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DYNAMICS OF A STOCHASTIC THREE SPECIES PREY-PREDATOR MODEL WITH INTRAGUILD PREDATION*

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Abstract Intraguild predation is ubiquitous in many ecological communities. This paper is concerned with a stochastic three species prey-predator model with intraguild predation. The model involves a prey, an intermediate predator which preys on only prey and an omnivorous top predator which preys on both prey and intermediate predator. First, we show the existence of a unique positive global solution of the model. Then we mainly establish the sufficient conditions for the extinction and persistence in the mean of each population. Moreover, we show that the model is stable in distribution. Finally, some numerical simulations are given to illustrate the main results.

Keywords Stochastic food-web model, predator-prey, intraguild predation, stability in distribution.

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1. Introduction

The dynamic relationship between predators and their preys has been universal in mathematical ecology (see [10, 23]). In recent years, omnivory, which is defined as feeding on more than one trophic level in food chain model, has received significant importance in ecology (see [1, 3, 5]). Intraguild predation is a special kind of omnivory, which is ubiquitous in many ecological communities (see [4]). As can be seen in [22], the three species food chain model with intraguild predation involves a resource, an intermediate predator which feeds upon only prey and a top predator which feeds upon both prey and intermediate predator. In [5], the authors investigated the following three species food web model

$$\begin{cases} \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = x_1(t)\left[r_1 - a_{11}x_1(t) - a_{12}x_2(t) - a_{13}x_3(t)\right], \\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = x_2(t)\left[-r_2 + a_{21}x_1(t) - a_{23}x_3(t)\right], \\ \frac{\mathrm{d}x_3(t)}{\mathrm{d}t} = x_3(t)\left[-r_3 + a_{31}x_1(t) + a_{32}x_2(t)\right], \end{cases} \tag{1.1}$$

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where x_1, x_2, and x_3 denote the sizes of prey, intermediate preadtor, and omnivorous top predator, respectively. r_1 is the growth rate of prey, r_i is the death rate of species x_i $(i = 2, 3)$. a_{11} is the intra-specific competition rate of prey. a_{12}, a_{13} and a_{23} are the capture rates; a_{21}, a_{31} and a_{32} denote the efficiency of food conversion. All coefficients are positive constants.

On the other hand, in the real world, population systems are always affected by the environmental noise. Recently, many authors have paid their attention to stochastic prey-predator models with white noise and revealed how the noise affect the population systems. To name a few, see [6–9, 11–17, 21] and the references therein. In [20], the authors investigated the stationary distribution and global asymptotic stability of the following stochastic three species prey-predator model with intraguild predation

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r_1 - a_{11}x_1(t) - a_{12}x_2(t) - a_{13}x_3(t)\right]\mathrm{d}t + \sigma_1 x_1(t)\mathrm{d}w_1(t), \\ \mathrm{d}x_2(t) = x_2(t)\left[-r_2 + a_{21}x_1(t) - a_{22}x_2(t) - a_{23}x_3(t)\right]\mathrm{d}t + \sigma_2 x_2(t)\mathrm{d}w_2(t), \\ \mathrm{d}x_3(t) = x_3(t)\left[-r_3 + a_{31}x_1(t) + a_{32}x_2(t) - a_{33}x_3(t)\right]\mathrm{d}t + \sigma_3 x_3(t)\mathrm{d}w_3(t), \end{cases} \tag{1.2}$$

with initial value $(x_1(0), x_2(0), x_3(0)) = (x_{10}, x_{20}, x_{30}) \in \mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_i > 0,\ i = 1, 2, 3\}$. All meanings of the parameters are exact to or similar as those for (1.1) except the following. Here $a_{ii} > 0$ is the intra-specific rate of species x_i $(i = 2, 3)$. $w(t) = \{w_1(t), w_2(t), w_3(t) : t \geq 0\}$ represents the three-dimensional standard Brownian motion defined on a compete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. σ_i^2 represents the intensity of noise $w_i(t)$, $i = 1, 2, 3$.

In [20], the authors only discussed the stationary distribution and global asymptotic stability of stochastic model (1.2). However, in this paper, we investigate the persistence, extinction and stability in distribution of the stochastic model (1.2). The complexity of model (1.2) is caused by the omnivorous top predator preying on prey and the intermediate predator. This also makes the analysis of model (1.2) more difficult than in [8, 12].

2. Preliminary

In this section, we give some useful preliminaries for the rest of the paper. Obviously, the corresponding deterministic model of (1.2) is

$$\begin{cases} \frac{\mathrm{d}x_1(t)}{\mathrm{d}t} = x_1(t)\left[r_1 - a_{11}x_1(t) - a_{12}x_2(t) - a_{13}x_3(t)\right], \\ \frac{\mathrm{d}x_2(t)}{\mathrm{d}t} = x_2(t)\left[-r_2 + a_{21}x_1(t) - a_{22}x_2(t) - a_{23}x_3(t)\right], \\ \frac{\mathrm{d}x_3(t)}{\mathrm{d}t} = x_3(t)\left[-r_3 + a_{31}x_1(t) + a_{32}x_2(t) - a_{33}x_3(t)\right], \end{cases} \tag{2.1}$$

with initial value $x_1(0) = x_{10}$, $x_2(0) = x_{20}$, $x_3(0) = x_{30}$. Denote

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ -a_{21} & a_{22} & a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{vmatrix},$$

$$\tilde{D}_1 = \begin{vmatrix} r_1 & a_{12} & a_{13} \\ -r_2 & a_{22} & a_{23} \\ -r_3 & -a_{32} & a_{33} \end{vmatrix}, \quad \tilde{D}_2 = \begin{vmatrix} a_{11} & r_1 & a_{13} \\ -a_{21} & -r_2 & a_{23} \\ -a_{31} & -r_3 & a_{33} \end{vmatrix}, \quad \tilde{D}_3 = \begin{vmatrix} a_{11} & a_{12} & r_1 \\ -a_{21} & a_{22} & -r_2 \\ -a_{31} & -a_{32} & -r_3 \end{vmatrix}.$$

$D > 0$, $\tilde{D}_i > 0$ $(i = 1, 2, 3)$ imply that all the populations in model (2.1) could coexist. Thus, model (2.1) has one interior equilibrium point $E_* = \big(\frac{\tilde{D}_1}{D}, \frac{\tilde{D}_2}{D}, \frac{\tilde{D}_3}{D}\big)$. Model (2.1) has one trivial equilibrium point $E_0 = (0, 0, 0)$ and one axial equilibrium point $E_1 = \big(\frac{r_1}{a_{11}}, 0, 0\big)$ irrespective of any parametric restriction. Two boundary equilibria $E_2 = \big(\frac{a_{33}r_1 + a_{13}r_3}{a_{11}a_{33} + a_{13}a_{31}}, 0, \frac{a_{31}r_1 - a_{11}r_3}{a_{11}a_{33} + a_{13}a_{31}}\big)$ and $E_3 = \big(\frac{a_{22}r_1 + a_{12}r_2}{a_{11}a_{22} + a_{12}a_{21}}, \frac{a_{21}r_1 - a_{11}r_2}{a_{11}a_{22} + a_{12}a_{21}}, 0\big)$ exist when $a_{31}r_1 - a_{11}r_3 > 0$ and $a_{21}r_1 - a_{11}r_2 > 0$ hold respectively.

Now we show that model (1.2) has a unique global positive solution. For the sake of simplification, we denote

$$\mathbb{R} = (-\infty, +\infty), \quad \langle u(t) \rangle = \frac{1}{t}\int_0^t u(s)\mathrm{d}s, \quad \kappa_1 = r_1 - \frac{\sigma_1^2}{2}, \quad \kappa_i = r_i + \frac{\sigma_i^2}{2} \quad (i = 2, 3).$$

Lemma 2.1. *For any initial value $(x_{10}, x_{20}, x_{30}) \in \mathbb{R}_+^3$, model (1.2) has a unique solution $(x_1(t), x_2(t), x_3(t))$ defined on $t \geq 0$ and the solution will remain in $\mathbb{R}_+^3$ with probability one. Moreover, for $p > 0$, if $a_{22} > a_{21}$ and $a_{33} > a_{31} + a_{32}$ there is a constant $K = K(p) > 0$ such that the solution of model (1.2) satisfies*

$$\limsup_{t\to\infty} \mathbb{E}\big[x_i^p(t)\big] \leq K, \quad i = 1, 2, 3. \tag{2.2}$$

Proof. Consider the following system

$$\begin{cases} \mathrm{d}X_1(t) = \big[\kappa_1 - a_{11}e^{X_1(t)} - a_{12}e^{X_2(t)} - a_{13}e^{X_3(t)}\big]\,\mathrm{d}t + \sigma_1 \mathrm{d}w_1(t), \\ \mathrm{d}X_2(t) = \big[-\kappa_2 + a_{21}e^{X_1(t)} - a_{22}e^{X_2(t)} - a_{23}e^{X_3(t)}\big]\,\mathrm{d}t + \sigma_2 \mathrm{d}w_2(t), \\ \mathrm{d}X_3(t) = \big[-\kappa_3 + a_{31}e^{X_1(t)} + a_{32}e^{X_2(t)} - a_{33}e^{X_3(t)}\big]\,\mathrm{d}t + \sigma_3 \mathrm{d}w_3(t), \end{cases} \tag{2.3}$$

with initial value $(X_1(0), X_2(0), X_3(0)) = (\ln x_{10}, \ln x_{20}, \ln x_{30})$. From [20], it follows that the coefficients of (2.3) are locally Lipschitz continuous. Thus, there is a unique maximal local solution $(X_1(t), X_2(t), X_3(t))$ of (2.3) for $t \in [0, \tau_e)$. Let $x_i(t) = e^{X_i(t)}$ $(i = 1, 2, 3)$. Using Itô formula, it follows that $(x_1(t), x_2(t), x_3(t)) = (e^{X_1(t)}, e^{X_2(t)}, e^{X_3(t)})$ is the unique positive local solution of (1.2) with initial value (x_{10}, x_{20}, x_{30}) for $t \in [0, \tau_e)$.

Next, we show that $(X_1(t), X_2(t), X_3(t))$ is a global solution of (2.3), that is $\tau_e = \infty$. Consider the following stochastic differential system

$$\begin{cases} \mathrm{d}\Phi_1(t) = \Phi_1(t)\big[r_1 - a_{11}\Phi_1(t)\big]\mathrm{d}t + \sigma_1\Phi_1(t)\mathrm{d}w_1(t), \\ \mathrm{d}\Phi_2(t) = \Phi_2(t)\big[-r_2 + a_{21}\Phi_1(t) - a_{22}\Phi_2(t)\big]\mathrm{d}t + \sigma_2\Phi_2(t)\mathrm{d}w_2(t), \\ \mathrm{d}\Phi_3(t) = \Phi_3(t)\big[-r_3 + a_{31}\Phi_1(t) + a_{32}\Phi_2(t) - a_{33}\Phi_3(t)\big]\mathrm{d}t + \sigma_3\Phi_3(t)\mathrm{d}w_3(t), \end{cases} \tag{2.4}$$

with initial value $(\Phi_1(0), \Phi_2(0), \Phi_3(0)) = (x_{10}, x_{20}, x_{30}) \in \mathbb{R}_+^3$. Thanks to Lemma

4.2 in [2], system (2.4) can be explicitly solved as follow

$$\begin{cases} \Phi_1(t) = \dfrac{\exp\left\{\kappa_1 t+\sigma_1 w_1(t)\right\}}{\frac{1}{x_{10}}+a_{11}\int_0^t \exp\left\{\kappa_1 s+\sigma_1 w_1(s)\right\}\mathrm{d}s}, \\ \Phi_2(t) = \dfrac{\exp\left\{-\kappa_2 t+\sigma_2 w_2(t)+a_{21}\int_0^t \Phi_1(s)\mathrm{d}s\right\}}{\frac{1}{x_{20}}+a_{22}\int_0^t \exp\left\{-\kappa_2 s+\sigma_2 w_2(s)+a_{21}\int_0^s \Phi_1(\tau)\mathrm{d}\tau\right\}\mathrm{d}s}, \\ \Phi_3(t) = \dfrac{\exp\left\{-\kappa_3 t+\sigma_3 w_3(t)+a_{31}\int_0^t \Phi_1(s)\mathrm{d}s+a_{32}\int_0^t \Phi_2(s)\mathrm{d}s\right\}}{\frac{1}{x_{30}}+a_{33}\int_0^t \exp\left\{-\kappa_3 s+\sigma_3 w_3(s)+a_{31}\int_0^s \Phi_1(\tau)\mathrm{d}\tau+a_{32}\int_0^s \Phi_2(\tau)\mathrm{d}\tau\right\}\mathrm{d}s}. \end{cases}$$

Further, construct the following stochastic differential system

$$\begin{cases} \mathrm{d}\phi_1(t) = \phi_1(t)\big[r_1 - a_{11}\phi_1(t) - a_{12}\Phi_2(t) - a_{13}\Phi_3(t)\big]\mathrm{d}t + \sigma_1\phi_1(t)\mathrm{d}w_1(t), \\ \mathrm{d}\phi_2(t) = \phi_2(t)\big[-r_2 + a_{21}\phi_1(t) - a_{22}\phi_2(t) - a_{23}\Phi_3(t)\big]\mathrm{d}t + \sigma_2\phi_2(t)\mathrm{d}w_2(t), \\ \mathrm{d}\phi_3(t) = \phi_3(t)\big[-r_3 + a_{31}\phi_1(t) + a_{32}\phi_2(t) - a_{33}\phi_3(t)\big]\mathrm{d}t + \sigma_3\phi_3(t)\mathrm{d}w_3(t), \end{cases} \tag{2.5}$$

with initial value $(\phi_1(0),\phi_2(0),\phi_3(0)) = (x_{10},x_{20},x_{30}) \in \mathbb{R}^3_+$. Similarly,

$$\begin{cases} \phi_1(t) = \dfrac{\exp\left\{\kappa_1 t+\sigma_1 w_1(t)-a_{12}\int_0^t \Phi_2(s)\mathrm{d}s-a_{13}\int_0^t \Phi_3(s)\mathrm{d}s\right\}}{\frac{1}{x_{10}}+a_{11}\int_0^t \exp\left\{\kappa_1 s+\sigma_1 w_1(s)-a_{12}\int_0^s \Phi_2(\tau)\mathrm{d}\tau-a_{13}\int_0^s \Phi_3(\tau)\mathrm{d}\tau\right\}\mathrm{d}s}, \\ \phi_2(t) = \dfrac{\exp\left\{-\kappa_2 t+\sigma_2 w_2(t)+a_{21}\int_0^t \phi_1(s)\mathrm{d}s-a_{23}\int_0^t \Phi_3(s)\mathrm{d}s\right\}}{\frac{1}{x_{20}}+a_{22}\int_0^t \exp\left\{-\kappa_2 s+\sigma_2 w_2(s)+a_{21}\int_0^s \phi_1(\tau)\mathrm{d}\tau-a_{23}\int_0^s \Phi_3(\tau)\mathrm{d}\tau\right\}\mathrm{d}s}, \\ \phi_3(t) = \dfrac{\exp\left\{-\kappa_3 t+\sigma_3 w_3(t)+a_{31}\int_0^t \phi_1(s)\mathrm{d}s+a_{32}\int_0^t \phi_2(s)\mathrm{d}s\right\}}{\frac{1}{x_{30}}+a_{33}\int_0^t \exp\left\{-\kappa_3 s+\sigma_3 w_3(s)+a_{31}\int_0^s \phi_1(\tau)\mathrm{d}\tau+a_{32}\int_0^s \phi_2(\tau)\mathrm{d}\tau\right\}\mathrm{d}s}. \end{cases}$$

Note that the local solution $(x_1(t),x_2(t),x_3(t))$ is positive on $[0,\tau_e)$. Then, from the stochastic comparison theorem (see [18]), it follows that for $t \in [0,\tau_e)$,

$$0 < \phi_i(t) \le x_i(t) \le \Phi_i(t) \quad \text{a.s.,} \quad i = 1,2,3.$$

Thus, for $t \in [0,\tau_e)$,

$$\ln\phi_i(t) \le X_i(t) \le \ln\Phi_i(t) \quad \text{a.s.,} \quad i = 1,2,3.$$

Since $\ln\phi_i(t)$ and $\ln\Phi_i(t)$ $(i = 1,2,3)$ exist for every $t \ge 0$, it follows that $\tau_e = \infty$. Thus, for any $(X_1(0),X_2(0),X_3(0)) = (\ln x_{10},\ln x_{20},\ln x_{30}) \in \mathbb{R}^3$, system (2.3) has a unique global solution $(X_1(t),X_2(t),X_3(t))$ on $[0,\infty)$ a.s. Therefore, for any initial value $(x_{10},x_{20},x_{30}) \in \mathbb{R}^3_+$, model (1.2) has a unique global positive solution $(x_1(t),x_2(t),x_3(t)) = (e^{X_1(t)},e^{X_2(t)},e^{X_3(t)})$ on $[0,\infty)$ a.s. Moreover, the above analysis shows that for any $t \in [0,\infty)$

$$0 < \phi_i(t) \le x_i(t) \le \Phi_i(t) \quad \text{a.s.,} \quad i = 1,2,3. \tag{2.6}$$

The proof of (2.2) is standard and hence is omitted (see [20]). Thus, the proof is complete. □

Lemma 2.2 (see [24]). *Assume* $u \in C(\Omega\times[0,+\infty),\mathbb{R}_+)$, $G \in C(\Omega\times[0,+\infty),\mathbb{R})$ *and* $\lim_{t\to\infty}\frac{G(t)}{t} = 0$ a.s.

(I) *If there are* $\varrho_0 > 0$, $T > 0$ *and* ϱ *satisfying*

$$\ln u(t) \le \varrho t - \varrho_0 \int_0^t u(s)\mathrm{d}s + G(t) \quad \text{a.s.}, \ t \ge T,$$

then

$$\begin{cases} \limsup\limits_{t\to\infty} \langle u(t)\rangle \le \frac{\varrho}{\varrho_0} \quad \text{a.s.}, & \text{if } \varrho > 0, \\ \lim\limits_{t\to\infty} \langle u(t)\rangle = 0 \quad \text{a.s.}, & \text{if } \varrho = 0, \\ \lim\limits_{t\to\infty} u(t) = 0 \quad \text{a.s.}, & \text{if } \varrho < 0. \end{cases}$$

(II) *If there exist* $\varrho > 0$, $\varrho_0 > 0$ *and* $T > 0$ *satisfying*

$$\ln u(t) \ge \varrho t - \varrho_0 \int_0^t u(s)\mathrm{d}s + G(t) \quad \text{a.s.}, \ t \ge T,$$

then $\liminf_{t\to\infty} \langle u(t)\rangle \ge \frac{\varrho}{\varrho_0}$ a.s.

Lemma 2.3 (see [7]). *Consider one-dimensional stochastic differential equation*

$$\mathrm{d}x(t) = x(t)\big[a - bx(t)\big]\mathrm{d}t + \sigma x(t)\mathrm{d}w(t), \tag{2.7}$$

where $a > 0$, $b > 0$, $\sigma > 0$, *and* $w(t)$ *is standard Brownian motion. For any* $x_0 > 0$, *let* $x(t)$ *be the solution of (2.7) with initial value* x_0. *If* $a - \frac{\sigma^2}{2} > 0$, *then*

$$\lim_{t\to\infty} \frac{\ln x(t)}{t} = 0, \qquad \lim_{t\to\infty} \langle x(t)\rangle = \frac{a - \frac{\sigma^2}{2}}{b} \quad \text{a.s.}$$

Assumption 1. $a_{21}\kappa_1 - a_{11}\kappa_2 > 0$, $a_{31}a_{22}\kappa_1 + a_{32}a_{21}\kappa_1 - a_{32}a_{11}\kappa_2 - a_{11}a_{22}\kappa_3 > 0$.

From Assumption 1, it is easy to see that $\kappa_1 > 0$.

Lemma 2.4. *Let* $(\Phi_1(t), \Phi_2(t), \Phi_3(t))$ *be the solution of (2.4) with any initial value* (x_{10}, x_{20}, x_{30}). *If Assumption 1 is satisfied, then*

$$\lim_{t\to\infty} \frac{\ln \Phi_i(t)}{t} = 0, \qquad \lim_{t\to\infty} \langle \Phi_i(t)\rangle = M_i \quad \text{a.s.}, \quad i = 1, 2, 3,$$

where

$$\begin{aligned} M_1 &= \frac{\kappa_1}{a_{11}}, \qquad M_2 = \frac{a_{21}\kappa_1 - a_{11}\kappa_2}{a_{11}a_{22}}, \\ M_3 &= \frac{a_{31}a_{22}\kappa_1 + a_{32}a_{21}\kappa_1 - a_{32}a_{11}\kappa_2 - a_{11}a_{22}\kappa_3}{a_{11}a_{22}a_{33}}. \end{aligned}$$

Proof. From Lemma 2.3 and Assumption 1, it follows that

$$\lim_{t\to\infty} \frac{\ln \Phi_1(t)}{t} = 0, \quad \lim_{t\to\infty} \langle \Phi_1(t)\rangle = \frac{\kappa_1}{a_{11}} = M_1 \quad \text{a.s.} \tag{2.8}$$

According to Itô formula, we have

$$\ln \Phi_2(t) = \ln x_{20} - \kappa_2 t + a_{21} \int_0^t \Phi_1(s)\mathrm{d}s - a_{22} \int_0^t \Phi_2(s)\mathrm{d}s + \sigma_2 w_2(t). \tag{2.9}$$

Set $H_1(t) = \ln x_{20} - \kappa_2 t + a_{21}\int_0^t \Phi_1(s)\mathrm{d}s$. Note that

$$\lim_{t\to\infty}\frac{H_1(t)}{t} = \frac{a_{21}\kappa_1 - a_{11}\kappa_2}{a_{11}} > 0 \text{ a.s.}$$

Thus, with Assumption 1, for any $0 < \varepsilon < \frac{a_{21}\kappa_1 - a_{11}\kappa_2}{a_{11}}$, there is a constant $T > 0$ such that $H_1(t) < \big(\frac{a_{21}\kappa_1 - a_{11}\kappa_2}{a_{11}} + \varepsilon\big)t$ and $H_1(t) > \big(\frac{a_{21}\kappa_1 - a_{11}\kappa_2}{a_{11}} - \varepsilon\big)t$ for any $t \geq T$. Therefore, from (2.9), for any $t \geq T$,

$$\ln\Phi_2(t) \leq \left(\frac{a_{21}\kappa_1 - a_{11}\kappa_2}{a_{11}} + \varepsilon\right)t - a_{22}\int_0^t \Phi_2(s)\mathrm{d} + \sigma_2 w_2(t).$$
$$\ln\Phi_2(t) \geq \left(\frac{a_{21}\kappa_1 - a_{11}\kappa_2}{a_{11}} - \varepsilon\right)t - a_{22}\int_0^t \Phi_2(s)\mathrm{d}s + \sigma_2 w_2(t).$$

Thus, from Lemma 2.2, Assumption 1 and arbitrariness of ε,

$$\liminf_{t\to\infty}\langle\Phi_2(t)\rangle \geq \frac{a_{21}\kappa_1 - a_{11}\kappa_2}{a_{11}a_{22}} = M_2 \text{ a.s.,}$$
$$\limsup_{t\to\infty}\langle\Phi_2(t)\rangle \leq \frac{a_{21}\kappa_1 - a_{11}\kappa_2}{a_{11}a_{22}} = M_2 \text{ a.s.}$$

That is,

$$\lim_{t\to\infty}\langle\Phi_2(t)\rangle = M_2 \text{ a.s.} \tag{2.10}$$

From (2.8)–(2.10),

$$\lim_{t\to\infty}\frac{\ln\Phi_2(t)}{t} = \lim_{t\to\infty}\left\{-\kappa_2 + a_{21}\langle\Phi_1(t)\rangle - a_{22}\langle\Phi_2(t)\rangle + \frac{\ln x_{20} + \sigma_2 w_2(t)}{t}\right\} = 0 \text{ a.s.}$$

Similarly, according to Itô formula, we have

$$\ln\Phi_3(t) = H_2(t) - a_{33}\int_0^t \Phi_3(s)\mathrm{d}s + \sigma_3 w_3(t), \tag{2.11}$$

where $H_2(t) = \ln x_{30} - \kappa_3 t + a_{31}\int_0^t \Phi_1(s)\mathrm{d}s + a_{32}\int_0^t \Phi_2(s)\mathrm{d}s$. Thus,

$$\lim_{t\to\infty}\frac{H_2(t)}{t} = \frac{a_{31}a_{22}\kappa_1 + a_{32}a_{21}\kappa_1 - a_{32}a_{11}\kappa_2 - a_{11}a_{22}\kappa_3}{a_{11}a_{22}} > 0 \text{ a.s.}$$

Thus, for any $0 < \varepsilon < \frac{a_{31}a_{22}\kappa_1 + a_{32}a_{21}\kappa_1 - a_{32}a_{11}\kappa_2 - a_{11}a_{22}\kappa_3}{a_{11}a_{22}}$, there is a constant $T > 0$ such that $H_2(t) < \big(\frac{a_{31}a_{22}\kappa_1 + a_{32}a_{21}\kappa_1 - a_{32}a_{11}\kappa_2 - a_{11}a_{22}\kappa_3}{a_{11}a_{22}} + \varepsilon\big)t$ and $H_2(t) > \big(\frac{a_{31}a_{22}\kappa_1 + a_{32}a_{21}\kappa_1 - a_{32}a_{11}\kappa_2 - a_{11}a_{22}\kappa_3}{a_{11}a_{22}} - \varepsilon\big)t$ for any $t \geq T$. Therefore, from (2.11), for any $t \geq T$,

$$\ln\Phi_3(t) \leq \left(\frac{a_{31}a_{22}\kappa_1 + a_{32}a_{21}\kappa_1 - a_{32}a_{11}\kappa_2 - a_{11}a_{22}\kappa_3}{a_{11}a_{22}} + \varepsilon\right)t$$
$$- a_{33}\int_0^t \Phi_3(s)\mathrm{d}s + \sigma_3 w_3(t).$$
$$\ln\Phi_3(t) \geq \left(\frac{a_{31}a_{22}\kappa_1 + a_{32}a_{21}\kappa_1 - a_{32}a_{11}\kappa_2 - a_{11}a_{22}\kappa_3}{a_{11}a_{22}} - \varepsilon\right)t$$

$$- a_{33}\int_0^t \Phi_3(s)\mathrm{d}s + \sigma_3 w_3(t).$$

Thus, from Lemma 2.2, Assumption 1 and the arbitrariness of ε, we get

$$\lim_{t\to\infty}\langle\Phi_3(t)\rangle = \frac{a_{31}a_{22}\kappa_1 + a_{32}a_{21}\kappa_1 - a_{32}a_{11}\kappa_2 - a_{11}a_{22}\kappa_3}{a_{11}a_{22}a_{33}} = M_3 \quad \text{a.s.} \tag{2.12}$$

From (2.8), (2.10)–(2.12),

$$\lim_{t\to\infty}\frac{\ln\Phi_3(t)}{t} = 0 \quad \text{a.s.}$$

The proof is therefore complete. □

From the proof of Lemma 2.4 and (2.6), we have the following result.

Corollary 2.1. *Let $(\Phi_1(t), \Phi_2(t), \Phi_3(t))$ and $(x_1(t), x_2(t), x_3(t))$ be the solution of system (2.4) and model (1.2) with initial value $(x_{10}, x_{20}, x_{30}) \in \mathbb{R}^3_+$, respectively.*

(i) *If $\kappa_1 > 0$, then*

$$\lim_{t\to\infty}\frac{\ln\Phi_1(t)}{t} = 0, \quad \lim_{t\to\infty}\langle\Phi_1(t)\rangle = M_1, \quad \limsup_{t\to\infty}\frac{\ln x_1(t)}{t} \le 0 \quad \text{a.s.}$$

(ii) *If $a_{21}\kappa_1 - a_{11}\kappa_2 > 0$, then*

$$\lim_{t\to\infty}\frac{\ln\Phi_i(t)}{t} = 0, \quad \lim_{t\to\infty}\langle\Phi_i(t)\rangle = M_i, \quad \limsup_{t\to\infty}\frac{\ln x_i(t)}{t} \le 0 \text{ a.s.}, \quad i = 1, 2.$$

(iii) *If Assumption 1 is satisfied, then*

$$\limsup_{t\to\infty}\frac{\ln x_i(t)}{t} \le 0 \quad \text{a.s.}, \quad i = 1, 2, 3.$$

Now, we introduce the following assumption.

Assumption 2. $a_{11} > a_{12} + a_{13}$, $a_{22} > a_{21} + a_{23}$, $a_{33} > a_{31} + a_{32}$.

Form Theorem 13 in [20], we have the following result.

Lemma 2.5. *Let Assumption 2 hold, then (1.2) is globally attractive. That is, for any $x_0 = (x_{10}, x_{20}, x_{30})$ and $\tilde{x}_0 = (\tilde{x}_{10}, \tilde{x}_{20}, \tilde{x}_{30}) \in \mathbb{R}^3_+$, let $x(t) = (x_1(t), x_2(t), x_3(t))$ and $\tilde{x}(t) = (\tilde{x}_1(t), \tilde{x}_2(t), \tilde{x}_3(t))$ be the solutions of model (1.2) with x_0 and $\tilde{x}_0$, respectively. If Assumption 2 is satisfied, then*

$$\lim_{t\to\infty}\mathbb{E}\big|x_i(t) - \tilde{x}_i(t)\big| = 0 \quad i = 1, 2, 3. \tag{2.13}$$

3. Main results

Denote

$$D_1 = \begin{vmatrix} \kappa_1 & a_{12} & a_{13} \\ -\kappa_2 & a_{22} & a_{23} \\ -\kappa_3 & -a_{32} & a_{33} \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_{11} & \kappa_1 & a_{13} \\ -a_{21} & -\kappa_2 & a_{23} \\ -a_{31} & -\kappa_3 & a_{33} \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_{11} & a_{12} & \kappa_1 \\ -a_{21} & a_{22} & -\kappa_2 \\ -a_{31} & -a_{32} & -\kappa_3 \end{vmatrix}.$$

Let A_{ij} be the algebraic cofactor of the $ij-$th element of D. Obviously, $A_{11} > 0$, $A_{21} < 0$, $A_{22} > 0$, $A_{32} < 0$, $A_{13} > 0$, $A_{33} > 0$.

Now, we introduce the following assumption.

Assumption 3. $D > 0$, $D_i > 0$ $(i = 1, 2, 3)$, $A_{23} \geq 0$, $A_{31} \leq 0$, $A_{12} \leq 0$.

Theorem 3.1. *For any $(x_{10}, x_{20}, x_{30}) \in \mathbb{R}^3_+$, let $(x_1(t), x_2(t), x_3(t))$ be the solution of model (1.2) with initial value (x_{10}, x_{20}, x_{30}).*

(i) *If $\kappa_1 < 0$, then*

$$\lim_{t\to\infty} x_i(t) = 0 \quad \text{a.s.}, \quad i = 1, 2, 3.$$

(ii) *If $\kappa_1 > 0$, $a_{21}\kappa_1 - a_{11}\kappa_2 < 0$ and $a_{31}\kappa_1 - a_{11}\kappa_3 < 0$, then*

$$\lim_{t\to\infty} \langle x_1(t)\rangle = \frac{\kappa_1}{a_{11}}, \quad \lim_{t\to\infty} x_2(t) = 0, \quad \lim_{t\to\infty} x_3(t) = 0 \quad \text{a.s.}$$

(iii) *If $a_{21}\kappa_1 - a_{11}\kappa_2 < 0$ and $a_{31}\kappa_1 - a_{11}\kappa_3 > 0$, then*

$$\lim_{t\to\infty} \langle x_1(t)\rangle = \frac{a_{33}\kappa_1 + a_{13}\kappa_3}{a_{11}a_{33} + a_{13}a_{31}}, \quad \lim_{t\to\infty} x_2(t) = 0, \quad \lim_{t\to\infty} \langle x_3(t)\rangle = \frac{a_{31}\kappa_1 - a_{11}\kappa_3}{a_{11}a_{33} + a_{13}a_{31}} \quad \text{a.s.}$$

(iv) *If $a_{21}\kappa_1 - a_{11}\kappa_2 > 0$ and $a_{31}a_{22}\kappa_1 + a_{32}a_{21}\kappa_1 - a_{32}a_{11}\kappa_2 - a_{11}a_{22}\kappa_3 < 0$, then*

$$\lim_{t\to\infty} \langle x_1(t)\rangle = \frac{a_{22}\kappa_1 + a_{12}\kappa_2}{a_{11}a_{22} + a_{12}a_{21}}, \quad \lim_{t\to\infty} \langle x_2(t)\rangle = \frac{a_{21}\kappa_1 - a_{11}\kappa_2}{a_{11}a_{22} + a_{12}a_{21}}, \quad \lim_{t\to\infty} x_3(t) = 0 \text{ a.s.}$$

(v) *If Assumptions 1 and 3 hold, then*

$$\lim_{t\to\infty} \langle x_i(t)\rangle = \frac{D_i}{D} \quad \text{a.s.}, \quad i = 1, 2, 3.$$

Proof. Applying the Itô formula to model (1.2) results in

$$\frac{\ln x_1(t)}{t} = \kappa_1 - a_{11}\langle x_1(t)\rangle - a_{12}\langle x_2(t)\rangle - a_{13}\langle x_3(t)\rangle + \frac{\sigma_1 w_1(t)}{t} + \frac{\ln x_{10}}{t}, \tag{3.1}$$

$$\frac{\ln x_2(t)}{t} = -\kappa_2 + a_{21}\langle x_1(t)\rangle - a_{22}\langle x_2(t)\rangle - a_{23}\langle x_3(t)\rangle + \frac{\sigma_2 w_2(t)}{t} + \frac{\ln x_{20}}{t}, \tag{3.2}$$

$$\frac{\ln x_3(t)}{t} = -\kappa_3 + a_{31}\langle x_1(t)\rangle + a_{32}\langle x_2(t)\rangle - a_{33}\langle x_3(t)\rangle + \frac{\sigma_3 w_3(t)}{t} + \frac{\ln x_{30}}{t}. \tag{3.3}$$

Now let us prove (i) firstly. From (3.1), it follows that

$$\ln x_1(t) \leq \kappa_1 t - a_{11}\int_0^t x_1(s)\mathrm{d}s + \sigma_1 w_1(t) + \ln x_{10}.$$

This, together with $\kappa_1 < 0$ and Lemma 2.2, yields

$$\lim_{t\to\infty} x_1(t) = 0 \quad \text{a.s.}$$

Applying L'Hospital's rule, we have $\lim_{t\to\infty}\langle x_1(t)\rangle = 0$ a.s. Thus,

$$\lim_{t\to\infty} \frac{1}{t}\left[a_{21}\int_0^t x_1(s)\mathrm{d}s + \sigma_2 w_2(t) + \ln x_{20}\right] = 0 \quad \text{a.s.} \tag{3.4}$$

From (3.2), we have

$$\ln x_2(t) \leq -\kappa_2 t - a_{22}\int_0^t x_2(s)\mathrm{d}s + \left[a_{21}\int_0^t x_1(s)\mathrm{d}s + \sigma_2 w_2(t) + \ln x_{20}\right].$$

Thus, from $-\kappa_2 < 0$, (3.4) and Lemma 2.2, it follows that

$$\lim_{t\to\infty} x_2(t) = 0 \ \text{ a.s.}$$

Applying L'Hospital's rule, we also have $\lim_{t\to\infty}\langle x_2(t)\rangle = 0$ a.s. Denote $H_3(t) = a_{31}\int_0^t x_1(s)\mathrm{d}s + a_{32}\int_0^t x_2(s)\mathrm{d}s + \sigma_3 w_3(t) + \ln x_{30}$. Thus,

$$\lim_{t\to\infty}\frac{H_3(t)}{t} = 0 \ \text{ a.s.} \tag{3.5}$$

It follows from (3.3) that

$$\ln x_3(t) = -\kappa_3 t - a_{33}\int_0^t x_3(s)\mathrm{d}s + H_3(t),$$

which, together with $\kappa_3 > 0$, Lemma 2.2 and (3.5), yields

$$\lim_{t\to\infty} x_3(t) = 0 \ \text{ a.s.}$$

Hence, (i) holds.

Now, let us prove (ii). Note that $\kappa_1 > 0$. Thus, from (2.9), we have

$$\frac{\ln\Phi_2(t)}{t} \leq -\kappa_2 + a_{21}\langle\Phi_1(t)\rangle + \frac{\ln x_{20}}{t} + \frac{\sigma_2 w_2(t)}{t}.$$

From Corollary 2.1, it follows that

$$\limsup_{t\to\infty}\frac{\ln\Phi_2(t)}{t} \leq -\kappa_2 + a_{21}\lim_{t\to\infty}\langle\Phi_1(t)\rangle = \frac{a_{21}\kappa_1 - a_{11}\kappa_2}{a_{11}} < 0.$$

This means $\lim_{t\to\infty}\Phi_2(t) = 0$, a.s. Applying L'Hospital's rule, we have

$$\lim_{t\to\infty}\langle\Phi_2(t)\rangle = 0 \ \text{ a.s.} \tag{3.6}$$

Similarly, from (2.11), we have

$$\frac{\ln\Phi_3(t)}{t} \leq -\kappa_3 + a_{31}\langle\Phi_1(t)\rangle + a_{32}\langle\Phi_2(t)\rangle + \frac{\ln x_{30}}{t} + \frac{\sigma_3 w_3(t)}{t}.$$

Thus, from (3.6) and Corollary 2.1, it follows that

$$\begin{aligned}\limsup_{t\to\infty}\frac{\ln\Phi_3(t)}{t} &\leq -\kappa_3 + a_{31}\lim_{t\to\infty}\langle\Phi_1(t)\rangle + a_{32}\lim_{t\to\infty}\langle\Phi_2(t)\rangle \\ &= \frac{a_{31}\kappa_1 - a_{11}\kappa_3}{a_{11}} < 0.\end{aligned}$$

This means $\lim_{t\to\infty}\Phi_3(t) = 0$ a.s. Therefore, from (2.6), we have

$$\lim_{t\to\infty} x_i(t) = 0 \ \text{ a.s.}, \quad i = 2, 3.$$

Applying L'Hospital's rule, it follows that

$$\lim_{t\to\infty}\langle x_i(t)\rangle = 0 \text{ a.s.}, \quad i = 2, 3.$$

Denote $H_4(t) = -a_{12}\int_0^t x_2(s)\mathrm{d}s - a_{13}\int_0^t x_3(s)\mathrm{d}s + \sigma_1 w_1(t) + \ln x_{10}$. Thus,

$$\lim_{t\to\infty}\frac{H_4(t)}{t} = 0 \text{ a.s.} \tag{3.7}$$

It follows from (3.1) that

$$\ln x_1(t) = \kappa_1 t - a_{11}\int_0^t x_1(s)\mathrm{d}s + H_4(t),$$

which, together with (3.7) and Lemma 2.2, yields

$$\lim_{t\to\infty}\langle x_1(t)\rangle = \frac{\kappa_1}{a_{11}} \text{ a.s.}$$

Hence, (ii) holds.

Next, we prove (iii). It follows from $a_{31}\kappa_1 - a_{11}\kappa_3 > 0$ that $\kappa_1 > 0$. From $a_{21}\kappa_1 - a_{11}\kappa_2 < 0$ and the proof of (ii), we have

$$\lim_{t\to\infty} x_2(t) = 0, \quad \lim_{t\to\infty}\langle x_2(t)\rangle = 0 \text{ a.s.} \tag{3.8}$$

Computing (3.1) $\times a_{31}$ + (3.3) $\times a_{11}$, one can derive that

$$a_{11}\frac{\ln x_3(t)}{t} =(a_{31}\kappa_1 - a_{11}\kappa_3) - (a_{11}a_{33} + a_{13}a_{31})\langle x_3(t)\rangle + H_5(t) - a_{31}\frac{\ln x_1(t)}{t},$$

where $H_5(t) = (a_{11}a_{32} - a_{12}a_{31})\langle x_2(t)\rangle + a_{31}\big(\frac{\sigma_1 w_1(t)}{t} + \frac{\ln x_{10}}{t}\big) + a_{11}\big(\frac{\sigma_3 w_3(t)}{t} + \frac{\ln x_{30}}{t}\big)$. From (3.8), it follows that $\lim_{t\to\infty} H_5(t) = 0$ a.s. Moreover, from Corollary 2.1, we have $\limsup_{t\to\infty}\frac{\ln x_1(t)}{t} \le 0$ a.s. Thus, for any $0 < \varepsilon < a_{31}\kappa_1 - a_{11}\kappa_3$, there is a constant $T > 0$ such that $a_{31}\frac{\ln x_1(t)}{t} < \varepsilon$ for $t \ge T$. Thus, for any $t \ge T$,

$$a_{11}\frac{\ln x_3(t)}{t} \ge(a_{31}\kappa_1 - a_{11}\kappa_3 - \varepsilon) - (a_{11}a_{33} + a_{13}a_{31})\langle x_3(t)\rangle + H_5(t).$$

This, together with Lemma 2.2 and the arbitrariness of ε, yields

$$\liminf_{t\to\infty}\langle x_3(t)\rangle \ge \frac{a_{31}\kappa_1 - a_{11}\kappa_3}{a_{11}a_{33} + a_{13}a_{31}} \text{ a.s.} \tag{3.9}$$

Namely, for every $0 < \varepsilon < \frac{a_{31}\kappa_1 - a_{11}\kappa_3}{a_{11}a_{33}+a_{13}a_{31}}$, there is a constant $T > 0$ such that $a_{13}\langle x_3(t)\rangle \ge a_{13}\frac{a_{31}\kappa_1 - a_{11}\kappa_3}{a_{11}a_{33}+a_{13}a_{31}} - \varepsilon$ for $t \ge T$. Thus, for any $t \ge T$, from (3.1),

$$\frac{\ln x_1(t)}{t} \le\frac{a_{11}(a_{33}\kappa_1 + a_{13}\kappa_3)}{a_{11}a_{33} + a_{13}a_{31}} + \varepsilon - a_{11}\langle x_1(t)\rangle + \frac{\sigma_1 w_1(t)}{t} + \frac{\ln x_{10}}{t}.$$

Applying Lemma 2.2 and the arbitrariness of ε, it follows that

$$\limsup_{t\to\infty}\langle x_1(t)\rangle \le \frac{a_{33}\kappa_1 + a_{13}\kappa_3}{a_{11}a_{33} + a_{13}a_{31}} \text{ a.s.} \tag{3.10}$$

This, together with (3.8), yields that for any $\varepsilon > 0$, there is a constant $T > 0$ such that $a_{32}\langle x_2(t)\rangle < \varepsilon$ and $a_{31}\langle x_1(t)\rangle < a_{31}\frac{a_{33}\kappa_1 + a_{13}\kappa_3}{a_{11}a_{33}+a_{13}a_{31}} + \varepsilon$ for $t \geq T$. Thus, for any $t \geq T$, from (3.3), it follows that

$$\frac{\ln x_3(t)}{t} \leq \frac{a_{33}(a_{31}\kappa_1 - a_{11}\kappa_3)}{a_{11}a_{33}+a_{13}a_{31}} + 2\varepsilon - a_{33}\langle x_3(t)\rangle + \frac{\sigma_3 w_3(t)}{t} + \frac{\ln x_{30}}{t}.$$

From Lemma 2.2 and the arbitrariness of ε, we have

$$\limsup_{t\to\infty}\langle x_3(t)\rangle \leq \frac{a_{31}\kappa_1 - a_{11}\kappa_3}{a_{11}a_{33}+a_{13}a_{31}} \quad \text{a.s.} \tag{3.11}$$

This, together with (3.8), yields that for any $0 < \varepsilon < \frac{a_{11}(a_{33}\kappa_1 + a_{13}\kappa_3)}{2(a_{11}a_{33}+a_{13}a_{31})}$, there is a positive constant T such that $a_{12}\langle x_2(t)\rangle < \varepsilon$ and $a_{13}\langle x_3(t)\rangle < a_{13}\frac{a_{31}\kappa_1 - a_{11}\kappa_3}{a_{11}a_{33}+a_{13}a_{31}} + \varepsilon$ for $t \geq T$. Thus, for any $t \geq T$, from (3.1), it follows that

$$\frac{\ln x_1(t)}{t} \geq \frac{a_{11}(a_{33}\kappa_1 + a_{13}\kappa_3)}{a_{11}a_{33}+a_{13}a_{31}} - 2\varepsilon - a_{11}\langle x_1(t)\rangle + \frac{\sigma_1 w_1(t)}{t} + \frac{\ln x_{10}}{t}.$$

Applying Lemma 2.2 and the arbitrariness of ε, we have

$$\liminf_{t\to\infty}\langle x_1(t)\rangle \geq \frac{a_{33}\kappa_1 + a_{13}\kappa_3}{a_{11}a_{33}+a_{13}a_{31}} \quad \text{a.s.} \tag{3.12}$$

Therefore, from (3.9)–(3.12), it follows that

$$\lim_{t\to\infty}\langle x_1(t)\rangle = \frac{a_{33}\kappa_1 + a_{13}\kappa_3}{a_{11}a_{33}+a_{13}a_{31}}, \quad \lim_{t\to\infty}\langle x_3(t)\rangle = \frac{a_{31}\kappa_1 - a_{11}\kappa_3}{a_{11}a_{33}+a_{13}a_{31}} \quad \text{a.s.}$$

Hence, (iii) holds. The proof of (iv) is similar to (iii), and hence is omitted.

At last, let us prove (v). Denote

$$F_i(t) = -\frac{\ln x_i(t)}{t} + \frac{\sigma_i w_i(t)}{t} + \frac{\ln x_{i0}}{t}, \quad i = 1, 2, 3.$$

Then, from (3.1)–(3.3), it follows that

$$\langle x_1(t)\rangle = \frac{D_1 + A_{11}F_1(t) + A_{21}F_2(t) + A_{31}F_3(t)}{D}, \tag{3.13}$$

$$\langle x_2(t)\rangle = \frac{D_2 + A_{12}F_1(t) + A_{22}F_2(t) + A_{32}F_3(t)}{D}, \tag{3.14}$$

$$\langle x_3(t)\rangle = \frac{D_3 + A_{13}F_1(t) + A_{23}F_2(t) + A_{33}F_3(t)}{D}. \tag{3.15}$$

It follows from (3.15) that

$$\frac{A_{33}}{D}\frac{\ln x_3(t)}{t} = \frac{D_3}{D} - \langle x_3(t)\rangle - \frac{A_{13}}{D}\frac{\ln x_1(t)}{t} - \frac{A_{23}}{D}\frac{\ln x_2(t)}{t} + H_6(t),$$

where $H_6(t) = \frac{A_{13}}{D}\left(\frac{\ln x_{10}}{t} + \frac{\sigma_1 w_1(t)}{t}\right) + \frac{A_{23}}{D}\left(\frac{\ln x_{20}}{t} + \frac{\sigma_2 w_2(t)}{t}\right) + \frac{A_{33}}{D}\left(\frac{\ln x_{30}}{t} + \frac{\sigma_3 w_3(t)}{t}\right)$. Clearly, $\lim_{t\to\infty} H_6(t) = 0$ a.s. Since Assumption 1 holds, Corollary 2.1 implies $\limsup_{t\to\infty}\frac{\ln x_i(t)}{t} \leq 0$ a.s., $i = 1, 2, 3$. Note that $A_{13} > 0$ and $A_{23} \geq 0$. Thus, for any $0 < \varepsilon < \frac{D_3}{D}$, there is a constant $T > 0$ such that $\frac{A_{13}}{D}\frac{\ln x_1(t)}{t} + \frac{A_{23}}{D}\frac{\ln x_2(t)}{t} < \varepsilon$ for $t \geq T$. Thus, for any $t \geq T$,

$$\frac{A_{33}}{D}\frac{\ln x_3(t)}{t} \geq \frac{D_3}{D} - \varepsilon - \langle x_3(t)\rangle + H_6(t).$$

This, together with Lemma 2.2 and the arbitrariness of ε, yields

$$\liminf_{t\to\infty}\langle x_3(t)\rangle \geq \frac{D_3}{D} \quad \text{a.s.} \tag{3.16}$$

From (3.13), it follows that

$$\frac{A_{11}}{D}\frac{\ln x_1(t)}{t} = \frac{D_1}{D} - \langle x_1(t)\rangle - \frac{A_{21}}{D}\frac{\ln x_2(t)}{t} - \frac{A_{31}}{D}\frac{\ln x_3(t)}{t} + H_7(t).$$

Here $H_7(t) = \frac{A_{11}}{D}\left(\frac{\ln x_{10}}{t} + \frac{\sigma_1 w_1(t)}{t}\right) + \frac{A_{21}}{D}\left(\frac{\ln x_{20}}{t} + \frac{\sigma_2 w_2(t)}{t}\right) + \frac{A_{31}}{D}\left(\frac{\ln x_{30}}{t} + \frac{\sigma_3 w_3(t)}{t}\right)$. Clearly, $\lim_{t\to\infty} H_7(t) = 0$ a.s. Note that $A_{21} < 0$ and $A_{31} \leq 0$. From Corollary 2.1, for any $\varepsilon > 0$, there is a constant $T > 0$ such that $-\frac{A_{21}}{D}\frac{\ln x_2(t)}{t} - \frac{A_{31}}{D}\frac{\ln x_3(t)}{t} < \varepsilon$ for $t \geq T$. Thus, for any $t \geq T$,

$$\frac{A_{11}}{D}\frac{\ln x_1(t)}{t} \leq \frac{D_1}{D} + \varepsilon - \langle x_1(t)\rangle + H_7(t).$$

This, together with Lemma 2.2 and the arbitrariness of ε, yields

$$\limsup_{t\to\infty}\langle x_1(t)\rangle \leq \frac{D_1}{D} \quad \text{a.s.} \tag{3.17}$$

Similarly, from $A_{12} \leq 0$ and $A_{32} < 0$, we also have

$$\limsup_{t\to\infty}\langle x_2(t)\rangle \leq \frac{D_2}{D} \quad \text{a.s.} \tag{3.18}$$

Namely, for every $\varepsilon > 0$, there is a constant $T > 0$ such that $a_{31}\langle x_1(t)\rangle \leq a_{31}\frac{D_1}{D} + \varepsilon$ and $a_{32}\langle x_2(t)\rangle \leq a_{32}\frac{D_2}{D} + \varepsilon$ for $t \geq T$. Thus, for any $t \geq T$, from (3.3),

$$\frac{\ln x_3(t)}{t} \leq a_{33}\frac{D_3}{D} + 2\varepsilon - a_{33}\langle x_3(t)\rangle + \frac{\sigma_3 w_3(t)}{t} + \frac{\ln x_{30}}{t}.$$

Applying Lemma 2.2 and the arbitrariness of ε, it follows that

$$\limsup_{t\to\infty}\langle x_3(t)\rangle \leq \frac{D_3}{D} \quad \text{a.s.} \tag{3.19}$$

From (3.18) and (3.19), for any $0 < \varepsilon < \frac{a_{11}D_1}{2D}$, there is a constant $T > 0$ such that $a_{12}\langle x_2(t)\rangle \leq a_{12}\frac{D_2}{D} + \varepsilon$ and $a_{13}\langle x_3(t)\rangle \leq a_{13}\frac{D_3}{D} + \varepsilon$ for $t \geq T$. Thus, for any $t \geq T$, from (3.1),

$$\frac{\ln x_1(t)}{t} \geq a_{11}\frac{D_1}{D} - 2\varepsilon - a_{11}\langle x_1(t)\rangle + \frac{\sigma_1 w_1(t)}{t} + \frac{\ln x_{10}}{t}.$$

Applying Lemma 2.2 and the arbitrariness of ε, it follows that

$$\liminf_{t\to\infty}\langle x_1(t)\rangle \geq \frac{D_1}{D} \quad \text{a.s.} \tag{3.20}$$

From (3.19) and (3.20), for any $0 < \varepsilon < \frac{a_{22}D_2}{2D}$, there is a constant $T > 0$ such that $a_{21}\langle x_1(t)\rangle \geq a_{21}\frac{D_1}{D} - \varepsilon$ and $a_{23}\langle x_3(t)\rangle \leq a_{23}\frac{D_3}{D} + \varepsilon$ for $t \geq T$. Thus, for any $t \geq T$, from (3.2),

$$\frac{\ln x_2(t)}{t} \geq a_{22}\frac{D_2}{D} - 2\varepsilon - a_{22}\langle x_2(t)\rangle + \frac{\sigma_2 w_2(t)}{t} + \frac{\ln x_{20}}{t}.$$

Applying Lemma 2.2 and the arbitrariness of ε, it follows that

$$\liminf_{t\to\infty}\langle x_2(t)\rangle \geq \frac{D_2}{D} \quad \text{a.s.} \tag{3.21}$$

From (3.16)–(3.21), we have the result. Hence, (v) holds. □

Theorem 3.2. *If Assumption 2 holds, then model (1.2) is stable in distribution.*

Proof. Let $A \in \mathcal{B}(\mathbb{R}^3_+)$, and $x(t;x_0)$ be the solution of (1.2) corresponding to $x(0) = x_0 \in \mathbb{R}^3_+$. Let $p(t, x_0, \cdot)$ be the transition probability of $x(t;x_0)$ and

$$P(t, x_0, A) = \mathbb{P}\{x(t;x_0) \in A\} = \int_A p(t, x_0, \mathrm{d}\eta).$$

Let $\mathcal{P}(\mathbb{R}^3_+)$ be the family of probability measures on the measurable space $(\mathbb{R}^3_+, \mathcal{B}(\mathbb{R}^3_+))$. For $P_1, P_2 \in \mathcal{P}(\mathbb{R}^3_+)$, define the metric

$$\mathrm{d}_{\mathbb{S}}(P_1, P_2) = \sup_{f\in\mathbb{S}}\left|\int_{\mathbb{R}^3_+} f(s)P_1(\mathrm{d}s) - \int_{\mathbb{R}^3_+} f(s)P_2(\mathrm{d}s)\right|,$$

where

$$\mathbb{S} = \left\{f : \mathbb{R}^3_+ \to \mathbb{R}\,\middle|\,|f(s_1) - f(s_2)| \leq |s_1 - s_2| \text{ and } |f(\cdot)| \leq 1 \text{ for } s_1, s_2 \in \mathbb{R}^3_+\right\}.$$

Thus, $(\mathcal{P}(\mathbb{R}^3_+), \mathrm{d}_{\mathbb{S}})$ is a complete metric space. For any $x_0 \in \mathbb{R}^3_+$, $f \in \mathbb{S}$ and $t, s > 0$,

$$\begin{aligned}|\mathbb{E}f(x(t+s;x_0)) - \mathbb{E}f(x(t;x_0))| &= \left|\int_{\mathbb{R}^3_+} \mathbb{E}f(x(t;y_0))p(s, x_0, \mathrm{d}y_0) - \mathbb{E}f(x(t;x_0))\right| \\ &\leq \int_{B_\theta} \big|\mathbb{E}f(x(t;x_0)) - \mathbb{E}f(x(t;y_0))\big|p(s, x_0, \mathrm{d}y_0) \\ &\quad + 2P(s, x_0, \bar{B}_\theta),\end{aligned} \tag{3.22}$$

where $\theta > 0$, $x_0 \in B_\theta = \{x \in \mathbb{R}^3_+ : \frac{1}{\theta} \leq |x| \leq \theta\}$ and $\bar{B}_\theta = \mathbb{R}^3_+ - B_\theta$. According to Chebyshev's inequality and Lemma 2.1, $\{p(t, x_0, \mathrm{d}\eta) : t \geq 0\}$ is tight. Thus, there is a sufficiently large θ satisfying

$$P(s, x_0, \bar{B}_\theta) \leq \frac{\varepsilon}{4}, \qquad s > 0. \tag{3.23}$$

From Lemma 2.5, for any $y_0 \in B_\theta$, there is $T > 0$ satisfying

$$\mathbb{E}\big|x(t;x_0) - x(t;y_0)\big| \leq \frac{\varepsilon}{2},\ t > T.$$

For any $f \in \mathbb{S}$ and $t > T$, from the inequality $|\mathbb{E}x| \leq \mathbb{E}|x|$,

$$\int_{B_\theta} \big|\mathbb{E}f(x(t;x_0)) - \mathbb{E}f(x(t;y_0))\big|p(s, x_0, \mathrm{d}y_0)$$

$$\leq \int_{B_\theta} \mathbb{E}\big|x(t;x_0) - x(t;y_0)\big| p(s, x_0, \mathrm{d}y_0)$$
$$\leq \frac{\varepsilon}{2} \int_{B_\theta} p(s, x_0, \mathrm{d}y_0) \leq \frac{\varepsilon}{2}. \tag{3.24}$$

From (3.22), (3.23), (3.24) and the arbitrariness of f, we derive immediately

$$\mathrm{d}_{\mathbb{S}}(p(t+s, x_0, \cdot), p(t, x_0, \cdot)) \leq \varepsilon, \quad \text{for any} \quad t > T, \ s > 0.$$

In other words, for any $x_0 \in \mathbb{R}^3_+$, $\{p(t, x_0, \cdot) : t \geq 0\}$ is a Cauchy sequence in $(\mathcal{P}(\mathbb{R}^3_+), \mathrm{d}_{\mathbb{S}})$. Hence $\{p(t, z_0, \cdot) : t \geq 0\}$ is a Cauchy sequence in $(\mathcal{P}(\mathbb{R}^3_+), \mathrm{d}_{\mathbb{S}})$, where $z_0 = (0.001, 0.001, 0.001)$. So, there is a unique $\nu(\cdot) \in \mathcal{P}(\mathbb{R}^3_+)$ satisfying

$$\lim_{t\to\infty} \mathrm{d}_{\mathbb{S}}(p(t, z_0, \cdot), \nu(\cdot)) = 0. \tag{3.25}$$

By Lemma 2.5, we have

$$\lim_{t\to\infty} \mathrm{d}_{\mathbb{S}}(p(t, x_0, \cdot), p(t, z_0, \cdot)) = 0. \tag{3.26}$$

Further, from triangle inequality, (3.25) and (3.26), it follows that

$$\limsup_{t\to\infty} \ \mathrm{d}_{\mathbb{S}}(p(t, x_0, \cdot), \nu(\cdot)) \leq \lim_{t\to\infty} \mathrm{d}_{\mathbb{S}}(p(t, x_0, \cdot), p(t, z_0, \cdot)) + \lim_{t\to\infty} \mathrm{d}_{\mathbb{S}}(p(t, z_0, \cdot), \nu(\cdot))$$
$$= 0.$$

Hence, $\lim_{t\to\infty} \mathrm{d}_{\mathbb{S}}(p(t, x_0, \cdot), \nu(\cdot)) = 0$, which is the desired assertion. □

Theorem 3.3. *For any $x_0 \in \mathbb{R}^3_+$, let $x(t)$ be the solution of model (1.2) with initial value x_0. Under Assumption 2, we have the following results:*

(i') *If $\kappa_1 < 0$, then*

$$\lim_{t\to\infty} x_i(t) = 0 \quad \text{a.s.}, \quad i = 1, 2, 3.$$

(ii') *If $\kappa_1 > 0$, $a_{21}\kappa_1 - a_{11}\kappa_2 < 0$ and $a_{31}\kappa_1 - a_{11}\kappa_3 < 0$, then* $\lim_{t\to\infty} x_i(t) = 0$ a.s., $i = 2, 3$, *and there is a unique ergodic invariant distribution μ_1 such that the distributions of $x_1(t)$ converge weakly to μ_1 and*

$$\lim_{t\to\infty} \langle x_1(t) \rangle = \int_{\mathbb{R}_+} z_1 \mu_1(\mathrm{d}z_1) = \frac{\kappa_1}{a_{11}} \quad \text{a.s.}$$

(iii') *If $a_{21}\kappa_1 - a_{11}\kappa_2 < 0$ and $a_{31}\kappa_1 - a_{11}\kappa_3 > 0$, then* $\lim_{t\to\infty} x_2(t) = 0$ a.s., *and there is a unique ergodic invariant distribution μ_2 such that the distributions of $(x_1(t), x_3(t))$ converge weakly to μ_2 and*

$$\lim_{t\to\infty} \langle x_1(t) \rangle = \int_{\mathbb{R}^2_+} z_1 \mu_2(\mathrm{d}z_1, \mathrm{d}z_3) = \frac{a_{33}\kappa_1 + a_{13}\kappa_3}{a_{11}a_{33} + a_{13}a_{31}} \quad \text{a.s.},$$
$$\lim_{t\to\infty} \langle x_3(t) \rangle = \int_{\mathbb{R}^2_+} z_3 \mu_2(\mathrm{d}z_1, \mathrm{d}z_3) = \frac{a_{31}\kappa_1 - a_{11}\kappa_3}{a_{11}a_{33} + a_{13}a_{31}} \quad \text{a.s.}$$

(iv') *If $a_{21}\kappa_1 - a_{11}\kappa_2 > 0$ and $a_{31}a_{22}\kappa_1 + a_{32}a_{21}\kappa_1 - a_{32}a_{11}\kappa_2 - a_{11}a_{22}\kappa_3 < 0$, then* $\lim_{t\to\infty} x_3(t) = 0$ a.s., *and there is a unique ergodic invariant distribution μ_3 such that the distributions of $(x_1(t), x_2(t))$ converge weakly to μ_3 and*

$$\lim_{t\to\infty} \langle x_1(t) \rangle = \int_{\mathbb{R}^2_+} z_1 \mu_3(\mathrm{d}z_1, \mathrm{d}z_2) = \frac{a_{22}\kappa_1 + a_{12}\kappa_2}{a_{11}a_{22} + a_{12}a_{21}} \quad \text{a.s.},$$

$$\lim_{t\to\infty}\langle x_2(t)\rangle = \int_{\mathbb{R}^2_+} z_1\mu_3(\mathrm{d}z_1,\mathrm{d}z_2) = \frac{a_{21}\kappa_1 - a_{11}\kappa_2}{a_{11}a_{22}+a_{12}a_{21}} \quad \text{a.s.}$$

(v') *If Assumptions 1 and 3 hold, then there is a unique ergodic invariant distribution μ_4 such that the distributions of solution $(x_1(t), x_2(t), x_3(t))$ converge weakly to μ_4 and*

$$\lim_{t\to\infty}\langle x_i(t)\rangle = \int_{\mathbb{R}^3_+} z_i\mu_4(\mathrm{d}z_1,\mathrm{d}z_2,\mathrm{d}z_3) = \frac{D_i}{D} \quad \text{a.s.},\ i=1,2,3.$$

Proof. First of all, let us prove (v'). In view of (v) in Theorem 3.1, we have

$$\lim_{t\to\infty}\langle x_i(t)\rangle = \frac{D_i}{D} \quad \text{a.s.}, \quad i=1,2,3. \tag{3.27}$$

At the same time, Theorem 3.2 means that model (1.2) is stable in distribution. Then, there is a unique probability measure, denoted by μ_4 such that $p(t, x_0, \cdot)$ of $(x_1(t), x_2(t), x_3(t))$ converges weakly to μ_4. An application of Kolmogorov-Chapman equation that μ_4 is invariant. From Corollary 3.4.3 in [19], it follows that μ_4 is strong mixing, and hence μ_4 is ergodic (see [19]). According to (3.3.2) in [19] and (3.27), we obtain

$$\lim_{t\to\infty}\langle x_i(t)\rangle = \int_{\mathbb{R}^3_+} z_i\mu_4(\mathrm{d}z_1,\mathrm{d}z_2,\mathrm{d}z_3) = \frac{D_i}{D} \quad \text{a.s.},\ i=1,2,3.$$

Now let us show (iv'). According to (iv) in Theorem 3.2,

$$\lim_{t\to\infty}\langle x_1(t)\rangle = \frac{a_{22}\kappa_1 + a_{12}\kappa_2}{a_{11}a_{22}+a_{12}a_{21}},\ \lim_{t\to\infty}\langle x_2(t)\rangle = \frac{a_{21}\kappa_1 - a_{11}\kappa_2}{a_{11}a_{22}+a_{12}a_{21}},\ \lim_{t\to\infty} x_3(t) = 0 \text{ a.s.}$$

Then model (1.2) reduces to the following predator-pery model

$$\begin{cases} \mathrm{d}\hat{x}_1(t) = \hat{x}_1(t)\left[r_1 - a_{11}\hat{x}_1(t) - a_{12}\hat{x}_2(t)\right]\mathrm{d}t + \sigma_1\hat{x}_1(t)\mathrm{d}w_1(t), \\ \mathrm{d}\hat{x}_2(t) = \hat{x}_2(t)\left[-r_2 + a_{21}\hat{x}_1(t) - a_{22}\hat{x}_2(t)\right]\mathrm{d}t + \sigma_2\hat{x}_2(t)\mathrm{d}w_2(t), \end{cases} \tag{3.28}$$

with initial value $\hat{x}_1(0) = x_{10}$, $\hat{x}_2(0) = x_{20}$. Similar to the proof of (v'), there is a unique ergodic invariant distribution denoted by μ_3 such that the transition probability of $(\hat{x}_1(t), \hat{x}_2(t))$ converges weakly to μ_3. Note that $\lim_{t\to\infty} x_3(t) = 0$ a.s. Thus, $(x_1(t), x_2(t))$ has the same asymptotic properties with the solution $(\hat{x}_1(t), \hat{x}_2(t))$ of (3.28). This completes the proof of (iv').

The proof of (iii') and (ii') are omitted for the same reason given above. By (i) in Theorem 3.2, (i') holds. The proof is therefore complete. □

If $a_{13} = a_{23} = a_{31} = a_{32} = a_{33} = r_3 = \sigma_3 \equiv 0$, then model (1.2) can be degraded into the following stochastic predator-prey model

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r_1 - a_{11}x_1(t) - a_{12}x_2(t)\right]\mathrm{d}t + \sigma_1 x_1(t)\mathrm{d}w_1(t), \\ \mathrm{d}x_2(t) = x_2(t)\left[-r_2 + a_{21}x_1(t) - a_{22}x_2(t)\right]\mathrm{d}t + \sigma_2 x_2(t)\mathrm{d}w_2(t), \end{cases} \tag{3.29}$$

with initial value $(x_1(0), x_2(0)) = (x_{10}, x_{20}) \in \mathbb{R}^2_+$. For model (3.29), from the proof of Theorem 3.1, we have the following result.

Corollary 3.1. *For any $(x_{10}, x_{20}) \in \mathbb{R}^2_+$, let $(x_1(t), x_2(t))$ be solution of (3.29) with initial value (x_{10}, x_{20}).*

(i) *If $\kappa_1 < 0$, then*

$$\lim_{t\to\infty} x_i(t) = 0 \quad \text{a.s.}, \quad i = 1, 2.$$

(ii) *If $\kappa_1 > 0$ and $a_{21}\kappa_1 - a_{11}\kappa_2 < 0$, then*

$$\lim_{t\to\infty} \langle x_1(t)\rangle = \frac{\kappa_1}{a_{11}}, \qquad \lim_{t\to\infty} x_2(t) = 0 \quad \text{a.s.}$$

(iii) *If $a_{21}\kappa_1 - a_{11}\kappa_2 > 0$, then*

$$\lim_{t\to\infty} \langle x_1(t)\rangle = \frac{a_{22}\kappa_1 + a_{12}\kappa_2}{a_{11}a_{22} + a_{12}a_{21}}, \quad \lim_{t\to\infty} \langle x_2(t)\rangle = \frac{a_{21}\kappa_1 - a_{11}\kappa_2}{a_{11}a_{22} + a_{12}a_{21}} \quad \text{a.s.}$$

Remark 3.1. If $\tau_1 = \tau_2 = 0$ in model (SM) in [13], then Corollary 3.1 is consistent with Theorem 1 in [13]. Moreover, if one considers a stochastic three species prey-predator model with intraguild predation, from Theorem 3.1, the conditions for population extinction and persistence will be more complicated.

Further, if $a_{13} = a_{31} \equiv 0$, then one can get the following stochastic food chain model

$$\begin{cases} \mathrm{d}x_1(t) = x_1\left[r_1 - a_{11}x_1(t) - a_{12}x_2(t)\right]\mathrm{d}t + \sigma_1 x_1(t)\mathrm{d}w_1(t), \\ \mathrm{d}x_2(t) = x_2\left[-r_2 + a_{21}x_1(t) - a_{22}x_2(t) - a_{23}x_3(t)\right]\mathrm{d}t + \sigma_2 x_2(t)\mathrm{d}w_2(t), \\ \mathrm{d}x_3(t) = x_3\left[-r_3 + a_{32}x_2(t) - a_{33}x_3(t)\right]\mathrm{d}t + \sigma_3 x_3(t)\mathrm{d}w_3(t), \end{cases} \tag{3.30}$$

with initial value $(x_1(0), x_2(0), x_3(0)) = (x_{10}, x_{20}, x_{30}) \in \mathbb{R}^3_+$. For model (3.30), from the proof of Theorem 3.1, we have the following result.

Corollary 3.2. *For any $(x_{10}, x_{20}, x_{30}) \in \mathbb{R}^3_+$, let $(x_1(t), x_2(t), x_3(t))$ be the solution of model (3.30) with initial value (x_{10}, x_{20}, x_{30}).*

(i) *If $\kappa_1 < 0$, then*

$$\lim_{t\to\infty} x_i(t) = 0 \quad \text{a.s.}, \quad i = 1, 2, 3.$$

(ii) *If $\kappa_1 > 0$ and $a_{21}\kappa_1 - a_{11}\kappa_2 < 0$, then*

$$\lim_{t\to\infty} \langle x_1(t)\rangle = \frac{\kappa_1}{a_{11}}, \quad \lim_{t\to\infty} x_2(t) = 0, \quad \lim_{t\to\infty} x_3(t) = 0 \quad \text{a.s.}$$

(iii) *If $a_{21}\kappa_1 - a_{11}\kappa_2 > 0$ and $a_{32}a_{21}\kappa_1 - a_{32}a_{11}\kappa_2 - a_{11}a_{22}\kappa_3 < 0$, then*

$$\lim_{t\to\infty} \langle x_1(t)\rangle = \frac{a_{22}\kappa_1 + a_{12}\kappa_2}{a_{11}a_{22} + a_{12}a_{21}}, \ \lim_{t\to\infty} \langle x_2(t)\rangle = \frac{a_{21}\kappa_1 - a_{11}\kappa_2}{a_{11}a_{22} + a_{12}a_{21}}, \ \lim_{t\to\infty} x_3(t) = 0 \text{ a.s.}$$

Remark 3.2. For model (3.30), it follows from Corollary 3.2 that the extinction of predator x_2 can lead to the extinction of predator x_3. However, for model (1.2), it follows from (iii) in Theorem 1 that under certain conditions, even if intermediate predator x_2 goes extinct, top predator x_3 can be persistent in mean. Thus, omnivory has great effects on the population dynamics.

Remark 3.3. In [25], the authors investigated the stability in the mean of a stochastic three species food chain model with general Lévy jumps. If $c_i(u) = 0$ $(i = 1, 2, 3)$, then we get stochastic model (3.30). Denote

$$E = \begin{vmatrix} a_{11} & a_{12} & 0 \\ -a_{21} & a_{22} & a_{23} \\ 0 & -a_{32} & a_{33} \end{vmatrix},$$

$$E_1 = \begin{vmatrix} \kappa_1 & a_{12} & 0 \\ -\kappa_2 & a_{22} & a_{23} \\ -\kappa_3 & -a_{32} & a_{33} \end{vmatrix}, \quad E_2 = \begin{vmatrix} a_{11} & \kappa_1 & 0 \\ -a_{21} & -\kappa_2 & a_{23} \\ 0 & -\kappa_3 & a_{33} \end{vmatrix}, \quad E_3 = \begin{vmatrix} a_{11} & a_{12} & \kappa_1 \\ -a_{21} & a_{22} & -\kappa_2 \\ 0 & -a_{32} & -\kappa_3 \end{vmatrix}.$$

From Theorem 3.1 in [25], if $E_i > 0$ $(i = 1, 2, 3)$, then model (3.30) is globally stable in the mean with probability one. That is, for any $(x_{10}, x_{20}, x_{30}) \in \mathbb{R}^3_+$, the solution $(x_1(t), x_2(t), x_3(t))$ of model (3.30) satisfies $\lim_{t\to\infty}\langle x_i(t)\rangle = \frac{E_i}{E}$ a.s., $i = 1, 2, 3$. This is consistent with Theorem 3.1.

4. Numerical simulations

In this section, we make numerical simulations to illustrate our results. Consider the following example

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[0.6 - 0.5x_1 - 0.2x_2 - 0.12x_3\right]\mathrm{d}t + \sigma_1 x_1(t)\mathrm{d}w_1(t), \\ \mathrm{d}x_2(t) = x_2(t)\left[-0.01 + 0.1x_1 - 0.35x_2 - 0.16x_3\right]\mathrm{d}t + \sigma_2 x_2(t)\mathrm{d}w_2(t), \\ \mathrm{d}x_3(t) = x_3(t)\left[-0.1 + 0.1x_1 + 0.05x_2 - 0.16x_3\right]\mathrm{d}t + \sigma_3 x_3(t)\mathrm{d}w_3(t), \end{cases} \tag{4.1}$$

with $x_{10} = 0.8$, $x_{20} = 0.3$, $x_{30} = 0.1$. It is easy to check that $D = 0.0368$, $\tilde{D}_1 = 0.0398$, $\tilde{D}_2 = 0.0083$, $\tilde{D}_3 = 0.0044$. Thus, the corresponding determination model has interior equilibrium point $E_* = (1.0815, 0.2255, 0.1196)$ (see Figure 1). Moreover, we have $A_{23} = 0.005 > 0$, $A_{31} = -0.01 < 0$ and $A_{12} = 0$.

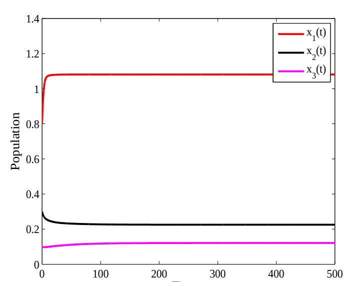


Figure 1. The solution $(x_1(t), x_2(t), x_3(t))$ of (4.1) with $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 0$.

Now, we introduce some numerical results to illustrate Theorem 3.3.

(i) In Figure 2, let $\sigma_1^2 = 1.3$, $\sigma_2^2 = 0.002$ and $\sigma_3^2 = 0.002$. Then, $\kappa_1 = -0.05 < 0$. Thus, the condition of (i') in Theorem 3.3 have been checked. From Theorem 3.3,

$$\lim_{t\to\infty} x_1(t) = 0 \quad \text{a.s.}, \quad i = 1, 2, 3.$$

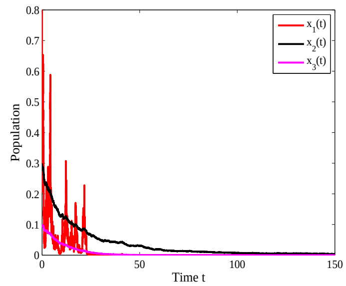
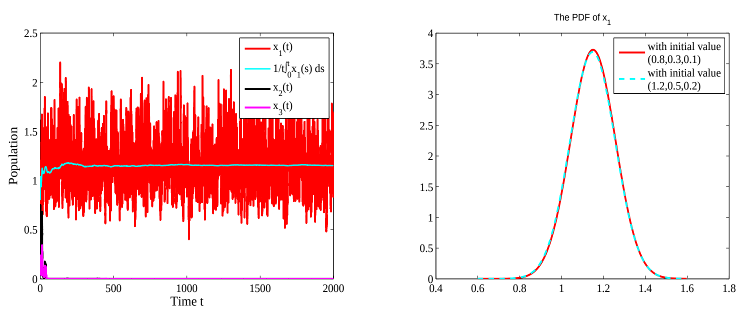


Figure 2. The solution $(x_1(t), x_2(t), x_3(t))$ of (4.1) with $\sigma_1^2 = 1.3$ and $\sigma_2^2 = \sigma_3^2 = 0.002$.

(ii) In Figure 3, choose $\sigma_1^2 = 0.05$, $\sigma_2^2 = 0.4$ and $\sigma_3^2 = 0.2$. Then, $\kappa_1 = 0.575 > 0$, $a_{21}\kappa_1 - a_{11}\kappa_2 = -0.0475 < 0$ and $a_{31}\kappa_1 - a_{11}\kappa_3 = -0.0925 < 0$. That is, all conditions of (ii') in Theorem 3.3 have been checked. Thus,

$$\lim_{t\to\infty} \langle x_1(t)\rangle = 1.15, \quad \lim_{t\to\infty} x_2(t) = 0 \quad \text{a.s.}, \quad i = 2, 3.$$



(a) (b)

Figure 3. The solution $(x_1(t), x_2(t), x_3(t))$ of (4.1) with $\sigma_1^2 = 0.05$, $\sigma_2^2 = 0.4$ and $\sigma_3^2 = 0.2$ (a) paths of $x_1(t)$, $x_2(t)$, $x_3(t)$ and $\langle x_1(t)\rangle$; (b) probability density functions of $x_1(t)$ at $t = 60000$.

(iii) In Figure 4, let $\sigma_1^2 = 0.05$, $\sigma_2^2 = 0.4$ and $\sigma_3^2 = 0.002$. Then, $\kappa_1 = 0.575 > 0$, $a_{21}\kappa_1 - a_{11}\kappa_2 = -0.0475 < 0$ and $a_{31}\kappa_1 - a_{11}\kappa_3 = 0.0070 > 0$. Thus, all conditions of (iii') in Theorem 3.3 have been checked. Therefore,

$$\lim_{t\to\infty} \langle x_1(t)\rangle = 1.1315, \quad \lim_{t\to\infty} x_2(t) = 0, \quad \lim_{t\to\infty} \langle x_3(t)\rangle = 0.0761 \quad \text{a.s.}$$

(iv) In Figure 5, set $\sigma_1^2 = 0.02$, $\sigma_2^2 = 0.002$ and $\sigma_3^2 = 0.2$. Then, $\kappa_1 = 0.59 > 0$, $a_{21}\kappa_1 - a_{11}\kappa_2 = 0.0535 > 0$ and $a_{31}a_{22}\kappa_1 + a_{32}a_{21}\kappa_1 - a_{32}a_{11}\kappa_2 - a_{11}a_{22}\kappa_3 = -0.0117 < 0$. Therefore, conditions of (iv') in Theorem 3.3 hold. Thus,

$$\lim_{t\to\infty} \langle x_1(t)\rangle = 1.0703, \quad \lim_{t\to\infty} \langle x_2(t)\rangle = 0.2744, \quad \lim_{t\to\infty} x_3(t) = 0 \quad \text{a.s.}$$

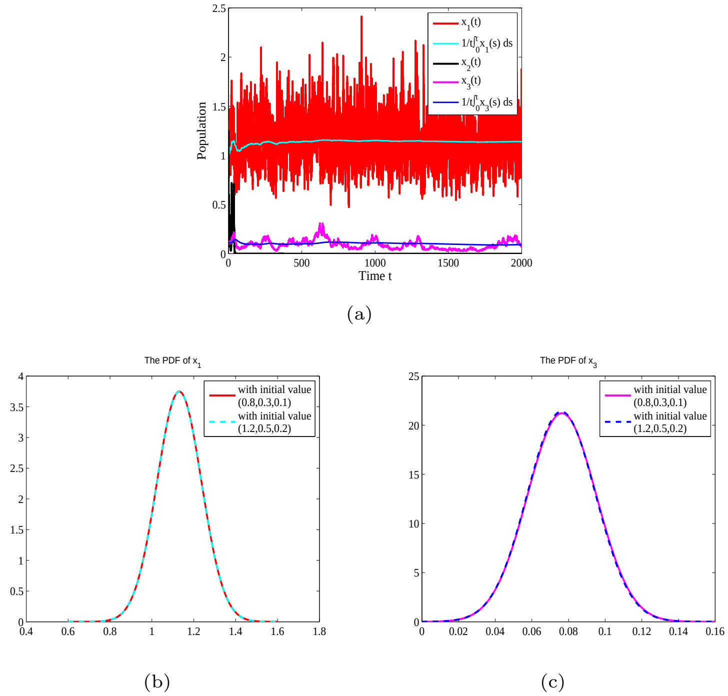


Figure 4. The solution $(x_1(t), x_2(t), x_3(t))$ of (4.1) with $\sigma_1^2 = 0.05$, $\sigma_2^2 = 0.4$ and $\sigma_3^2 = 0.002$ (a) paths of $x_1(t)$, $x_2(t)$, $x_3(t)$, $\langle x_1(t)\rangle$ and $\langle x_3(t)\rangle$; (b) probability density functions of $x_1(t)$ at $t = 60000$; (c) probability density functions of $x_3(t)$ at $t = 60000$.

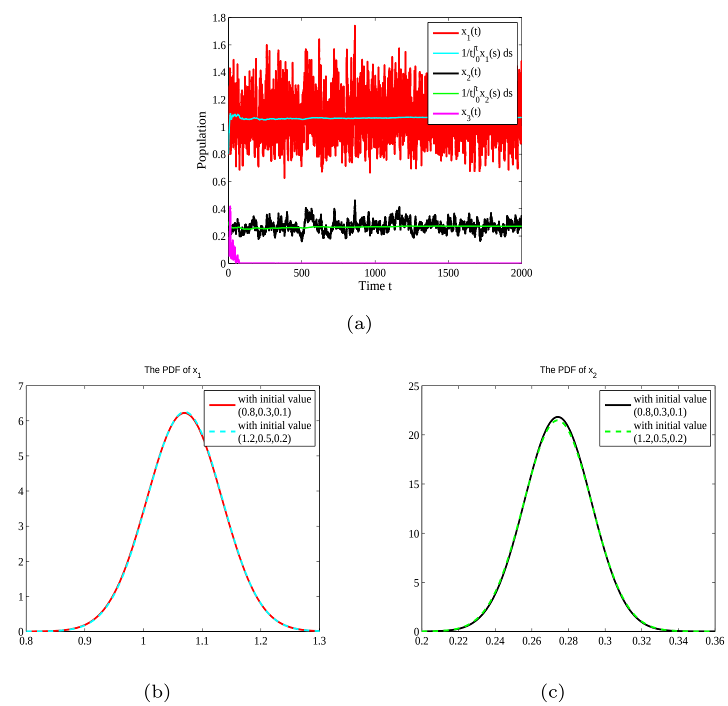


Figure 5. The solution $(x_1(t), x_2(t), x_3(t))$ of (4.1) with $\sigma_1^2 = 0.02$, $\sigma_2^2 = 0.002$ and $\sigma_3^2 = 0.2$. (a) paths of $x_1(t)$, $x_2(t)$, $x_3(t)$, $\langle x_1(t)\rangle$ and $\langle x_2(t)\rangle$; (b) probability density functions of $x_1(t)$ at $t = 60000$; (c) probability density functions of $x_2(t)$ at $t = 60000$.

(v) In Figure 6, set $\sigma_1^2 = 0.002$, $\sigma_2^2 = 0.002$, $\sigma_3^2 = 0.002$. Then, $\kappa_1 = 0.599 > 0$, $a_{21}\kappa_1 - a_{11}\kappa_2 = 0.0544 > 0$, $a_{31}a_{22}\kappa_1 + a_{32}a_{21}\kappa_1 - a_{32}a_{11}\kappa_2 - a_{11}a_{22}\kappa_3 = 0.006 > 0$, $D_1 = 0.0392$, $D_2 = 0.0083$ and $D_3 = 0.0038$. Therefore all conditions of (v') in Theorem 3.3 have been checked. Thus,

$$\lim_{t\to\infty}\langle x_1(t)\rangle = 1.0805, \quad \lim_{t\to\infty}\langle x_2(t)\rangle = 0.2250, \quad \lim_{t\to\infty}\langle x_3(t)\rangle = 0.1144 \quad \text{a.s.}$$

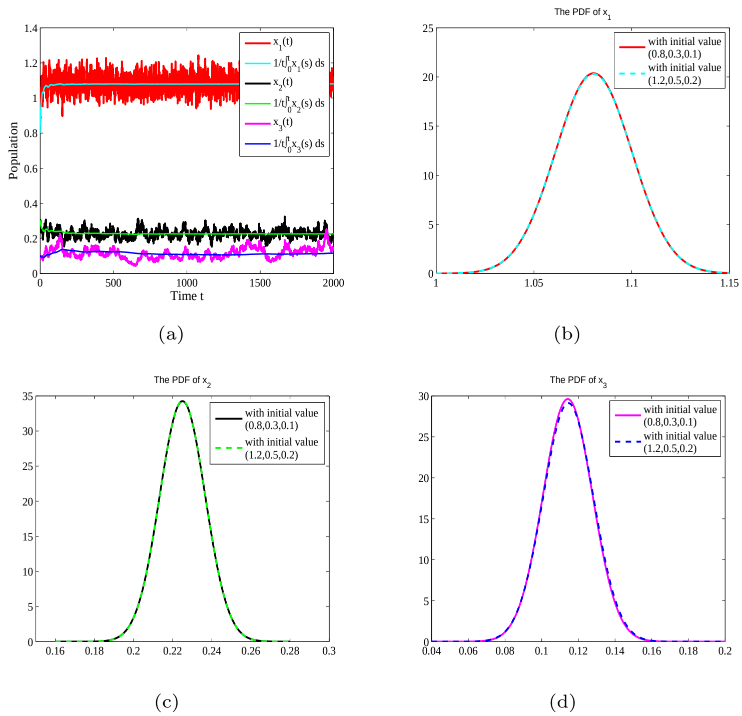


Figure 6. The solution $(x_1(t), x_2(t), x_3(t))$ of (4.1) with $\sigma_1^2 = 0.002$, $\sigma_2^2 = 0.002$ and $\sigma_3^2 = 0.002$. (a) paths of $x_1(t)$, $x_2(t)$, $x_3(t)$, $\langle x_1(t)\rangle$, $\langle x_2(t)\rangle$ and $\langle x_3(t)\rangle$; (b) probability density functions of $x_1(t)$ at $t = 60000$; (c) probability density functions of $x_2(t)$ at $t = 60000$; (d) probability density functions of $x_3(t)$ at $t = 60000$.

As can be seen from Figure 2 that if noise intensity σ_1^2 is large, then all the populations in model (4.1) go to extinction. From Figure 3, we can see that great noise intensity σ_i^2 $(i = 2, 3)$ can make predator x_i extinction. Moreover, if noise intensity σ_1^2 is small, then prey x_1 is persistent in mean. As can be seen from Figure 4 that prey x_1 and top predator x_3 are persistent in mean while intermediate predator x_2 goes to extinction. From Figure 5, we know that prey x_1 and predator x_2 are persistent in mean while predator x_3 goes to extinction. It can be seen from Figure 6 that all the populations in model (4.1) are persistent in mean.

From the above numerical simulations, we see can that the originally persist species x_1, x_2 and x_3 in the deterministic model (see Figure 1) has emerged the possibility of extinction under the noise disturbance (see Figure 2). This means that noise intensity has great influence on population dynamics.

5. Conclusions and discussions

In this paper, we consider a stochastic three-species food-web model with intraguild predation. The main result is Theorem 3.3, which establishes the sufficient conditions for the persistence and extinction of each population in model (1.2).

Theorem 3.1 explains the effects of white noise and omnivores on the population dynamics. From Theorem 3.1, we have the following results.

(i) If the noise intensity σ_1^2 is large, that is, $r_1 < \frac{\sigma_1^2}{2}$, then prey x_1 will become extinct. Moreover, extinction of prey will make intermediate predator x_2 and top predator x_3 extinction.

(ii) For predator x_i $(i = 2, 3)$, if $\frac{a_{i1}}{a_{11}} < \frac{\kappa_i}{\kappa_1}$, then x_i becomes extinct. Note that $a_{i1} > 0$ $(i = 1, 2, 3)$ and $\kappa_i > 0$ $(i = 2, 3)$. Thus, $\kappa_1 > 0$. Further, prey x_1 is persistent in mean. From the proof of Theorem 3.1, we know that great noise intensity σ_i^2 $(i = 2, 3)$ can make predator x_i extinction regardless of the size of prey. Further, if noise intensity σ_1^2 is small, that is, $r_1 > \frac{\sigma_1^2}{2}$, then prey x_1 is persistent in mean.

(iii) If $\frac{a_{21}}{a_{11}} < \frac{\kappa_2}{\kappa_1}$ and $\frac{a_{31}}{a_{11}} > \frac{\kappa_3}{\kappa_1}$, then predator x_2 will go to extinction, and prey x_1 and predator x_3 will be persistent in mean. This means that great noise intensity σ_2^2 can make x_2 extinction. Further, if the noise intensities σ_1^2 and σ_3^2 are small, then prey x_1 and predator x_3 are persistent in mean.

(iv) If $a_{31}a_{22}\kappa_1 + a_{32}a_{21}\kappa_1 - a_{32}a_{11}\kappa_2 - a_{11}a_{22}\kappa_3 < 0$, then predator x_3 will go to extinction. That is, great noise intensity σ_3^2 can make predator x_3 extinction. Further, if the noise intensities σ_1^2 and σ_2^2 are small, then prey x_1 and predator x_2 are persistent in mean.

(v) If the intensities σ_1^2, σ_2^2 and σ_3^2 are small, then all populations in model (1.2) will be persistent in mean.

If $a_{13} = a_{31} \equiv 0$, that is, top predator x_3 only feeds on intermediate predator x_2, then stochastic food-web model (1.2) can be reduced to stochastic food-chain model (3.30). It is clear that $a_{31}\kappa_1 - a_{11}\kappa_3 < 0$. Thus, for the stochastic food-chain model (3.30), from Corollary 3.2, if intermediate predator x_2 becomes extinct, then top predator x_3 must go to extinction. This is consistent with the result in [11]. However, for stochastic food-web model (1.2), from Remark 3.2, we can see that top predator x_3 can be persistent in mean even if intermediate predator x_2 goes extinct. This makes sense, because top predator x_3 can feed upon prey x_1. This results show that omnivory has great effects on the population dynamics.

Some interesting problems deserve further consideration. As done in [12, 13], one can introduce time delays in model (1.2). Moreover, one can study model with other perturbations, such as Markovian switching or Lévy jumps. We leave this for future consideration.

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