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HOMEOMORPHISMS RELATED TO THE POLYNOMIAL-LIKE ITERATIVE EQUATION ON $\mathbb{S}^1$ *

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Abstract In this paper we study all homeomorphisms on the unit circle $\mathbb{S}^1$, whose lifts are C^0 solutions of a class of nonhomogeneous polynomial-like iterative equation. By an auxiliary equation, we present all those homeomorphisms and illustrate our results by examples.

Keywords Unit circle, homeomorphism, iterative equation, C^0 solution.

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1. Introduction

Let X be a nonempty subset of $\mathbb{R}$, the n-th iterate of a self-mapping $f: X \to X$ is defined by $f^n(x) = f(f^{n-1}(x))$ and $f^0(x) = x$ for all $x \in X$ inductively. The origin of the iterative functional equation can be traced back to 1815, C. Babbage wrote iterative roots problem ([1]). As a weak version of embedding flows ([5]), iterative roots problem attracts many people in both dynamical systems and functional equations (e.g. [2,9,10,26,38]). It is known that the problem of iterative roots is to solve the elementary iterative equation

$$f^k(x) = F(x), \ \forall x \in X, \tag{1.1}$$

where $F : X \to X$ is a given map and $f : X \to X$ is an unknown map. Even for simple $F(x)$, the equation (1.1) is not solved entirely ([14,16]).

A more general form is the polynomial-like iterative equation

$$\lambda_n f^n(x) + \lambda_{n-1} f^{n-1}(x) + ... + \lambda_1 f(x) = F(x), \ \forall x \in X, \tag{1.2}$$

an equation of the linear dependence of iterates, which becomes one of the most favorite objects for those people being interested in iterative equations. For nonlinear F, the existence, uniqueness and the stability of the C^0 solutions of Eq.(1.2) have been investigated ([13,18,34,40]) and further results such as smooth ([32]), analyticity ([21]) and convexity ([6,27,37]) were also given. Higher dimensional cases

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and multivalued cases refer to the references [11, 25, 33] and [12, 28], respectively. Linear F ([20, 39]), i.e.,

$$f^n(x) + \lambda_{n-1} f^{n-1}(x) + ... + \lambda_1 f(x) + \lambda_0 x = c, \quad \forall x \in X,\ c \in \mathbb{R},$$

even the homogeneous equation

$$f^n(x) + \lambda_{n-1} f^{n-1}(x) + ... + \lambda_1 f(x) + \lambda_0 x = 0 \tag{1.3}$$

is attractive, which has been investigated extensively (see [3,4,7,15,19,23,24,29,36] and some references therein). Substituting $f(x) = rx$ ($r \in \mathbb{C}$) in (1.3), we get the characteristic equation

$$P(r) := r^n + \lambda_{n-1} r^{n-1} + ... + \lambda_1 r + \lambda_0 = 0 \tag{1.4}$$

and r is called characteristic root. Under restrictive conditions on (1.4), the mentioned references present the C^0 solutions of Eq.(1.3) by all those linear solutions $f(x) = rx$.

It is also interesting to investigate iterative equation on the unit circle $\mathbb{S}^1$. As we all know that a homeomorphism $P:\ \mathbb{S}^1 \to \mathbb{S}^1$ has a unique lift $\varphi:\ \mathbb{R} \to \mathbb{R}$ such that

$$P(e^{i2\pi x}) = e^{i2\pi\varphi(x)} \tag{1.5}$$

and the lift φ satisfies

$$\varphi(x+1) = \varphi(x) + k, \quad k \in \{-1, 1\}. \tag{1.6}$$

We call P is orientation-preserving if $k = 1$ and orientation-reversing if $k = -1$. By lifting maps on $\mathbb{S}^1$ to the whole line $\mathbb{R}$, many results on iterative roots and iteration groups on $\mathbb{S}^1$ are given ([8, 17, 22, 30, 35]). In 2007, M. C. Zdun and W. Zhang ([31]) considered the C^0 solutions of the general iterative equation

$$\Phi(f(z), f^2(z), ..., f^n(z)) = F(z), \quad z \in \mathbb{S}^1 \tag{1.7}$$

and proved the existence, uniqueness and stability in the set

$$H_1^0(\mathbb{S}^1, \mathbb{S}^1) = \{f \in C^0(\mathbb{S}^1, \mathbb{S}^1):\ f(\mathbb{S}^1) = \mathbb{S}^1 \ \text{ homeomorphically } \ \text{and } f(\mathbf{1}) = \mathbf{1}\}$$

using fixed point theorems, where $\mathbf{1}$ indicates the point $(1, 0)$ in the complex plane $\mathbb{C}$. We say that Lemma 3.2 plays an important role in Ref. [31]. Let $\tilde{F}$ and $\tilde{\Phi}$ be the lifts of F and Φ, respectively, this lemma shows that Eq.(1.7) is equivalent to

$$\tilde{\Phi}(\tilde{f}(x), \tilde{f}^2(x), ..., \tilde{f}^n(x)) = \tilde{F}(x), \quad x \in \mathbb{R}$$

under **the assumptions that** $\tilde{\Phi}(0, ..., 0) = 0$ **and** $\tilde{F}(0) = 0$.

Removing the condition $\tilde{F}(0) = 0$, in this paper we consider all homeomorphisms $P:\ \mathbb{S}^1 \to \mathbb{S}^1$ whose lifts $\varphi:\ \mathbb{R} \to \mathbb{R}$ are C^0 solutions of the nonhomogeneous polynomial-like iterative equation

$$\varphi^n(x) + \lambda_{n-1}\varphi^{n-1}(x) + ... + \lambda_1 \varphi(x) + \lambda_0 x = c, \quad x \in \mathbb{R}. \tag{1.8}$$

For this purpose, we study the C^0 solutions φ which satisfy Eq.(1.6) and Eq.(1.8) simultaneously, and then we construct all those homeomorphisms P by using (1.5).

2. Preliminaries

We first give a lemma used in the proofs of Lemma 2.2 and Lemma 2.3.

Lemma 2.1 (Lemma 3, [39]). *Suppose that all roots $\gamma_j \neq 0$ $(j = 1, 2, ..., n)$ of the characteristic equation (1.4) are real and none of them is equal to* 1*. Then Eq.(1.8) can be reduced to*

$$g^n(x) + \lambda_{n-1}g^{n-1}(x) + ... + \lambda_1 g(x) + \lambda_0 x = 0 \tag{2.1}$$

by the substitution $g(x) = \varphi(x+\eta) - \eta$, where $\eta := c/\prod_{j=1}^{n}(1-\gamma_j)$, and vice versa.

Eq.(2.1) is called an auxiliary equation in the present paper. We say that Lemma 2.2 and Lemma 2.3 are important in the proof of theorems.

Lemma 2.2. *Assume that none of roots of the characteristic equation (1.4) equals* 0 *and* 1*. If a homeomorphism $\varphi: \mathbb{R} \to \mathbb{R}$, satisfying $\varphi(x+1) = \varphi(x) + 1$, is a C^0 solution of Eq.(1.8), then $g: \mathbb{R} \to \mathbb{R}$ is a C^0 solution of Eq.(2.1), where $g(x) = \varphi(x + \eta_1 - 1) - \eta_1 + 1$ and $\eta_1 = (c + 1 + \sum_{j=0}^{n-1}\lambda_j)/\prod_{j=1}^{n}(1-\gamma_j)$.*

Proof. Note that

$$\varphi(x) = \varphi(x+1) - 1, \quad x \in \mathbb{R}.$$

By induction we have

$$\varphi^j(x) = \varphi^j(x+1) - 1 \quad \text{for all} \quad j \in \mathbb{N}^0,$$

then Eq.(1.8) can be rewritten as

$$\varphi^n(x+1) + \lambda_{n-1}\varphi^{n-1}(x+1) + ... + \lambda_1\varphi(x+1) + \lambda_0 x = c + 1 + \sum_{j=1}^{n-1}\lambda_j. \tag{2.2}$$

Let $t := x + 1$, then Eq.(2.2) is equivalent to the equation

$$\varphi^n(t) + \lambda_{n-1}\varphi^{n-1}(t) + ... + \lambda_1\varphi(t) + \lambda_0 t = c + 1 + \sum_{j=0}^{n-1}\lambda_j.$$

Consider the auxiliary equation (2.1). From Lemma 2.1, using the translation transformation

$$g(t) := \varphi(t + \eta_1) - \eta_1,$$

we have

$$\varphi(t) = g(t - \eta_1) + \eta_1,$$

where

$$\eta_1 := (c + 1 + \sum_{i=0}^{n-1}\lambda_i)/\prod_{i=1}^{n}(1-\gamma_i).$$

Then

$$\begin{aligned}\varphi(x) &= \varphi(x+1) - 1\\ &= \varphi(t) - 1\end{aligned}$$

$$= g(t - \eta_1) + \eta_1 - 1$$
$$= g(x - \eta_1 + 1) + \eta_1 - 1,$$

that is

$$g(x) = \varphi(x + \eta_1 - 1) - \eta_1 + 1.$$

This completes the proof. □

Lemma 2.3. *Assume that none of roots of the characteristic equation (1.4) equals* 0 *and* 1*. If a homeomorphism* $\varphi : \mathbb{R} \to \mathbb{R}$*, satisfying* $\varphi(x+1) = \varphi(x) - 1$*, is a* C^0 *solution of Eq.(1.8), then* $g : \mathbb{R} \to \mathbb{R}$ *is a* C^0 *solution of Eq.(2.1), where* $g(x) = \varphi(x+\eta_2-1) - \eta_2 - 1$ *and* $\eta_2 := (c + (-1)^n + \sum_{j=0}^{n-1}(-1)^j\lambda_j)/\prod_{j=1}^{n}(1-\gamma_j)$.

Proof. From the condition

$$\varphi(x+1) = \varphi(x) - 1, \ x \in \mathbb{R},$$

by induction we have

$$\begin{cases} \varphi^j(x) = \varphi^j(x+1) - 1, & j \text{ is even}, \\ \varphi^j(x) = \varphi^j(x+1) + 1, & j \text{ is odd}. \end{cases} \tag{2.3}$$

Using (2.3), we rewrite Eq.(1.8) as

$$\varphi^n(x+1) + \lambda_{n-1}\varphi^{n-1}(x+1) + ... + \lambda_1\varphi(x+1) + \lambda_0 x = c + (-1)^n + \sum_{j=1}^{n-1}(-1)^j\lambda_j,$$

Let $t := x + 1$, we have

$$\varphi^n(t) + \lambda_{n-1}\varphi^{n-1}(t) + ... + \lambda_1\varphi(t) + \lambda_0 t = c + (-1)^n + \sum_{j=0}^{n-1}(-1)^j\lambda_j.$$

Consider the auxiliary equation (2.1). By Lemma 2.1, using the translation transformation

$$g(t) := \varphi(t + \eta_2) - \eta_2,$$

we get

$$\varphi(t) = g(t - \eta_2 - 1) + \eta_2,$$

where

$$\eta_2 := (c + (-1)^n + \sum_{j=0}^{n-1}(-1)^j\lambda_j)/\prod_{j=1}^{n}(1-\gamma_j).$$

Then

$$\begin{aligned} \varphi(x) &= \varphi(x+1) + 1 \\ &= \varphi(t) + 1 \\ &= g(t - \eta_2) + \eta_2 + 1 \\ &= g(x - \eta_2 + 1) + \eta_2 + 1, \end{aligned}$$

thus,

$$g(x) = \varphi(x + \eta_2 - 1) - \eta_2 - 1.$$

This completes the proof. □

3. Main results

Theorem 3.1. *If the characteristic equation (1.4) has roots* $1 < \gamma_1 < ... < \gamma_n$ *(or* $0 < \gamma_1 < ... < \gamma_n < 1$*). Then every homeomorphism* $P: \mathbb{S}^1 \to \mathbb{S}^1$*, whose lift* φ *is a* C^0 *solution of Eq.(1.8), is orientation-preserving and can be constructed by using (1.5).*

Proof. Under the condition that $1 < \gamma_1 < ... < \gamma_n$ (or $0 < \gamma_1 < ... < \gamma_n < 1$), each C^0 solution $g: \mathbb{R} \to \mathbb{R}$ of Eq.(2.1) is strictly increasing and can be constructed by using Theorem 2 in Ref. [29].

Let $\varphi(x) := g(x - \eta_1 + 1) + \eta_1 - 1$. If φ satisfies $\varphi(x+1) = \varphi(x) + 1$, from Lemma 2.2 we find all orientation-preserving homeomorphisms $P: \mathbb{S}^1 \to \mathbb{S}^1$ by using $P(e^{i2\pi x}) = e^{i2\pi\varphi(x)}$. This completes the proof. □

Theorem 3.2. *If the characteristic equation (1.4) has roots* $\gamma_1 < ... < \gamma_n < -1$ *(or* $-1 < \gamma_1 < ... < \gamma_n < 0$*). Then every homeomorphism* $P: \mathbb{S}^1 \to \mathbb{S}^1$*, whose lift* φ *is a* C^0 *solution of Eq.(1.8), is orientation-reversing and can be constructed by using (1.5).*

Proof. Under the condition that $\gamma_1 < ... < \gamma_n < -1$ (or $-1 < \gamma_1 < ... < \gamma_n < 0$), each C^0 solution $g: \mathbb{R} \to \mathbb{R}$ of Eq.(2.1) is orientation-reversing homeomorphism and can be constructed by using Theorem 4 in Ref. [29].

Let $\varphi(x) := g(x - \eta_2 + 1) + \eta_2 + 1$. If φ satisfies $\varphi(x+1) = \varphi(x) - 1$, using Lemma 2.3 we get all orientation-reversing homeomorphisms $P: \mathbb{S}^1 \to \mathbb{S}^1$ by using $P(e^{i2\pi x}) = e^{i2\pi\varphi(x)}$. This completes the proof. □

Theorem 3.3. *If the characteristic equation (1.4) has roots* $1 < -\gamma_1 < ... < -\gamma_p < \gamma_{p+1} < ... < \gamma_n$ *(or* $0 < -\gamma_1 < ... < -\gamma_p < \gamma_{p+1} < ... < \gamma_n < 1$*). Then each homeomorphism* $P: \mathbb{S}^1 \to \mathbb{S}^1$*, whose lift* φ *is a* C^0 *solution of Eq.(1.8), can be constructed by using (1.5). There are two cases:*

(i) $P: \mathbb{S}^1 \to \mathbb{S}^1$*, whose lift* φ *is a* C^0 *solution of a lower equation with characteristic roots* $\gamma_1, ..., \gamma_p$*, is orientation-reversing.*

(ii) $P: \mathbb{S}^1 \to \mathbb{S}^1$*, whose lift* φ *is a* C^0 *solution of a lower equation with characteristic roots* $\gamma_{p+1}, ..., \gamma_n$*, is orientation-preserving.*

Proof. **(i)** If $1 < -\gamma_1 < ... < -\gamma_p < \gamma_{p+1} < ... < \gamma_n$ and g is an orientation-reversing homeomorphism of Eq.(2.1). By using the method provided in Theorem 4.1 of Ref. [36], we can remove the characteristic roots $\gamma_n, \gamma_{n-1}, ..., \gamma_{p+1}$ one after another and eventually change Eq.(2.1) into the p-th order iterative equation

$$g^p(x) + \lambda'_{p-1} g^{p-1}(x) + ... + \lambda'_1 g(x) + \lambda'_0 x = 0. \tag{3.1}$$

Repeating the progress as that of Theorem 3.2, each C^0 solution $g: \mathbb{R} \to \mathbb{R}$ of Eq.(3.1) can be constructed. Now let $\varphi(x) := g(x - \eta_3 + 1) + \eta_3 + 1$, where

$$\eta_3 := (c + (-1)^p + \sum_{j=0}^{p-1} (-1)^j \lambda_j) / \prod_{j=1}^{p} (1 - \gamma_j).$$

If φ satisfies $\varphi(x+1) = \varphi(x) - 1$, by using Lemma 2.3 we find all those orientation-reversing homeomorphisms $P: \mathbb{S}^1 \to \mathbb{S}^1$ by using $P(e^{i2\pi x}) = e^{i2\pi\varphi(x)}$.

(ii) If $1 < -\gamma_1 < ... < -\gamma_p < \gamma_{p+1} < ... < \gamma_n$ and g is an orientation-preserving homeomorphism of Eq.(2.1), we consider the dual equation of Eq.(2.1). By the same method as that of the case (i), we remove $\gamma_1, \gamma_2, ..., \gamma_p$ in turn and eventually change the dual equation into the k-th order iterative equation ($k := n - p$)

$$g^k(x) + \lambda''_{k-1} g^{k-1}(x) + ... + \lambda''_1 g(x) + \lambda''_0 x = 0. \tag{3.2}$$

Repeating the process as that of Theorem 3.1, each C^0 solution $g: \mathbb{R} \to \mathbb{R}$ of Eq.(3.2) can be constructed. Now let $\varphi(x) := g(x - \eta_4 + 1) + \eta_4 - 1$, where

$$\eta_4 = (c + 1 + \sum_{j=0}^{k-1} \lambda_j) / \prod_{j=1}^{k} (1 - \gamma_j).$$

If φ satisfies $\varphi(x+1) = \varphi(x) + 1$, using Lemma 2.2 we find all those orientation-reversing homeomorphisms $P: \mathbb{S}^1 \to \mathbb{S}^1$ by using $P(e^{i2\pi x}) = e^{i2\pi\varphi(x)}$. This completes the proof. □

Example 3.1. Consider an iterative equation

$$\varphi^2(x) - \frac{5}{3}\varphi(x) - \frac{2}{3}x = 1. \tag{3.3}$$

Clearly, the characteristic equation

$$r^2(x) - \frac{5}{3}r - \frac{2}{3} = 0$$

has two roots $r_1 = -\frac{1}{3},\ r_2 = 2$ satisfying

$$0 < \frac{1}{3} < 1 < 2,$$

and the auxiliary function

$$g^2(x) - \frac{5}{3}g(x) - \frac{2}{3}x = 0.$$

has two characteristic solutions $g_1(x) = 2x$ and $g_2(x) = -\frac{1}{3}x$.

Let $\varphi_1(x) = g_1(x - \omega_1 + 1) + \omega_1 - 1$, where

$$\begin{aligned}\omega_1 &= (c + 1 + \sum_{j=0}^{1} \lambda_j) / \prod_{j=1}^{2} (1 - \gamma_j) \\ &= (1 + 1 - \frac{7}{3})/(-\frac{4}{3}) \\ &= \frac{1}{4},\end{aligned}$$

we get

$$\varphi_1(x) = 2x + \frac{3}{4}.$$

Let $\varphi_2(x) = g_2(x - \omega_2 + 1) + \omega_2 + 1$, in which

$$\omega_2 = (c + (-1)^2 + \sum_{j=0}^{1} (-1)^j \lambda_j) / \prod_{j=1}^{2} (1 - \gamma_j)$$

$$
\begin{aligned}
&= (1+1+1)/(-\frac{4}{3}) \\
&= -\frac{9}{4},
\end{aligned}
$$

we have

$$\varphi_2(x) = -\frac{1}{3}x - \frac{7}{3}.$$

Neither of $\varphi_1(x+1) = \varphi_1(x) + 1$ and $\varphi_2(x+1) = \varphi_2(x) - 1$ holds, by using Theorem 3.3 we have no homeomorphisms P whose lifts φ satisfy Eq.(3.3).

4. Further discussion

All characteristic roots having same sign are real and inside (or outside) the unit circle $\mathbb{S}^1$, Theorem 3.1 and Theorem 3.2 give all those homeomorphisms on unit circle with lifts being C^0 solutions of Eq.(1.8). Theorem 3.3 illustrate a case that characteristic roots have different sign. In fact, by using the C^0 solutions of the auxiliary equation (2.1) and Eq.(1.6), we can discuss the more general case. We give an example.

Example 4.1. Consider the 3rd-order iterative equation

$$\varphi^3(x) + \frac{1}{2}\varphi^2(x) - \frac{13}{2}\varphi(x) + 3x = 5.$$

Clearly, the characteristic equation

$$r^3 + \frac{1}{2}r^2 - \frac{13}{2}r + 3 = 0$$

has three real roots $r_1 = \frac{1}{2},\ r_2 = 2,\ r_3 = -3$. So the auxiliary function

$$g^3(x) + \frac{1}{2}g^2(x) - \frac{13}{2}g(x) + 3x = 0 \tag{4.1}$$

has three characteristic solutions $g_1(x) := \frac{1}{2}x,\ g_2(x) := 2x,\ g_3(x) := -3x$ and all C^0 solutions g of Eq.(4.1) can be constructed by using Theorem 4.2 in Ref. [36].

For convenience, here we only consider $\varphi_1, \varphi_2, \varphi_3$ yielded by g_1, g_2, g_3, respectively. Let $\varphi_j(x) = g_j(x - \tau_1 + 1) + \tau_1 - 1$ $(j = 1, 2)$, where

$$
\begin{aligned}
\tau_1 &= (c + 1 + \sum_{j=0}^{2}\lambda_j)/\prod_{j=1}^{3}(1-\gamma_j) \\
&= (5+1-3)/(-2) \\
&= -\frac{3}{2},
\end{aligned}
$$

then

$$\varphi_1(x) = \frac{1}{2}x - \frac{5}{4}, \quad \varphi_2(x) = 2x + \frac{5}{2}. \tag{4.2}$$

Let $\varphi_3(x) = g_3(x - \tau_2 + 1) + \tau_2 + 1$, where

$$\begin{aligned}\tau_2 &= (c + (-1)^3 + \sum_{j=0}^{2}(-1)^j\lambda_j)/\prod_{j=1}^{3}(1-\gamma_j)\\ &= (5-1+10)/(-2)\\ &= -7,\end{aligned}$$

then

$$\varphi_3(x) = -3x - 30. \tag{4.3}$$

Neither (4.2) satisfy $\varphi(x+1) = \varphi(x) + 1$ nor (4.3) satisfies $\varphi(x+1) = \varphi(x) - 1$, so we have no orientation-preserving homeomorphisms P on unit circle with lift φ_1 or φ_2, and have no orientation-reversing homeomorphism P on unit circle with lift φ_3.

In Lemma 2.1-2.3, we assume that all roots $\gamma_j \neq 0$ $(j = 1, 2, ..., n)$ of the characteristic equation (1.4) are real and none of them is equal to 1. The more general cases involving complex characteristic roots have no result, such as the characteristic equation (1.4) has simple roots

$$\gamma_1, \gamma_2, ..., \gamma_p \in \mathbb{R},\ \gamma_{p+1}, ..., \gamma_s, \bar{\gamma}_{p+1}, ..., \bar{\gamma}_s \in \mathbb{C}\backslash\mathbb{R},\ \gamma_{s+1}, \gamma_{s+2}, ..., \gamma_t \in \mathbb{R},$$

where $s - p + t = n$, which satisfy

$$0 < -\gamma_1 < ... < -\gamma_p < |\gamma_{p+1}| < ... < 1 < ... < |\gamma_s| < \gamma_{s+1} < ... < \gamma_t. \tag{4.4}$$

How to construct the homeomorphism $P:\ \mathbb{S}^1 \to \mathbb{S}^1$ under condition (4.4), whose lift φ is a C^0 solution of Eq.(1.8), is unsolved.

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