced from a new discrete matrix spectral problem by using the discrete zero curvature equation. Moreover, using the discrete trace identity, the bi-Hamiltonian structures of the obtained hierarchy are established. In Section 3, a higher-order binary Bargmann symmetry constraint is involved to nonlinearize the new Lax pairs and the adjoint Lax pairs of the resulting lattice. Finally, some conclusions and remarks are concluded.

2. A new integrable hierarchy and its bi-Hamiltonian structures

We first introduce the shift operator E and two difference operators D and Δ by

$$\begin{aligned} &(Ef)(n) = f(n+1), (E^{-1}f)(n) = f(n-1), n \in Z, \\ &(Df)(n) = f(n+1) - f(n), (\Delta f)(n) = f(n+1) - f(n-1), n \in Z, \\ &\Delta = E - E^{-1},\ (1-E)^{-1} = -(1+E^{-1})\Delta^{-1},\ (1-E^{-1})^{-1} = (1+E)\Delta^{-1}, \end{aligned} \tag{2.1}$$

where f is a lattice function from Z to R .

We consider the following isospectral problem

$$E\varphi_n = U_n(u_n,\lambda)\varphi_n, \quad U_n(u_n,\lambda) = \begin{pmatrix} \frac{u_n}{\lambda} & \frac{1+v_n}{\lambda} \\ u_n & 1 \end{pmatrix}, \tag{2.2}$$

where $\varphi_n = (\varphi_n^1, \varphi_n^2)^T$, λ is a constant spectral parameter of $\lambda_t = 0$. In order to derive the hierarchy of lattice equations associated with (2.2), we first solve the stationary discrete zero curvature equation

$$(EV_n^{[m]})U_n - U_n V_n^{[m]} = 0. \tag{2.3}$$

Here, $V_n^{[m]}$ is a solution of (4) given by

$$V_n^{[m]} = \begin{pmatrix} a_n & b_n \\ \lambda c_n & -a_n \end{pmatrix}.$$

The form feature of $V_n^{[m]}$ is associated with the isospectral problem (2.2). Then (2.3) is equivalent to

$$\begin{cases} \dfrac{u_n}{\lambda}(a_{n+1} - a_n) + u_n b_{n+1} - (1 + v_n)c_n = 0, \\ \dfrac{1 + v_n}{\lambda}(a_{n+1} + a_n) + b_{n+1} - \dfrac{u_n}{\lambda} b_n = 0, \\ -u_n(a_{n+1} + a_n) + u_n c_{n+1} - \lambda c_n = 0, \\ (1 + v_n)c_{n+1} - u_n b_n - (a_{n+1} - a_n) = 0. \end{cases} \tag{2.4}$$

Taking $a_n^{(0)} = -\frac{1}{2}, b_n^{(0)} = 0, c_n^{(0)} = 0$, and substituting the expanding expressions

$$a_n = \sum_{m \geq 0} a_n^{(m)} \lambda^{-m}, b_n = \sum_{m \geq 0} b_n^{(m)} \lambda^{-m}, c_n = \sum_{m \geq 0} c_n^{(m)} \lambda^{-m} \tag{2.5}$$

into (2.4), we obtain the following recursion relations

$$\begin{cases} u_n(a_{n+1}^{(m)} - a_n^{(m)}) + u_n b_{n+1}^{(m+1)} - (1 + v_n)c_n^{(m+1)} = 0, \\ (1 + v_n)(a_{n+1}^{(m)} + a_n^{(m)}) - u_n b_n^{(m)} + b_{n+1}^{(m+1)} = 0, \\ -u_n(a_{n+1}^{(m)} + a_n^{(m)}) + u_n c_{n+1}^{(m)} - c_n^{(m+1)} = 0, \\ (1 + v_n)c_{n+1}^{(m)} - u_n b_n^{(m)} - (a_{n+1}^{(m)} - a_n^{(m)}) = 0. \end{cases} \tag{2.6}$$

Under the initial-value conditions of selecting zero constants for the inverse operation of the difference operator D in computing $a_n^{(m)}, m \geq 1$, the recursion relations (2.6) uniquely determine $a_n^{(m)}, b_n^{(m)}, c_n^{(m)}, m \geq 1$ and the first few quantities are given by

$$\begin{aligned} &a_n^{(1)} = u_n(1 + v_{n-1}), \quad b_{n+1}^{(1)} = 1 + q_n, \quad c_n^{(1)} = u_n, \\ &a_n^{(2)} = -u_n^2(1 + v_{n-1})^2 - u_n u_{n-1} v_{n-1}(1 + v_{n-2}) - u_n u_{n+1} v_n(1 + v_{n-1}), \\ &b_{n+1}^{(2)} = -u_{n+1}(1 + v_n)^2 - u_n v_n(1 + v_{n-1}), \\ &c_n^{(2)} = -u_n u_{n+1} v_n - u_n^2(1 + v_{n-1}), \cdots. \end{aligned}$$

For any integer $m \geq 0$, we choose

$$V_n^{[m]} = \begin{pmatrix} \sum_{i=0}^m a_i \lambda^{m-i} & \sum_{i=0}^m b_i \lambda^{m-i} \\ \sum_{i=0}^m c_i \lambda^{m+1-i} & -\sum_{i=0}^m a_i \lambda^{m-i} \end{pmatrix} \tag{2.7}$$

and make a modification

$$\Delta_n^{(m)} = \begin{pmatrix} -2a_n^{(m)} + c_n^{(m)} & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.8}$$

Then, we set

$$V_n^{(m)} = V_n^{[m]} + \Delta_n^{(m)},$$

and the following equation holds:

$$E(V_n^{(m)})U_n - U_n V_n^{(m)} = \begin{pmatrix} \frac{u_n}{\lambda}(a_n^{(m)} - a_{n+1}^{(m)}) + \frac{u_n}{\lambda}(c_{n+1}^{(m)} - c_n^{(m)}) & \frac{v_n}{\lambda}(a_n^{(m)} - a_{n+1}^{(m)}) \\ u_n(a_n^{(m)} - a_{n+1}^{(m)}) + u_n(c_{n+1}^{(m)} - c_n^{(m)}) & 0 \end{pmatrix}. \tag{2.9}$$

We introduce the following continuous time evolution equations

$$\varphi_{n_{t_m}} = V_n^{(m)} \varphi_n. \tag{2.10}$$

Then the compatibility conditions of (2.2) and (2.10) give rise to a hierarchy of discrete zero curvature equations

$$U_{n_{t_m}} = E(V_n^{(m)})U_n - U_n V_n^{(m)}, \quad m \geq 0, \tag{2.11}$$

which implies the following hierarchy of differential-difference equations

$$\begin{cases} u_{n_{t_m}} = u_n(a_n^{(m)} - a_{n+1}^{(m)}) + u_n(c_{n+1}^{(m)} - c_n^{(m)}), \\ v_{n_{t_m}} = v_n(a_n^{(m)} - a_{n+1}^{(m)}). \end{cases} \tag{2.12}$$

When $m = 1$, (2.12) be reduced to (1.1).

Bi-Hamiltonian structures of equation (1.1) in the hierarchy (2.11) may be established by applying the trace identity proposed by Tu. First, we define $R_n = \Gamma_n U_n^{-1}$ and $< M, N >= tr(MN)$, where M and N are the same order square matrices, then a direct calculations gives:

$$< R_n, \frac{\partial u_n}{\partial \lambda} >= -a_{n+1}, \; < R_n, \frac{\partial u_n}{\partial u_n} >= \frac{a_n}{u_n}, \; < R_n, \frac{\partial u_n}{\partial v_n} >= \frac{a_{n+1} - c_{n+1}}{v_n}.$$

By virtue of the discrete trace identity

$$\frac{\delta}{\delta u_n} \sum_{n \in Z} \langle R_n, \frac{\partial U_n}{\partial \lambda} \rangle = \left(\lambda^{-\varepsilon} \left(\frac{\partial}{\partial \lambda} \right) \lambda^{\varepsilon} \right) \langle R_n, \frac{\partial U_n}{\partial u_n^i} \rangle, \quad i = 1, 2, \tag{2.13}$$

we have

$$\frac{\delta}{\delta u_n} \sum_{n \in Z} (-a_{n+1}) = \lambda^{-\varepsilon} (\frac{\partial}{\partial \lambda}) \lambda^{\varepsilon} \begin{pmatrix} \frac{a_n}{u_n} \\ \frac{a_{n+1} - c_{n+1}}{v_n} \end{pmatrix}. \tag{2.14}$$

The substitution of (2.5) into (2.14) and comparing the coefficients of λ^{-m-1} in both sides of the resulting equations yield

$$\frac{\delta}{\delta u_n} \sum_{n \in Z} (-a_n^{(m+1)}) = (\varepsilon - m) \begin{pmatrix} \frac{a_n^{(m)}}{u_n} \\ \frac{a_{n+1}^{(m)} - c_{n+1}^{(m)}}{v_n} \end{pmatrix}. \tag{2.15}$$

We set $m = 0$ in (2.15), and find that $\varepsilon = 0$. Therefore, we obtain

$$U_{n_{t_m}} = \begin{pmatrix} u_n \\ v_n \end{pmatrix}_{t_m} = J \frac{\delta H_n^{(m)}}{\delta u_n} = K \frac{\delta H_n^{(m-1)}}{\delta u_n}. \tag{2.16}$$

Here,

$$J=\begin{pmatrix} 0 & u_n(E^{-1}-1)v_n \\ v_n(1-E)u_n & 0 \end{pmatrix}, \tag{2.17}$$

and

$$K=\begin{pmatrix} u_nv_nEu_n^2-u_n^2E^{-1}u_nv_n & u_nv_nEu_nv_n-u_n^2E^{-1}(v_n+v_n^2)+u_n^2v_n \\ (v_n+v_n^2)Eu_n^2-u_nv_nE^{-1}u_nv_n-u_n^2v_n & (v_n+v_n^2)Eu_nv_n-u_nv_nE^{-1}(v_n+v_n^2) \end{pmatrix}. \tag{2.18}$$

Using (2.6), we get the recursion structures

$$\frac{\delta H_n^{(m)}}{\delta u_n}=\Phi\frac{\delta H_n^{(m-1)}}{\delta u_n}, \tag{2.19}$$

where the recursion operator

$$\Phi=\begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}, \tag{2.20}$$

$$\begin{cases} \Phi_{11}=-\dfrac{1}{u_n}(E-1)^{-1}(1+v_n)Ep_n^2+\dfrac{1}{u_n}(E-1)^{-1}u_nE^{-1}u_nv_n+\dfrac{1}{u_n}(1-E)^{-1}u_n^2, \\ \Phi_{12}=-\dfrac{1}{u_n}(E-1)^{-1}(1+v_n)Eu_nv_n+\dfrac{1}{u_n}(E-1)^{-1}u_nE^{-1}(v_n+v_n^2), \\ \Phi_{21}=\dfrac{1}{v_n}(1-E^{-1})^{-1}u_nE^{-1}u_nv_n-\dfrac{1}{v_n}(1-E^{-1})^{-1}v_nEu_n^2, \\ \Phi_{22}=\dfrac{1}{v_n}(1-E^{-1})^{-1}u_nE^{-1}(v_n+v_n^2)-\dfrac{1}{v_n}(1-E^{-1})^{-1}v_nEu_nv_n \\ \qquad -\dfrac{1}{v_n}(1-E^{-1})^{-1}u_nv_n, \end{cases}$$

and

$$H_n^{(0)}=-\frac{1}{2}\sum_{n\in z}(ln(u_nv_n)),\quad H_n^{(m)}=\sum_{n\in z}-\frac{a_n^{(m+1)}}{m},m>0. \tag{2.21}$$

Based on above discussion, we can give the following propositions and theorems:

Proposition 2.1. *The Hamiltonian functions* $\{H_n^{(m)}\}_{m=0}^{\infty}$ *defined by (2.21) forms an infinite set of conserved functionals of the hierarchy (2.16), which has infinitely many common commuting symmetries* $\left\{J_1\frac{\delta H_n^{(m)}}{\delta u_n}\right\}_{m=1}^{\infty}$. $\{H_n^{(m)}\}_{m=0}^{\infty}$ *are in involution in pairs with respect to the Poisson bracket.*

Proof. It is clear that

$$(J\Phi)^*=-J\Phi.$$

Namely,

$$\Phi^*J=J\Phi.$$

We find that

$$\{H_n^{(m)},H_n^{(l)}\}_J=\langle\frac{\delta H_n^{(m)}}{\delta u_n},J\frac{\delta H_n^{(l)}}{\delta u_n}\rangle=\langle\Phi^{m-1}\frac{\delta H_n^{(1)}}{\delta u_n},J\Phi^{l-1}\frac{\delta H_n^{(1)}}{\delta u_n}\rangle$$

$$= \langle \Phi^{m-1}\frac{\delta H_n^{(1)}}{\delta u_n}, \Phi^* J\Phi^{l-2}\frac{\delta H_n^{(1)}}{\delta u_n}\rangle = \langle \Phi^{m}\frac{\delta H_n^{(1)}}{\delta u_n}, J\Phi^{l-2}\frac{\delta H_n^{(1)}}{\delta u_n}\rangle$$
$$= \{H_n^{(m+1)}, H_n^{(l-1)}\}_J = \cdots = \{H_n^{(m+1-1)}, H_n^{(1)}\}_J.$$

Repeating the above process gives

$$\{H_n^{(l)}, H_n^{(m)}\}_J = \{H_n^{(m+1-1)}, H_n^{(1)}\}_J,$$

this implies that

$$\{H_n^{(m)}, H_n^{(l)}\}_J = -\{H_n^{(l)}, H_n^{(m)}\}_J. \tag{2.22}$$

Therefore

$$\{H_n^{(l)}, H_n^{(m)}\}_J = 0, m, l \geq 1$$

and

$$(H_n^{(m)})_{t_l} = \langle\frac{\delta H_n^{(m)}}{\delta u_n}, u_{nt_l}\rangle = \langle\frac{\delta H_n^{(m)}}{\delta u_n}, J\frac{\delta H_n^{(l)}}{\delta u_n}\rangle = \{H_n^{(m)}, H_n^{(l)}\}_J = 0, m, l \geq 1.$$

From (2.22), we have

$$[J_1\frac{\delta H_n^{(m)}}{\delta u}, J_1\frac{\delta H_n^{(l)}}{\delta u_n}] = J_1\frac{\delta\{H_n^{(m)}, H_n^{(l)}\}_{J_1}}{\delta u_n} = 0, m, l \geq 0.$$

Which is crucial to show the existence of infinite involutive conserved functionals. The proof is completed. □

Theorem 2.1. *The integrable equations in (2.16) are all discrete Liouville integrable Hamiltonian systems.*

3. Implicit Bagmann symmetry constraint

Let us consider the adjoint spectral problem of spectral problem (2.2)

$$E^{-1}\psi_n = (E^{-1}U_n^T(u,\lambda)\psi_n), \psi_n = \begin{pmatrix} \psi_n^1 \\ \psi_n^2 \end{pmatrix} \tag{3.1}$$

and the continuous time evolution equations

$$\psi_{nt_m} = -(\bar{V}^{(m)}(u,\lambda))^T\psi_n. \tag{3.2}$$

From $(E^{-1}\psi_n)_{t_m} = E^{-1}\psi_{nt_m}$, we can find

$$E^{-1}U_{t_m}^T = (E^{-1}U_n^T)(\bar{V}_n^{(m)})^T - (E^{-1}\bar{V}_n^{(m)})^T(E^{-1}U_n^T) \tag{3.3}$$

and (3.3) can be rewritten as

$$U_{nt_m} = (E\bar{V}_n^{(m)})U_n - U_n\bar{V}_n^{(m)}.$$

Therefore, the integrable differential-difference equation (2.12) has another discrete zero curvature representation (3.3). The adjoint Lax pairs of (2.12), which is (3.1) and (3.2), they can help us determine the variational derivative of the spectral parameter λ with respect to the potential u_n.

Let $\lambda_1, \lambda_2, ..., \lambda_N$ be N distinct eigenvalues of spectral problem (2.2) and $\lambda_j \neq 0, j = 1, 2, ..., N$, we have

$$\begin{pmatrix} E\varphi_n^{1j} \\ E\varphi_n^{2j} \end{pmatrix} = U_n(u_n, \lambda_j) \begin{pmatrix} \varphi_n^{1j} \\ \varphi_n^{2j} \end{pmatrix}, \begin{pmatrix} E^{-1}\psi_n^{1j} \\ E^{-1}\psi_n^{2j} \end{pmatrix} = (E^{-1}U_n^T(u_n, \lambda_j)) \begin{pmatrix} \psi_n^{1j} \\ \psi_n^{2j} \end{pmatrix}, \tag{3.4}$$

$$\begin{pmatrix} \varphi_n^{1j} \\ \varphi_n^{2j} \end{pmatrix}_{t_m} = \bar{V}_n^{(m)}(u_n, \lambda_j) \begin{pmatrix} \varphi_n^{1j} \\ \varphi_n^{2j} \end{pmatrix}, \begin{pmatrix} \psi_n^{1j} \\ \psi_n^{2j} \end{pmatrix}_{t_m} = -(\bar{V}_n^{(m)}(u_n, \lambda_j))^T \begin{pmatrix} \psi_n^{1j} \\ \psi_n^{2j} \end{pmatrix}, \tag{3.5}$$

$$(E\varphi_n^{1j}, E\varphi_n^{2j}) = (\varphi_n^{1j}, \varphi_n^{2j}) U_n^T(u_n, \lambda_j)^T, \tag{3.6}$$

$$(E\psi_n^{1j}, E\psi_n^{2j}) = (\psi_n^{1j}, \psi_n^{2j}) U_n^T(u_n, \lambda_j)^{-1}. \tag{3.7}$$

From [10], we know that

$$\frac{\delta \lambda_j}{\delta u_n} = \alpha_j (\psi_{n+1}^{1j}, \psi_{n+1}^{2j}) \frac{\partial U_n(u_n, \lambda_j)}{\partial u_n} (\varphi_n^{1j}, \varphi_n^{2j})^T. \tag{3.8}$$

It is evident that (3.8) will play an important role in the binary nonlinearization method.

We suppose that

$$\begin{pmatrix} \frac{\delta \lambda_j}{\delta u_n} \\ \frac{\delta \lambda_j}{\delta v_n} \end{pmatrix} = \alpha_j \begin{pmatrix} \frac{1}{u_n} \varphi_n^{1j} \psi_n^{1j} \\ \frac{1}{u_n v_n} (u_n \varphi_n^{2j} \psi_n^{2j} - \varphi_n^{2j} \psi_n^{1j}) \end{pmatrix}, \tag{3.9}$$

where $\frac{\delta \lambda_j}{\delta u_n}$ is a variational derivative for eigenvalue λ_j, and α_j are constants. The eigenfunctions $\varphi_i, \psi_i, i = 1, 2 \cdots N$ are required to be rapidly vanishing at the infinity.

Putting

$$\Phi_n^i = (\varphi_n^{i1}, \varphi_n^{i2}, ..., \varphi_n^{iN}), \Psi_n^i = (\psi_n^{i1}, \psi_n^{i2}, ..., \psi_n^{iN}), i = 1, 2,$$

$$\wedge = diag(\lambda_1, \lambda_2, ..., \lambda_N),$$

we consider the discrete symmetry constraint

$$J \frac{\delta H_n^{(1)}}{\delta u_n} = J \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u_n}, \tag{3.10}$$

where $\alpha_j = 1, j = 1, 2, ..., N$. Because of $\lambda_{tm} = 0, J(\delta\lambda_j/\delta u_n), 1 \leq j \leq N$, are all symmetries of $u_{ntm} = J(\delta H_n^{(m)}/\delta u_n), m \geq 0$, then (3.10) is a symmetry constraint. From (3.8), we can obtain

$$\frac{\delta H_n^{(1)}}{\delta u} = \begin{pmatrix} 1 + v_{n-1} \\ u_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{< \Phi_n^1, \Psi_n^1 >}{u_n} \\ \frac{1}{u_n v_n} (u_n < \Phi_n^2, \Psi_n^2 > - < \Phi_n^2, \Psi_n^1 >) \end{pmatrix}. \tag{3.11}$$

Then, we select the dependent variables

$$1+v_{n-1}=\varphi_n^{2N+1},\quad v_n=\psi_n^{2N+1},$$

the symmetry constraint (3.11) yields implicit expressions

$$\begin{cases} u_n=\dfrac{<\Phi_n^1,\Psi_n^1>}{\varphi_n^{2N+1}}, \\ v_n=\psi_n^{2N+1}. \end{cases} \tag{3.12}$$

Because it is possible to solve (3.11) for p_n and q_n depending on $\Phi_i,\Psi_i,i=1,2$, so (3.10) or (3.12) is called a binary Bargmann constraint. Applying (3.4) and (3.12), we obtain a discrete flow

$$\begin{aligned}
&E\varphi_n^{1j}=\frac{<\Phi_n^1,\Psi_n^1>}{\lambda_j\varphi_n^{N+1}}\varphi_n^{1j}+\frac{1+\varphi_n^{2N+1}}{\lambda_j}\varphi_n^{2j},\quad 1\le j\le N,\\
&E\varphi_n^{2j}=\frac{<\Phi_n^1,\Psi_n^1>}{\varphi_n^{2N+1}}\varphi_n^{1j}+\varphi_n^{2j},\quad 1\le j\le N,\\
&E\varphi_n^{2N+1}=1+\psi_n^{2N+1},\\
&E\psi_n^{1j}=-\frac{\lambda_j\varphi_n^{2N+1}}{<\Phi_n^1,\Psi_n^1>\psi_n^{2N+1}}\psi_n^{1j}+\frac{\lambda_j}{\psi_n^{2N+1}}\psi_n^{2j},\quad 1\le j\le N,\\
&E\psi_n^{2j}=\frac{(1+\psi_n^{2N+1})\varphi_n^{2N+1}}{<\Phi_n^1,\Psi_n^1>\psi_n^{2N+1}}\psi_n^{1j}-\frac{1}{\psi_n^{2N+1}}\psi_n^{2j},\quad 1\le j\le N,\\
&E\psi_n^{2N+1}=\frac{<\Phi_n^2,\Psi_n^2>}{\psi_n^{2N+1}}-\frac{<\Phi_n^2,\Psi_n^1>\varphi_n^{2N+1}}{<\Phi_n^1,\Psi_n^1>\psi_n^{2N+1}}.
\end{aligned} \tag{3.13}$$

Setting

$$\begin{aligned}
&\chi_n^i=\chi_n^i(\Phi_n^1,\Phi_n^2,\varphi_n^{2N+1},\Psi_n^1,\Psi_n^2,\psi_n^{2N+1}),\\
&\zeta_n^i=\zeta_n^i(\Phi_n^1,\Phi_n^2,\varphi_n^{2N+1},\Psi_n^1,\Psi_n^2,\psi_n^{2N+1}),1\le i\le 2N+1,
\end{aligned}$$

we have

$$\begin{aligned}
&\chi_n^j=-\frac{\lambda_j\varphi_n^{2N+1}}{<\Phi_n^1,\Psi_n^1>\psi_n^{2N+1}}\psi_n^{1j}+\frac{\lambda_j}{\psi_n^{2N+1}}\psi_n^{2j},\quad 1\le j\le N,\\
&\chi_n^{N+j}=\frac{(1+\psi_n^{2N+1})\varphi_n^{2N+1}}{<\Phi_n^1,\Psi_n^1>\psi_n^{2N+1}}\psi_n^{1j}-\frac{1}{\psi_n^{2N+1}}\psi_n^{2j},\quad 1\le j\le N,\\
&\chi_n^{2N+1}=\frac{<\Phi_n^2,\Psi_n^2>}{\psi_n^{2N+1}}-\frac{<\Phi_n^2,\Psi_n^1>\varphi_n^{2N+1}}{<\Phi_n^1,\Psi_n^1>\psi_n^{2N+1}},\\
&\zeta_n^j=\frac{<\Phi_n^1,\Psi_n^1>}{\lambda_j\varphi_n^{2N+1}}\varphi_n^{1j}+\frac{1+\varphi_n^{2N+1}}{\lambda_j}\varphi_n^{2j},\quad 1\le j\le N,\\
&\zeta_n^{N+j}=\frac{<\Phi_n^1,\Psi_n^1>}{\varphi_n^{2N+1}}\varphi_n^{1j}+\varphi_n^{2j},\quad 1\le j\le N,\\
&\zeta_n^{2N+1}=1+\psi_n^{2N+1}.
\end{aligned} \tag{3.14}$$

Set

$$P = (\varphi_n^{11}, \varphi_n^{12}, \cdots, \varphi_n^{1N}, \varphi_n^{21}, \varphi_n^{22}, \cdots, \varphi_n^{2N}, \varphi_n^{2N+1})^T,$$
$$Q = (\psi_n^{11}, \psi_n^{12}, \cdots, \psi_n^{1N}, \psi_n^{21}, \psi_n^{22}, \cdots, \psi_n^{2N}, \psi_n^{2N+1})^T$$

and

$$\frac{\partial}{\partial P} = (\frac{\partial}{\partial \varphi_n^{11}}, \frac{\partial}{\partial \varphi_n^{12}}, \cdots, \frac{\partial}{\partial \varphi_n^{1N}}, \frac{\partial}{\partial \varphi_n^{21}}, \cdots \frac{\partial}{\partial \varphi_n^{2N}}, \frac{\partial}{\partial \varphi_n^{2N+1}})^T,$$
$$\frac{\partial}{\partial Q} = (\frac{\partial}{\partial \psi_n^{11}}, \frac{\partial}{\partial \psi_n^{12}}, \cdots, \frac{\partial}{\partial \psi_n^{1N}}, \frac{\partial}{\partial \psi_n^{21}}, \cdots \frac{\partial}{\partial \psi_n^{2N}}, \frac{\partial}{\partial \varphi_n^{2N+1}})^T.$$

The Poisson bracket of two function χ and ζ in symplectic apace R^{4N+2} is defined by

$$\{\chi, \zeta\} = \langle \frac{\partial \chi}{\partial P}, \frac{\partial \zeta}{\partial Q} \rangle - \langle \frac{\partial \zeta}{\partial P}, \frac{\partial \chi}{\partial Q} \rangle = (\frac{\partial \chi}{\partial P})^T (\frac{\partial \zeta}{\partial Q}) - (\frac{\partial \zeta}{\partial P})^T (\frac{\partial \chi}{\partial Q}). \tag{3.15}$$

This is skew-symmetric, bilinear, and satisfies the Jacobi identity. In particular, χ and ζ are called involutive if $\{\chi, \zeta\} = 0$.

Proposition 3.1. *Equation* (3.10) *defines a map* H,

$$H(P^T, Q^T) = (EP^T, EQ^T). \tag{3.16}$$

It is symplectic.

Proof. Through direct and tedious computation, we get

$$\{\chi_i, \chi_j\} = \{\zeta_i, \zeta_j\} = 0, \{\chi_i, \zeta_j\} = \delta_{ij}, 1 \le i, j \le 2N + 1$$

and

$$d(EP) \wedge d(EQ) = dP \wedge dQ.$$

Therefore, (3.16) determines a symplectic map. The proof is finished.

In what follows, we would like to solve (2.6). When $m > 1$, we have

$$U_{n_{t_m}} = \begin{pmatrix} u_n \\ v_n \end{pmatrix}_{t_m} = J \frac{\delta H_n^{(m)}}{\delta u_n} = J \Phi^{m-1} \frac{\delta H_n^{(1)}}{\delta u_n} = J \sum_{j=1}^{N} \lambda_j^{m-1} \frac{\delta \lambda_j}{\delta u_n}.$$

That is to say,

$$\begin{pmatrix} u_n(a_n^{(m)} - a_{n+1}^{(m)}) + u_n(c_{n+1}^{(m)} - c_n^{(m)}) \\ v_n(a_n^{(m)} - a_{n+1}^{(m)}) \end{pmatrix} = J \begin{pmatrix} \frac{1}{u_n} \sum_{j=1}^{N} \lambda_j^{m-1} \varphi_n^{1j} \psi_n^{1j} \\ \frac{1}{u_n v_n} \sum_{j=1}^{N} \lambda_j^{m-1} (u_n \varphi_n^{2j} \psi_n^{2j} - \varphi_n^{2j} \psi_n^{1j}) \end{pmatrix}. \tag{3.17}$$

From (3.17), we can conclude

$$c_n^{(m)} =< \wedge^{m-2} \Phi_n^2, \Psi_n^1 > .$$

Moreover, $a_n^{(m)}$ can be decided

$$a_n^{(m)} = \frac{1}{2} (< \wedge^{m-1} \Phi_n^1, \Psi_n^1 > - < \wedge^{m-1} \Phi_n^2, \Psi_n^2 >).$$

Substituting $c_n^{(m)}, a_n^{(m)}$ into the first equation of (7), it is clear that

$$b_n^{(m)} =< \wedge^{m-2}\Phi_n^1, \Psi_n^2 > .$$

In summary, we choose

$$\begin{aligned}
&\tilde{a}_n^{(0)} = -\frac{1}{2}, \quad \tilde{b}_n^{(0)} = 0, \quad \tilde{c}_n^{(0)} = 0,\\
&\tilde{a}_n^{(1)} =< \Phi_n^1, \Psi_n^1 >, \quad \tilde{b}_n^{(1)} = \varphi_n^{2N+1}, \quad \tilde{c}_n^{(1)} = \frac{< \Phi_n^1, \Psi_n^1 >}{\varphi_n^{2N+1}},\\
&\tilde{a}_n^{(m)} = \frac{1}{2}(< \wedge^{m-2}\Phi_n^1, \Psi_n^1 > - < \wedge^{m-2}\Phi_n^2, \Psi_n^2 >), \quad m \geq 2,\\
&\tilde{b}_n^{(m)} =< \wedge^{m-2}\Phi_n^1, \Psi_n^2 >, \quad \tilde{c}_n^{(m)} =< \wedge^{m-2}\Phi_n^2, \Psi_n^1 >, \quad m \geq 2.
\end{aligned} \tag{3.18}$$

Setting

$$\tilde{F}_m = \sum_{i=0}^{m}(\tilde{a}_i\tilde{a}_{m-i} + \tilde{b}_i\tilde{c}_{m-i}), \quad m \geq 0, \tag{3.19}$$

we have

$$\begin{aligned}
&\tilde{F}_n^0 = (\tilde{a}_n^{(0)})^2 = \frac{1}{4}, \quad \tilde{F}_n^1 = -\tilde{a}_n^{(1)} = - < \Phi_n^1, \Psi_n^1 >,\\
&\tilde{F}_n^2 = -\frac{1}{2}(< \Phi_n^1, \Psi_n^1 > - < \Phi_n^2, \Psi_n^2 >) + < \Phi_n^1, \Psi_n^1 >^2 + < \Phi_n^1, \Psi_n^2 >,\\
&\tilde{F}_n^3 = -\frac{1}{2}(< \wedge\Phi_n^1, \Psi_n^1 > - < \wedge\Phi_n^2, \Psi_n^2 >) < \Phi_n^1, \Psi_n^1 > \cdot(< \Phi_n^1, \Psi_n^1 > -\\
&\qquad + < \Phi_n^2, \Psi_n^2 >) + < \Phi_n^2, \Psi_n^1 > \varphi_n^{2N+1} + < \Phi_n^1, \Psi_n^2 > \frac{< \Phi_n^1, \Psi_n^2 >}{\varphi_n^{2N+1}},\\
&\tilde{F}_n^m = -\frac{1}{2}(< \wedge^{m-2}\Phi_n^1, \Psi_n^1 > - < \wedge^{m-2}\Phi_n^2, \Psi_n^2 >)\\
&\qquad + < \wedge^{m-2}\Phi_n^1, \Psi_n^1 > \cdot(< \Phi_n^1, \Psi_n^1 > - < \Phi_n^2, \Psi_n^2 >)\\
&\qquad + < \wedge^{m-3}\Phi_n^2, \Psi_n^1 > \varphi_n^{2N+1} + < \wedge^{m-3}\Phi_n^1, \Psi_n^2 > \frac{< \Phi_n^1, \Psi_n^2 >}{\varphi_n^{2N+1}}\\
&\qquad + \textstyle\sum_{i=0}^{m-4}(\frac{< \wedge^i\Phi_n^1, \Psi_n^1 > - < \wedge^i\Phi_n^2, \Psi_n^2 >}{2}\\
&\qquad \cdot \frac{< \wedge^{m-i-4}\Phi_n^1, \Psi_n^1 > - < \wedge^{m-i-4}\Phi_n^2, \Psi_n^2 >}{2}\\
&\qquad + < \wedge^i\Phi_n^1, \Psi_n^2 > \cdot < \wedge^{m-i-4}\Phi_n^2, \Psi_n^1 >), \quad m \geq 4.
\end{aligned} \tag{3.20}$$

Proposition 3.2.

$$D\tilde{F}_n^m = 0, m \geq 1.$$

Proof. Because expressions (3.18) are the solutions of (2.6), we have

$$\tilde{a}_n = \sum_{m=0}^{\infty}\tilde{a}_n^{(m)}\lambda^{-m}, \quad \tilde{b}_n = \sum_{m=0}^{\infty}\tilde{b}_n^{(m)}\lambda^{-m}, \quad \tilde{c}_n = \sum_{m=0}^{\infty}\tilde{c}_n^{(m)}\lambda^{-m},$$

which is a solution set of (2.4), thus

$$D(\tilde{a}_n^2 + \tilde{b}_n\tilde{c}_n) = 0, \tag{3.21}$$

this implies $D\tilde{F}_n^m = 0, m \geq 1$. This completes the proof. $\square$

In the following paragraph, we would like to discuss the Louville integrability on the nonlinearized temporal parts of the Lax pairs and adjoint Lax pairs.

Under the control of (3.12), by substituting (3.20) into (3.5), the temporal parts of the Lax pairs and the adjoint Lax pairs become

$$\begin{aligned}
(\varphi_n^{1j}, \varphi_n^{2j})_{t_m}^T &= \bar{V}_n^{(m)}|_B(\varphi_n^{1j}, \varphi_n^{2j})^T, j = 1, 2, \cdots, N, \\
(\psi_n^{1j}, \psi_n^{2j})_{t_m}^T &= -(\bar{V}_n^{(m)})^T|_B(\psi_n^{1j}, \psi_n^{2j})^T, j = 1, 2, \cdots, N.
\end{aligned}$$

Here the subscript B means substitution of (3.12) into the expression.

Proposition 3.3. *The temporal parts of the nonlinearized Lax pairs (3.2) and the adjoint Lax pairs (3.5) may be rewritten as*

$$\begin{aligned}
&\Phi_{it_m} = \frac{\partial \tilde{F}_n^{m+2}}{\partial \Psi_i}, \quad \Psi_{it_m} = -\frac{\partial \tilde{F}_n^{m+2}}{\partial \Phi_i}, i = 1, 2, \\
&(\varphi_n^{2N+1})_{t_m} = \frac{\partial \tilde{F}_n^{m+2}}{\partial \psi_n^{2N+1}}, \quad (\psi_n^{2N+1})_{t_m} = -\frac{\partial \tilde{F}_n^{m+2}}{\partial \varphi_n^{2N+1}}.
\end{aligned} \tag{3.22}$$

Proof. Through a direct calculation, it can be verified that

$$\begin{aligned}
\frac{\partial \tilde{F}_n^{m+2}}{\partial \psi_n^{1j}} &= \textstyle\sum_{i=0}^{m+2}(2\tilde{a}_n^i \frac{\partial}{\partial \psi_n^{1j}} \tilde{a}_n^{m+2-i} + \tilde{b}_n^i \frac{\partial}{\partial \psi_n^{1j}} \tilde{c}_n^{m+2-i}) \\
&= \textstyle\sum_{i=0}^{m}(2\tilde{a}_n^i \cdot \frac{1}{2}\lambda_j^{m-i}\varphi_n^{1j} + \tilde{b}_n^i \lambda_j^{m-i}\varphi_n^{2j}) + 2\tilde{a}_n^{m+1}\varphi_n^{1j} + \tilde{c}_n^{m+2}\varphi_n^{1j} \\
&= (\varphi_n^{1j})_{t_m}.
\end{aligned}$$

The other equations can also be concluded in this way. The proof is finished. $\square$

Proposition 3.4. *$\tilde{F}_n^m, m \geq 0$ are in involution in pairs with respect to the Poisson bracket, and constitutes a hierarchy of integrals of motion in involution for (3.22).*

Proof. Through a direct calculation, we can get

$$\tilde{\Gamma}_{t_m} = [\bar{V}^{(m)}, \tilde{\Gamma}],$$

where

$$\tilde{\Gamma} = \begin{pmatrix} \tilde{a}_n & \tilde{b}_n \\ \tilde{c}_n & -\tilde{a}_n \end{pmatrix} = \begin{pmatrix} \sum\limits_{m=0}^{\infty} \tilde{a}_n^m \lambda^{-m} & \sum\limits_{m=0}^{\infty} \tilde{b}_n^m \lambda^{-m} \\ \sum\limits_{m=0}^{\infty} \tilde{c}_n^m \lambda^{-m+1} & -\sum\limits_{m=0}^{\infty} \tilde{a}_n^m \lambda^{-m} \end{pmatrix}.$$

This yields

$$2\frac{d}{d_{t_m}}(\tilde{a}^2 + \tilde{b}\tilde{c}) = \frac{d}{d_{t_m}} Tr\tilde{\Gamma}^2 = \frac{d}{d_{t_m}}[\tilde{V}^{(m)}, \tilde{\Gamma}^2] = 0, \tag{3.23}$$

that is to say,

$$\begin{aligned}
\frac{d}{d_{t_m}}\tilde{F}_n^l &= \sum_{i=0}^{N}(\frac{\partial \tilde{F}_n^l}{\partial \varphi_n^{1j}} \cdot \frac{\partial \varphi_n^{1j}}{\partial t_m} + \frac{\partial \tilde{F}_n^l}{\partial \varphi_n^{2j}} \cdot \frac{\partial \varphi_n^{2j}}{\partial t_m} + \frac{\partial \tilde{F}_n^l}{\partial \psi_n^{1j}} \cdot \frac{\partial \psi_n^{1j}}{\partial t_m} + \frac{\partial \tilde{F}_n^l}{\partial \psi_n^{2j}} \cdot \frac{\partial \psi_n^{2j}}{\partial t_m}) \\
&= \sum_{i=0}^{N}(\frac{\partial \tilde{F}_n^{m+2}}{\partial \psi_n^{1j}} \cdot \frac{\partial \tilde{F}_n^l}{\partial \varphi_n^{1j}} - \frac{\partial \tilde{F}_n^{m+2}}{\partial \varphi_n^{1j}} \cdot \frac{\partial \tilde{F}_n^l}{\psi_n^{1j}} + \frac{\partial \tilde{F}_n^{m+2}}{\partial \psi_n^{2j}} \cdot \frac{\partial \tilde{F}_n^l}{\partial \varphi_n^{2j}} - \frac{\partial \tilde{F}_n^{m+2}}{\partial \varphi_n^{2j}} \cdot \frac{\partial \tilde{F}_n^l}{\partial \psi_n^{2j}})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{N} (\langle \frac{\partial \tilde{F}_n^{m+2}}{\partial \Psi_n^i}, \frac{\partial \tilde{F}_n^l}{\partial \Phi_n^i} \rangle - \langle \frac{\partial \tilde{F}_n^{m+2}}{\partial \Phi_n^i}, \frac{\partial \tilde{F}_n^l}{\partial \Psi_n^i} \rangle) \\
&= \{\tilde{F}_n^l, \tilde{F}_n^m\} = 0, \quad l, m \geq 1.
\end{aligned}
$$

This completes the proof. □

This elucidates that $\tilde{F}_n^m, m \geq 1$ constitutes a hierarchy of involution integrals of motion for (3.22).

We define

$$
\bar{F}_n^j = \varphi_n^{1j}\psi_n^{1j} + \varphi_n^{2j}\psi_n^{2j}, 1 \leq j \leq N. \tag{3.24}
$$

It is easy to verify that

$$
\begin{aligned}
&D\bar{F}_n^j = 0, \quad \frac{d}{dt_m}\bar{F}_n^j = 0, \quad 1 \leq j \leq N, \\
&\{\tilde{F}_n^i, \bar{F}_n^j\} = 0, \quad 1 \leq i, j \leq N, \quad \{\bar{F}_n^j, \bar{F}_n^m\} = 0, \quad m \geq 0, 1 \leq j \leq N.
\end{aligned} \tag{3.25}
$$

Proposition 3.5. $\bar{F}_n^j, 1 \leq j \leq N, \quad \tilde{F}_n^{m+2}, 0 \leq m \leq N$ *are functionally independent over some region of* R^{4N+2}.

Proof. Using the ε technique proposed, a positive real number ε is given. We let

$$
\begin{aligned}
\tau =&(\varphi_n^{11}, \varphi_n^{12}, \cdots, \varphi_n^{1N}, \varphi_n^{21}, \varphi_n^{22}, \cdots, \varphi_n^{2N}, \varphi_n^{2N+1}, \psi_n^{11}, \psi_n^{12}, \cdots, \psi_n^{1N}, \\
&\psi_n^{21}, \psi_n^{22}, \cdots, \psi_n^{2N}, \psi_n^{2N+1})
\end{aligned}
$$

be a point of R^{4N+2} satisfying

$$
\varphi_n^{ij} = \psi_n^{2j} = \varepsilon, \quad \psi_n^{1j} = -\varepsilon, \ i = 1, 2, \ j = 1, 2, \cdots, N.
$$

From expressions (3.24), we obtain

$$
\frac{\partial \bar{F}_n^j}{\partial \varphi_n^{jl}} = \psi_n^{ij}\delta_{jl}, \ i = 1, 2, \ j, l = 1, 2, \cdots, N.
$$

By using of (3.19), it is clearly that

$$
\begin{aligned}
&\frac{\partial \tilde{F}_n^{m+2}}{\partial \Phi_n^1} = \textstyle\sum_{i=0}^m \tilde{a}_n^{(i)}\Lambda^{m-i}\Psi_n^1 + \sum_{i=0}^m \tilde{c}_n^{(i)}\Lambda^{m-i+1}\Psi_n^2 - 2\tilde{a}_n^{(m)}\Psi_n^1 + \tilde{c}_n^{(m)}\Psi_n^1, \\
&\frac{\partial \tilde{F}_n^{m+2}}{\partial \Phi_n^2} = \textstyle\sum_{i=0}^m \tilde{b}_n^{(i)}\Lambda^{m-i}\Psi_n^1 - \sum_{i=0}^m \tilde{a}_n^{(i)}\Lambda^{m-i}\Psi_n^2.
\end{aligned} \tag{3.26}
$$

Thus, it gives that

$$
\frac{\partial \tilde{F}_n^{m+2}}{\partial \Phi_n^{1j}}|_\tau = \frac{1}{2}\lambda_j^m\varepsilon + o(\varepsilon^2), \quad \frac{\partial \tilde{F}_n^{m+2}}{\partial \Phi_n^{2j}}|_\tau = -\frac{1}{2}\lambda_j^m\varepsilon + o(\varepsilon^2), \ m \geq 0. \tag{3.27}
$$

Assume that the result on the functional independence is not true. Then there exist $2N$ constants $\xi_1, \xi_2, \ldots \xi_N, \eta_1, \eta_2, \ldots, \eta_N$ satisfying

$$
\sum_{i=1}^{N} \xi_i \bar{F}_n^j + \sum_{j=1}^{N} \eta_j \tilde{F}_n^{j+2} \neq 0. \tag{3.28}
$$

We can compute that

$$
\begin{vmatrix}
\frac{\partial \bar{F}_n^1}{\partial \Phi_n^1} \cdots \frac{\partial \bar{F}_n^N}{\partial \Phi_n^1} & \frac{\partial \tilde{F}_n^1}{\partial \Phi_n^i} \cdots \frac{\partial \tilde{F}_n^N}{\partial \Phi_n^i} \\
\frac{\partial \bar{F}_n^1}{\partial \Phi_n^2} \cdots \frac{\partial \bar{F}_n^N}{\partial \Phi_n^2} & \frac{\partial \tilde{F}_n^1}{\partial \Phi_n^2} \cdots \frac{\partial \tilde{F}_n^N}{\partial \Phi_n^2}
\end{vmatrix}_{\tau,\ \varphi_n^{2N+1}=0,\ \psi_n^{2N+1}=0}
$$

$$
= \begin{vmatrix}
\psi_n^{11} & \dots & 0 & \frac{1}{2}\lambda_1\psi_n^{11} & \dots & \frac{1}{2}\lambda_1^N\psi_n^{11} \\
\vdots & \ddots & \vdots & & \ddots & \vdots \\
0 & \dots & \psi_n^{1N} & \frac{1}{2}\lambda_N\psi_n^{1N} & \dots & \frac{1}{2}\lambda_N^N\psi_n^{1N} \\
\psi_n^{21} & \dots & 0 & -\frac{1}{2}\lambda_1\psi_n^{21} & \dots & -\frac{1}{2}\lambda_1^N\psi_n^{21} \\
\vdots & \ddots & \vdots & & \ddots & \vdots \\
0 & \dots & \psi_n^{2N} & -\frac{1}{2}\lambda_N\psi_n^{2N} & \dots & -\frac{1}{2}\lambda_N^N\psi_{n^2N} \\
0 & \cdots & 0 & 0 & \cdots & 0
\end{vmatrix} \tag{3.29}
$$

$$
= \begin{vmatrix}
-\varepsilon & \dots & 0 & -\frac{1}{2}\lambda_1\varepsilon & \dots & -\frac{1}{2}\lambda_1^N\varepsilon \\
\vdots & \ddots & \vdots & & \ddots & \vdots \\
0 & \dots & -\varepsilon & -\frac{1}{2}\lambda_N\varepsilon & \dots & -\frac{1}{2}\lambda_N^N\varepsilon \\
\varepsilon & \dots & 0 & -\frac{1}{2}\lambda_1\varepsilon & \dots & -\frac{1}{2}\lambda_1^N\varepsilon \\
\vdots & \ddots & \vdots & & \ddots & \vdots \\
0 & \dots & \varepsilon & -\frac{1}{2}\lambda_N\varepsilon & \dots & -\frac{1}{2}\lambda_N^N\varepsilon
\end{vmatrix}
= \varepsilon^{2N}\left(\prod_{j=1}^{N}\lambda_j\right)
\begin{vmatrix}
1 & \lambda_1 & \dots & \lambda_1^N \\
1 & \lambda_2 & \dots & \lambda_2^N \\
\dots & \dots & & \\
1 & \lambda_N & \dots & \lambda_N^N
\end{vmatrix}.
$$

In light of the Vandermode determinant $V(\lambda_1,\dots,\lambda_N) \neq 0$, when $\psi_n^{1j} \neq 0, j = 1,2,\dots,N$, we can find that all ξ_i, η_i must be zero. So, the function $\bar{F}_n^j, \tilde{F}_n^{j+2}, j = 1,2,\dots,N$ are functionally independent at least on certain region of R^{4N+2}. The proof is completed. □

In conclusion, by the proposition 3 $\sim$ proposition 6, we have the following theorem:

Theorem 3.1. *Symplectic map (3.16) is integrable and the temporal parts equation (11) of constrained flows are all completely integrable finite-dimensional systems in the Liouville sense.*

Theorem 3.2. *Under the control of (3.12)*

$$
U(n,t_m) = \begin{pmatrix} u(n,t_m) \\ v(n,t_m) \end{pmatrix} = \begin{pmatrix} \frac{<\Phi_n^1(n,t_m),\Psi_n^1(n,t_m)>}{\varphi_n^{2N+1}(n,t_m)} \\ \psi_n^{2N+1}(n,t_m) \end{pmatrix}, \tag{3.30}
$$

we solves the lattice soliton (1.1) or the discrete Hamiltonian system (2.12). As a result of proposition 5, construction of solution (3.30) of equation (1.1) is split into obtaining $\Phi_i(n,t_m), \Psi_i(n,t_m), i = 1,2, \varphi_n^{2N+1}(n,t_m), \psi_n^{2N+1}(n,t_m)$ to the integrable symplectic map (3.16) and the finite-dimensional completely integrable *systems (3.22). In fact, (3.12) is a Bäcklund transformation between the lattice equation (1.1) and the integrable symplectic map (3.16).*

4. Conclusions and Remarks

In this paper, we have deduced a new family of differential-difference equations (1.1) through the discrete zero curvature equation (2.3) associated with the discrete spectral problem (2.2). Moreover, under the Bargmann constraint (3.10) between the potentials and the eigenfunctions, the binary nonlinearization of the Lax pairs and the adjoint Lax pairs of the obtained family is presented.

In addition, using the Lax pairs equations (2.2) and (2.10), we can discuss many other solutions for the family (2.12), such as, lump and interaction solutions [5, 29–33, 35, 37, 47, 48, 51, 52], Hirota bilinear technology [4, 16, 46], Darboux transformation [6, 20, 43, 44, 50, 54], algebro-geometric solutions [2, 14, 41], symmetry analysis [21, 42], and so on. Furthermore, it is also interesting and meaningful to discuss some integrable properties [11, 12, 15, 18, 22, 23, 39, 53] and super-integrable coupled equations [19]. These problems will be discussed in the future.

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