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FLOCKING AND COLLISION AVOIDANCE OF A CUCKER-SMALE TYPE SYSTEM WITH SINGULAR WEIGHTS*

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Abstract The dynamical behavior of a flock model with a singular communication rate and extra interaction terms is investigated in this paper. A rigorous theoretical proof of collision avoidance between any two agents is obtained which guarantees the existence of global solutions. Moreover, a sufficient condition for the existence of time-asymptotic flocking is also acquired and numerical simulations verified these results which show that a compact equilibrium configuration may emerge.

Keywords Time-asymptotic flocking, singular weights, collision avoidance.

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1. Introduction

Emergent collective behavior includes flocking [1, 2], consensus [9, 14, 19] and synchronization [15, 16], which are ubiquitous in nature, such as flocks of birds and fish migration. Inspired by the above phenomenon, many scholars have developed a keen interest in the study of the mathematical mechanism of flocking. Flocking can describe the phenomenon of group collaboration, that is, relying on limited environmental information and simple interaction rules to change from disordered state to ordered one [1, 2].

In recent years, synergistic behavior of self-organizing groups in biology, robotics, sociology, economics and other researches have attracted the attention of scholars. In 1986, Reynolds introduced three heuristic rules in [13], that is, collision avoidance, velocity matching and flock centering. These regulations are meant to illustrate how a single agent operates according to the location and speed of nearby flockmates. Vicsek et al. proposed a simple model of self-ordered motion in systems of particles with motivated interaction in [17], 1995. Subsequently, many of models based on local interaction were proposed and studied from both theoretical and numerical perspectives. In these discussions, the well-known Cucker-Smale model has been constructed in [1, 2] which built a new platform to study flocking behavior and inspired follow-up work. The C-S system relies on a simple rule that can be traced back to [13] to describe how the agents interact to align with their neighbors. The

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motion of the ith agent is characterized by its position and velocity, which are expressed as $x_i(t) \in \mathbb{R}^d$ and $v_i(t) \in \mathbb{R}^d$ respectively. The evolution of each agent based on limited environmental information is described as follows:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} x_i(t) = v_i(t) \\ \frac{\mathrm{d}}{\mathrm{d}t} v_i(t) = \alpha \sum_{j=1}^{N} a_{ij}(x)(v_j(t) - v_i(t)), \quad i = 1, 2, ..., N, \end{cases} \tag{1.1}$$

where α depicts the interaction strength and $x = (x_1, \cdots, x_N)^T$. Moreover, $a_{ij}(x)$ characterize the intensity of interaction between j and i, which is defined as

$$a_{ij}(x) = \frac{K}{(\sigma^2 + \|x_i - x_j\|^2)^\beta}, \tag{1.2}$$

where $K > 0,\ \sigma > 0,\ \beta \geq 0$. The notation $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$.

Based on the original work of Cucker and Smale(see [1, 2]) and Ha-Liu(see [6]), various improvements and modifications of the classical C-S system have been made in several directions during the recent years, such as collision avoidance [3, 8], introduction of time delay [4, 5, 10, 18], and even adapted out of its original realm to explain emergence of cultural classes [7].

In this paper, we consider a more realistic requirement, that is, flocking behavior can avoid collision while evolving. Inspired by the work in [11], we describe a flock model with a singular communication rate and extra interaction terms between agents as follows

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} x_i(t) = v_i(t),\ i = 1, 2, \cdots, N \\ \frac{\mathrm{d}}{\mathrm{d}t} v_i(t) = \alpha \sum_{j=1, j\neq i}^{N} \psi(r_{ij})(v_j(t) - v_i(t)) + \sum_{j=1}^{N} f(r_{ij}), \end{cases} \tag{1.3}$$

with $\alpha > 0$ and

$$f(r_{ij}) = \frac{\|x_i - x_j\| - R}{\|x_i - x_j\|}(x_j(t) - x_i(t)) \text{ for } 1 \leq i,\ j \leq N. \tag{1.4}$$

The associated initial conditions are

$$(x_i, v_i)(0) = (x_{i0}, v_{i0}), \quad i = 1, 2, \cdots, N. \tag{1.5}$$

In (1.3), $r_{ij} = \|x_i - x_j\|,\ i, j \in \{1, 2, \cdots, N\}$. R is a preset distance to control the inter-agent distance. The right end of the second equation of (1.3), $\psi(r)$ is singular at $r = 0$, that is, $\psi(0) = +\infty$; the second term is an extra interaction term that produces an attractive or repulsive effect. We list the main assumptions on the communication rate $\psi(\cdot)$.

Assumption 1.1. $\psi(\cdot)$ is non-negative, non-increasing, Lipschitz continuous on $(0, \infty)$ with Lipschitz constant $L > 0$ and $\int_0^\delta \psi(r)dr = +\infty$ for some $\delta > 0$.

One of our intentions is to establish sufficient conditions to guarantee that system (1.3) converges to a flock. A flock, presented by Ha and Liu [6], is defined as follows.

Definition 1.1. let $\{x_i(t), v_i(t)\}$, $i \in \{1, 2, \cdots, N\}$ be the solution to (1.1), a time-asymptotic flocking can be achieved if and only if the system satisfies the following

two conditions:
(i) (velocity alignment) The velocity fluctuations go to zero asymptotically:

$$\lim_{t\to+\infty} \|v_i(t) - v_j(t)\| = 0, \text{ for } i, j = 1, 2, \cdots, N.$$

(ii) (forming a group) The position fluctuations are uniformly bounded in time t:

$$\sup_{0\le t<+\infty} \|x_i(t) - x_j(t)\| < +\infty, \text{ for } i, j = 1, 2, \cdots, N.$$

For the sake of follow-up discussion, we introduce macroscopic variables

$$x_c(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t) \text{ and } v_c(t) = \frac{1}{N}\sum_{i=1}^{N} v_i(t), \tag{1.6}$$

which represent the center and the average velocity of the system respectively. Consider the second equation of (1.3), by the symmetry of the indices, we have

$$\begin{aligned} &\alpha\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\psi(r_{ij})(v_j(t)-v_i(t)) + \sum_{i=1}^{N}\sum_{j=1}^{N} f(r_{ij}) \\ = &-\alpha\sum_{j=1}^{N}\sum_{i=1,i\neq j}^{N}\psi(r_{ij})(v_j(t)-v_i(t)) - \sum_{j=1}^{N}\sum_{i=1}^{N} f(r_{ij}), \end{aligned} \tag{1.7}$$

which yields

$$2\sum_{i=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}v_i(t) = \sum_{i=1}^{N}\left(2\sum_{j=1,j\neq i}^{N}\psi(r_{ij})(v_j(t)-v_i(t)) + 2\sum_{j=1}^{N} f(r_{ij})\right) = 0. \tag{1.8}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}x_c(t) = v_c(t) \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}v_c(t) = 0. \tag{1.9}$$

The deviation variables $\hat{x}_i(t)$, $\hat{v}_i(t)$ are denoted by $\hat{x}_i(t) := x_i(t) - x_c(t)$, $\hat{v}_i(t) := v_i(t) - v_c(t)$ respectively, and then the error system of (1.3) can be written as

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\hat{x}_i(t) = \hat{v}_i(t), \ i = 1, 2, \cdots, N \\ \frac{\mathrm{d}}{\mathrm{d}t}\hat{v}_i(t) = \alpha\sum_{j=1,j\neq i}^{N}\psi(\hat{r}_{ij})(\hat{v}_j(t) - \hat{v}_i(t)) + \sum_{j=1}^{N} f(\hat{r}_{ij}). \end{cases} \tag{1.10}$$

The second equation of (1.3) is rewritten due to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\hat{v}_i(t) &= \frac{\mathrm{d}}{\mathrm{d}t}v_i(t) - \frac{\mathrm{d}}{\mathrm{d}t}v_c(t) \\ &= \alpha\sum_{j=1,j\neq i}^{N}\psi(r_{ij})(v_j(t)-v_i(t)) + \sum_{j=1}^{N} f(r_{ij}) \\ &= \alpha\sum_{j=1,j\neq i}^{N}\psi(\|x_i - x_c - (x_j - x_c)\|)(v_j(t) - v_c(t) - (v_i(t) - v_c(t))) \end{aligned}$$

$$+\sum_{j=1}^{N} f(\|x_i - x_c - (x_j - x_c)\|)$$

$$= \alpha \sum_{j=1, j\neq i}^{N} \psi(\hat{r}_{ij})(\hat{v}_j(t) - \hat{v}_i(t)) + \sum_{j=1}^{N} f(\hat{r}_{ij}), \tag{1.11}$$

where $\hat{r}_{ij} = \|\hat{x}_i - \hat{x}_j\|$. Thus, $\hat{x}_i(t),\ \hat{v}_i(t)$ satisfy (1.10) and $\hat{x}_c(t) = 0,\ \hat{v}_c(t) = 0$.

Thus, to study the flocking behavior of (1.3), we only need to consider the dynamic behavior of (1.10). Specifically, for System (1.10), we need to prove that $\lim_{t\to+\infty} \hat{v}_i(t) = 0$ and $\lim_{t\to+\infty} \hat{r}_{ij}(t) < +\infty$.

This paper is organized as follows. We offer a rigorous collision-free mechanism analysis of system (1.3) in Section 2 and further indicating the existence of global solutions. Asymptotic convergence of system (1.3) are given in Section 3. Some numerical simulation experiments are given in Section 4 to elucidate the availability of the theoretical results. Finally, the conclusion is drawn and further work is stated briefly in Section 5.

2. A collision-avoiding condition

A collision avoidance condition of System (1.3) is proposed, based on which the existence of the global solution is also stated. To carry out this, we first propose the following auxiliary proposition.

Proposition 2.1. *Let $\{x_i(t), v_i(t)\}$, $i \in \{1, 2, \cdots, N\}$ be the solution to (1.3)-(1.5). Then, for $t \geq 0$, we have*

$$\sup_{1\leq i\leq N} \|v_i(t)\| \leq M \textit{ and } \sup_{0\leq t<+\infty} ||x_i(t) - x_j(t)|| \leq x_M, \tag{2.1}$$

where $x_M > 0$, $M > 0$ are constants.

Proof. Motivated by the work of [11], the energy of System (1.3) is expressed as follows

$$Q(t) = \frac{1}{2}\sum_{i=1}^{N} \|v_i(t)\|^2 + \frac{1}{4}\sum_{i,j=1}^{N} (\|x_i(t) - x_j(t)\| - R)^2,\ t \geq 0. \tag{2.2}$$

The derivative of $Q(t)$ with respect to t along the trajectory of (1.3)-(1.5) is expressed as

$$\begin{aligned} \left.\frac{\mathrm{d}Q}{\mathrm{d}t}\right|_{(1.3)} &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N} \|v_i(t)\|^2\right) + \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i,j=1}^{N} (\|x_i(t) - x_j(t)\| - R)^2 \\ &= \sum_{i=1}^{N} \left\langle v_i(t), \frac{\mathrm{d}}{\mathrm{dt}} v_i(t)\right\rangle + \frac{1}{2}\sum_{i,j=1}^{N} (\|x_i(t) - x_j(t)\| - R)\frac{\mathrm{d}}{\mathrm{d}t}\|x_i(t) - x_j(t)\|. \end{aligned} \tag{2.3}$$

More specific,

$$\sum_{i=1}^{N}\left\langle v_i(t),\frac{\mathrm{d}}{\mathrm{d}t}v_i(t)\right\rangle=\sum_{i=1}^{N}\langle v_i(t),\alpha\sum_{j=1}^{N}\psi(r_{ij})(v_j(t)-v_i(t))+\sum_{j=1}^{N}f(r_{ij})\rangle$$
$$=-\frac{\alpha}{2}\sum_{i,j=1}^{N}\psi(r_{ij})\|v_j-v_i\|^2-\frac{1}{2}\sum_{i,j=1}^{N}\langle v_j(t)-v_i(t),f(r_{ij})\rangle. \tag{2.4}$$

Noting the fact that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x_i(t)-x_j(t)\|=\frac{\langle x_i(t)-x_j(t),v_i(t)-v_j(t)\rangle}{\|x_i(t)-x_j(t)\|}. \tag{2.5}$$

Put (2.4) and (2.5) in (2.3), and then (2.3) is reduced to

$$\left.\frac{\mathrm{d}Q}{\mathrm{d}t}\right|_{(1.3)}=-\frac{\alpha}{2}\sum_{i,j=1}^{N}\psi(r_{ij})\|v_j(t)-v_i(t)\|^2\le 0,\ \text{for } t\ge 0, \tag{2.6}$$

which implies that for all $t\ge 0$,

$$\frac{1}{2}\sum_{i=1}^{N}\|v_i(t)\|^2+\frac{1}{4}\sum_{i,j=1}^{N}(\|x_i(t)-x_j(t)\|-R)^2=Q(t)\le Q(0). \tag{2.7}$$

Further, $\sum_{i,j=1}^{N}(\|x_i(t)-x_j(t)\|-R)^2\le 4Q(0)$. Hence, it follows from the symmetry of the indices i and j that $\sup_{0\le t<+\infty}\|x_i(t)-x_j(t)\|\le R+\sqrt{2Q(0)}=:x_M$, which means that $\|x_i(t)-x_j(t)\|$ is uniformly bounded by x_M, for all $i,j\in\{1,2,\dots,N\}$ and $t\ge 0$. Carrying out (2.7) again, there exists a bound M such that $\|v_i(t)\|\le M, i=1,2,\cdots,N,\ t\ge 0$. This completes the proof if Proposition 2.1. □

Assisted by Proposition 2.1, a collision avoidance condition will be proposed in the following theorem.

Theorem 2.1. *Let $\{x_i(t),v_i(t)\},\ i\in\{1,2,\cdots,N\}$ be the solution to (1.3)-(1.5). Assume that $\psi(\cdot)$ satisfies the Assumption 1.1 and the initial configuration (1.5) satisfy $\|x_{i0}-x_{j0}\|>0,\ for\ i\ne j$. Then $||x_i(t)-x_j(t)||>0$ for all $t>0$ and $i\ne j$, that is, collision avoidance can be maintained.*

Proof. To prove that $||x_i(t)-x_j(t)||>0$ for all $i,j\in\{1,2,\cdots,N\}$, $i\ne j$ and $t>0$, we just show that it is impossible to collide on $[0,T]$ for any $T>0$. Otherwise, there is $a\in(0,T)$ such that a is the first time of collision of any agents. Clearly, due to the Cauchy-Lipschitz theorem, System (1.3) admits a unique and smooth solution on $[0,a)$. Then, it follows from the definition of a that there exists an index $s\in\{1,2,\cdots,N\}$ such that the i-th agent collides with some others. Denote $S:=\{\,i\in\{1,2,\cdots,N\}\mid\lim_{t\to a^-}\|x_i(t)-x_s(t)\|=0\,\}$. This provides the fact that $\lim_{t\to a^-}\|x_i(t)-x_j(t)\|=0$ for all $i,j\in S$.

Let $\|x(t)\|_S^2:=\sum_{i,j\in S}||x_i(t)-x_j(t)||^2$ and $\|v(t)\|_S^2:=\sum_{i,j\in S}||v_i(t)-v_j(t)||^2$. By Proposition 2.1, it is straightforward to get that $\|x(t)\|_S^2\le 2|S|^2x_M^2$ and $\|v(t)\|_S^2\le 2|S|^2M^2$. Then from

$$\pm\frac{\mathrm{d}\|x(t)\|_S^2}{\mathrm{d}t}\le 2\|x(t)\|_S\cdot\|v(t)\|_S,\ a.e.\ t\in[0,a),$$

we have

$$\left|\frac{\mathrm{d}\|x(t)\|_S}{\mathrm{d}t}\right| \le \|v(t)\|_S,\ a.e.\ t\in[0,a). \tag{2.8}$$

Furthermore, it follows from the second equation of (1.3) that

$$\begin{aligned}\frac{\mathrm{d}\|v(t)\|_S^2}{\mathrm{d}t} &= 2\sum_{i,j\in S}\langle v_i(t)-v_j(t),\dot v_i(t)-\dot v_j(t)\rangle\\ &= 2\sum_{i,j\in S}\langle v_i(t)-v_j(t),\alpha\sum_{l=1}^N\psi(r_{il})(v_l(t)-v_i(t))+\sum_{l=1}^N f(r_{il})\rangle\\ &\quad -2\sum_{i,j\in S}\langle v_i(t)-v_j(t),\alpha\sum_{l=1}^N\psi(r_{jl})(v_l(t)-v_j(t))+\sum_{l=1}^N f(r_{jl})\rangle\\ &= 2\alpha\sum_{i,j\in S}\langle v_i(t)-v_j(t),\sum_{l=1}^N[\psi(r_{il})(v_l(t)-v_i(t))-\psi(r_{jl})(v_l(t)-v_j(t))]\rangle\\ &\quad +2\sum_{i,j\in S}\langle v_i(t)-v_j(t),\sum_{l=1}^N(f(r_{il})-f(r_{jl}))\rangle.\end{aligned} \tag{2.9}$$

For convenience, we denote

$$\begin{aligned}\mathrm{I} &= 2\alpha\sum_{i,j\in S}\langle v_i(t)-v_j(t),\sum_{l=1}^N[\psi(r_{il})(v_l(t)-v_i(t))-\psi(r_{jl})(v_l(t)-v_j(t))]\rangle,\\ \mathrm{II} &= 2\sum_{i,j\in S}\langle v_i(t)-v_j(t),\sum_{l=1}^N(f(r_{il})-f(r_{jl}))\rangle.\end{aligned} \tag{2.10}$$

Below we discuss I, II separately. For the former, we rewrite it as follows

$$\mathrm{I} = 2\alpha\sum_{i,j\in S}\langle v_i(t)-v_j(t),\left(\sum_{l\in S}+\sum_{l\notin S}\right)[\psi(r_{il})(v_l(t)-v_i(t))-\psi(r_{jl})(v_l(t)-v_j(t))]\rangle. \tag{2.11}$$

For one thing, by monotonicity of $\psi(\cdot)$, $\|x_i(t)-x_l(t)\|_S\le\|x(t)\|_S$, and $\|x_j(t)-x_l(t)\|_S\le\|x(t)\|_S$, one can be obtained that

$$\begin{aligned}&2\alpha\sum_{i,j\in S}\langle v_i(t)-v_j(t),\sum_{l\in S}[\psi(r_{il})(v_l(t)-v_i(t))-\psi(r_{jl})(v_l(t)-v_j(t))]\rangle\\ &= -\alpha\sum_{i,j,l\in S}\langle v_i(t)-v_j(t),(\psi(r_{il})+\psi(r_{jl}))(v_i(t)-v_j(t))\rangle\\ &\quad +\alpha\sum_{i,j,l\in S}(\langle v_i(t)-v_j(t),(\psi(r_{il})-\psi(r_{jl}))(v_l(t)-v_i(t))\rangle\\ &\quad +\alpha\sum_{i,j,l\in S}\langle v_i(t)-v_j(t),(\psi(r_{il})-\psi(r_{jl}))(v_l(t)-v_j(t))\rangle\end{aligned}$$

$$
\begin{aligned}
&\leq -2\alpha|S|\psi(||x||_S)\cdot||v||_S^2+2\alpha\sum_{i,j,l\in S}\langle v_i(t)-v_j(t),(\psi(r_{il})-\psi(r_{jl}))(v_l(t)-v_j(t))\rangle\\
&\leq -2\alpha|S|\psi(||x||_S)\cdot||v||_S^2+4\alpha LM|S|\sqrt{(|S|-1)|S|}\ \|x\|_S\cdot\|v\|_S,
\end{aligned}
\tag{2.12}
$$

where $|S|$ is the number of elements in S. It is obvious that $|S|>1$ according to the definition of S. For another thing,

$$
\begin{aligned}
&2\alpha\sum_{i,j\in S}\langle v_i(t)-v_j(t),\sum_{l\notin S}[\psi(r_{il})(v_l(t)-v_i(t))-\psi(r_{jl})(v_l(t)-v_j(t))]\rangle\\
&=-\alpha\sum_{i,j\in S,l\notin S}(\psi(r_{il})+\psi(r_{jl}))\cdot\|v_i(t)-v_j(t)\|_S^2\\
&\quad+\alpha\sum_{i,j\in S,l\notin S}\langle v_i(t)-v_j(t),(\psi(r_{il})-\psi(r_{jl}))(v_l(t)-v_j(t))\rangle\\
&\quad+\alpha\sum_{i,j\in S,l\notin S}\langle v_i(t)-v_j(t),(\psi(r_{il})-\psi(r_{jl}))(v_l(t)-v_i(t))\rangle\\
&\leq 2\alpha\sum_{i,j\in S,l\notin S}\langle v_i(t)-v_j(t),(\psi(r_{il})-\psi(r_{jl}))(v_l(t)-v_i(t))\rangle\\
&\leq 2\alpha L\sum_{i,j\in S,l\notin S}\|v_i(t)-v_j(t)\|_S\cdot\|x_i(t)-x_j(t)\|_S\cdot\|v_l(t)-v_i\|\\
&\leq 4\alpha LM(N-|S|)\sum_{i,j\in S}\|v_i(t)-v_j(t)\|_S\cdot\|x_i(t)-x_j(t)\|_S.
\end{aligned}
\tag{2.13}
$$

Using the Cauchy-Schwarz inequality and noting the fact that

$$
\begin{aligned}
&\left(\sum_{i,j\in S}\|v_i(t)-v_j(t)\|_S\cdot\|x_i(t)-x_j(t)\|_S\right)^2\\
\leq&(|S|-1)|S|\sum_{i,j\in S}\|v_i(t)-v_j(t)\|_S^2\cdot\|x_i(t)-x_j(t)\|_S^2,
\end{aligned}
\tag{2.14}
$$

we have a bound $4\alpha LM(N-|S|)\sqrt{(|S|-1)|S|}\ \|x\|_S\cdot\|v\|_S$ of (2.13). Hence,

$$
\mathrm{I}\leq -2\alpha|S|\psi(\|x\|_S)\cdot\|v\|_S^2+4\alpha NLM\sqrt{(|S|-1)|S|}\ \|x\|_S\cdot\|v\|_S. \tag{2.15}
$$

Similarly, we consider II below. Combining (1.4) and II, we have

$$
\begin{aligned}
\mathrm{II}=&\,2\sum_{i,j\in S}\left\langle v_i(t)-v_j(t),\left(\sum_{l\in S}+\sum_{l\notin S}\right)\frac{\|x_i-x_l\|-R}{\|x_i-x_l\|}(x_l(t)-x_i(t))\right\rangle\\
&-2\sum_{i,j\in S}\left\langle v_i(t)-v_j(t),\left(\sum_{l\in S}+\sum_{l\notin S}\right)\frac{\|x_j-x_l\|-R}{\|x_j-x_l\|}(x_l(t)-x_j(t))\right\rangle\\
\leq&\,4\sum_{i,j\in S}\|v_i(t)-v_j(t)\|\left(\sum_{l\in S}+\sum_{l\notin S}\right)\|x_l(t)-x_i(t))\|,
\end{aligned}
\tag{2.16}
$$

where we used the Cauchy-Schwarz inequality again. Assisted by the following auxiliary inequality,

$$\begin{aligned}&\left(\sum_{i,j,l\in S}\|v_i(t)-v_j(t)\|\cdot\|x_l(t)-x_i(t)\|\right)^2\\&\le \mathrm{C}^1_{|S|}(\mathrm{C}^1_{|S|-1})^2\sum_{i,j,l\in S}\|v_i(t)-v_j(t)\|^2\cdot\|x_l(t)-x_i(t)\|^2\\&\le |S|^3(|S|-1)^2\|x(t)\|_S\cdot\|v(t)\|_S,\end{aligned}\tag{2.17}$$

one can be got that

$$\mathrm{II}\le 4(|S|-1)|S|\sqrt{|S|}\ \|x\|_S\cdot\|v\|_S+4(N-|S|)\sqrt{(|S|-1)|S|}\ x_M\cdot\|v\|_S.\tag{2.18}$$

Hence, reviewing (2.9) and the above analysis yields

$$\begin{aligned}\frac{d\|v\|_S^2}{dt}\le&-2\alpha|S|\cdot\psi(\|x\|_S)\|v\|_S^2\\&+4\left(\alpha NLM\sqrt{2(|S|-1)|S|}+(|S|-1)|S|\sqrt{|S|}\right)\|x\|_S\cdot\|v\|_S\\&+4(N-|S|)\sqrt{(|S|-1)|S|}\ x_M\cdot\|v\|_S.\end{aligned}\tag{2.19}$$

For convenience, let $c_0:=\alpha|S|$, $c_1:=2\left(\alpha NLM\sqrt{2(|S|-1)|S|}+(|S|-1)|S|\sqrt{|S|}\right)$ and $c_2:=2(N-|S|)\sqrt{2(|S|-1)|S|}\cdot x_M$, further, it follows that

$$\frac{\mathrm{d}\|v(t)\|_S}{\mathrm{d}t}\le -c_0\psi(\|x(t)\|_S)\|v(t)\|_S+c_1\|x(t)\|_S+c_2,\tag{2.20}$$

where $c_0>0,\ c_1>0,\ c_2>0$. Applying (2.8) to (2.20) produces

$$\begin{aligned}\frac{\mathrm{d}\|v(t)\|_S}{\mathrm{d}t}\le&-c_0\psi(\|x(t)\|_S)\|v\|_S+c_1\|x(t)\|_S+c_2\\\le&-c_0\ \psi(\|x(t)\|_S)\left(-\frac{\mathrm{d}\|x(t)\|_S}{\mathrm{d}t}\right)+c_1\|x(t)\|_S+c_2.\end{aligned}\tag{2.21}$$

And integrating both sides of (2.21) from 0 to $t(t\in[0,a))$, we obtain

$$\|v(t)\|_S-\|v(0)\|_S\le c_0\int_{\|x(0)\|_S}^{\|x(t)\|_S}\psi(r)dr+c_1\int_0^t\|x(\theta)\|_S d\theta+c_2\int_0^t d\theta,$$

that is,

$$c_0\int_{\|x(t)\|_S}^{\|x(0)\|_S}\psi(r)dr\le\|v(0)\|_S-\|v(t)\|_S+\int_0^t c_1\|x(\theta)\|_S d\theta+c_2a.\tag{2.22}$$

The right end of (2.22) is bounded as $t\to a^-$ by Proposition 2.1, while

$$\lim_{t\to a^-}\int_{\|x(t)\|_S}^{\|x(0)\|_S}\psi(r)dr=\int_0^{\|x(0)\|_S}\psi(r)dr=+\infty.$$

This yields a contradiction. Hence, $||x_i(t)-x_j(t)||>0$ for $t\ge 0$, $i,j\in\{1,2,\cdots,N\}$, $i\ne j$, i.e. collision avoidance can be achieved. □

Corollary 2.1. *Under the conditions of the Theorem 2.1, then the existence and uniqueness of the global solution to system (1.3)-(1.5) can be guaranteed.*

Proof. Let ϑ be the maximal time when the local solution can be took in $[0, \vartheta)$. We claim that $\vartheta = +\infty$. If not, suppose that $\vartheta < +\infty$, then there exists $i \neq j \in \{1, 2, \cdots, N\}$ such that $r_{ij}(t) = \|x_i(t) - x_j(t)\| \to 0$ as $t \to \vartheta^-$, which implies that $\lim_{t\to\vartheta^-} \psi(r_{ij}(t)) = +\infty$ and the right hand of the second equation of (1.10) is infinite as $t \to \vartheta^-$. Nevertheless, as a direct application of Proposition 2.1 and Theorem 2.1, we can get the fact that $\left|\frac{\mathrm{d}v_i(t)}{\mathrm{d}t}\right|$ for all $i \in \{1, 2, \cdots, N\}$ and $t \geq 0$ is bounded. This yields a contradiction. □

3. Convergence to a flock

The Lyapunov function approach is developed to get the asymptotic convergence results of System (1.3).

Theorem 3.1. *Suppose $\psi(\cdot)$ satisfy the Assumption 1.1. Let $\{x_i(t), v_i(t)\}$, $i \in \{1, 2, \cdots, N\}$ be the solution to (1.3)-(1.5). If $\|x_{i0} - x_{j0}\| > 0$ for all $i \neq j$, then this system converges to a flock, that is, $\sup_{0\leq t<+\infty} ||x_i(t) - x_j(t)|| \leq x_M$ and $\lim_{t\to+\infty} ||v_i(t) - v_j(t)|| = 0$ for all $i, j \in \{1, 2, \cdots, N\}$ and $i \neq j$, where x_M is expressed in Proposition 2.1.*

Proof. Consider System (1.3)-(1.5), it has been obtained that $\sup_{0\leq t<+\infty} ||x_i(t) - x_j(t)|| \leq x_M := R+\sqrt{2Q(0)}$ in Proposition 2.1. We next show that $\lim_{t\to+\infty} ||v_i(t) - v_j(t)|| = 0$ for all $i, j \in \{1, 2, \cdots, N\}$. We just need to prove that $\lim_{t\to+\infty} \|\hat{v}_i(t)\| = 0$ for $i \in \{1, 2, \cdots, N\}$ for System (1.10).

Take $Q(t)$, which is defined by (2.2), as a candidate Lyapunov function $V(t)$ of the System (1.10). Clearly, $V(t)$ is a continuous and positive definite function. Since $\psi(\cdot)$ is non-increasing and $\hat{r}_{ij}(t) = r_{ij}(t) \leq x_M$ for all $i,\ j \in \{1, 2, \cdots, N\}$ and $t \geq 0$, we have $\psi(\hat{r}_{ij}(t)) \geq \psi(x_M)$ for $t \geq 0$, further,

$$
\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t}\bigg|_{(1.10)} &= -\frac{\alpha}{2}\sum_{i,j=1}^{N}\psi(\hat{r}_{ij})\|\hat{v}_j(t) - \hat{v}_i(t)\|^2 \\
&\leq -\frac{\alpha}{2}\psi(x_M)\sum_{i,j=1}^{N}\|\hat{v}_j(t) - \hat{v}_i(t)\|^2 \\
&= -\alpha N\psi(x_M)\|\hat{v}_i(t)\|^2.
\end{aligned} \tag{3.1}
$$

Subsequently, integrating both sides of (3.1) from 0 to t, we have

$$\alpha N\psi(x_M)\int_0^t ||\hat{v}_i(s)||^2 ds \leq V(0) - V(t) \leq V(0),$$

which means $\int_0^t \|\hat{v}_i(s)\|^2 ds \leq \frac{V(0)}{\alpha N\psi(x_M)}$ for $t \geq 0$. Furthermore, let t tend to $+\infty$, thus $\int_0^{+\infty} \|\hat{v}_i(s)\|^2 ds$ has bound $\frac{V(0)}{\alpha N\psi(x_M)}$. Moreover, according to Proposition 2.1 and Cauchy-Schwarz inequality, we obtain that $\left|\frac{\mathrm{d}\|\hat{v}_i(t)\|^2}{\mathrm{d}t}\right|$ is bounded for $t > 0$ due to the following fact.

$$\left|\frac{\mathrm{d}\|\hat{v}_i(t)\|^2}{\mathrm{d}t}\right| = \left|2\left\langle \hat{v}_i(t), \frac{\mathrm{d}\hat{v}_i(t)}{\mathrm{d}t}\right\rangle\right| \leq 2\|\hat{v}_i(t)\| \cdot \|\dot{\hat{v}}_i(t)\|.$$

It means that $\|\hat{v}_i(t)\|^2$ is uniformly continuous on $(0, +\infty)$. Due to the Barbalat Lemma in [12], we have

$$\lim_{t\to+\infty} ||\hat{v}_i(t)||^2 = \lim_{t\to+\infty} ||v_i(t) - v_c(0)||^2 = 0 \text{ for all } i \in \{1, 2, \cdots, N\}.$$

Therefore, $\lim_{t\to+\infty} ||v_i(t)|| = v_c(0)$ for $i \in \{1, 2, \cdots, N\}$ for System (1.3), further, $\lim_{t\to+\infty} ||v_i(t) - v_j(t)|| = 0$ for $i, j \in \{1, 2, \cdots, N\}$ and the proof is complete. □

Remark 3.1. Proposition 2.1 and Theorem 3.1 show that the flock radius is determined by the initial energy of System (1.3) and the preset relative distance between agents R. The asymptotic flocking velocity is exactly the average velocity of this system at the initial moment. Therefore, the appropriate R and initial configurations can be set so that the system is controlled within a certain range and reaches the expected synchronization velocity.

4. Numerical Simulation

In Section 3 the theoretical results of the asymptotic flocking of System (1.3) have been established. In this section we illustrate the above analysis with several numerical simulations and explore other possible dynamics behavior of System (1.3).

In all simulation experiments, we set the communication rate $\psi(r) = r^{-1}$ that satisfies the Assumption 1.1. The initial positions and velocities are determined randomly from the interval $[0, 10]$ and $[0, 1]$ respectively, and the initial position satisfy $||x_i(0) - x_j(0)|| > 0$ for $i \neq j,\ i,\ j \in \{1, 2, \cdots, N\}$. We will present two graphs to characterize the ordered flocking behavior, which is the velocity convergence and the equilibrium configuration of all agents.

Notice that the initial conditions of each experiment are randomly generated. Therefore, for a fixed system, the equilibrium configuration will be different through repeated experiments with random initial data.

Below we consider the cases of $N = 5$ and $N = 10$. It can be seen intuitively from Figure 1 to Figure 4 that under the conditions of Proposition 2.1 and Theorem 3.1, the velocity of System (1.3) will be synchronized and all agents will be controlled within a certain range.

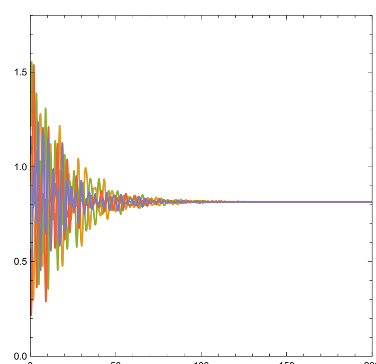
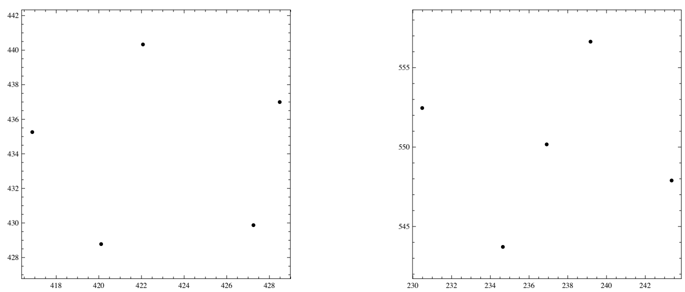


Figure 1. Trajectories of the velocities of the 5 agents with $\alpha = 0.2,\ R = 5$.



(a) (b)

Figure 2. The equilibrium configurations have the following two forms: (a): the five particles at the apex of the regular pentagon; (b): all agents are distributed at the four vertices of a square and its center.

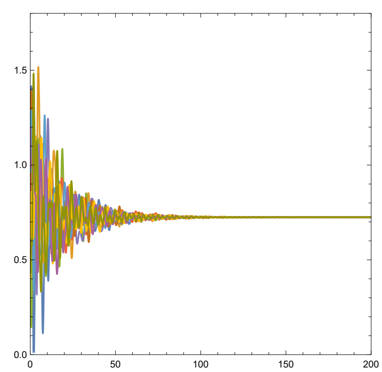
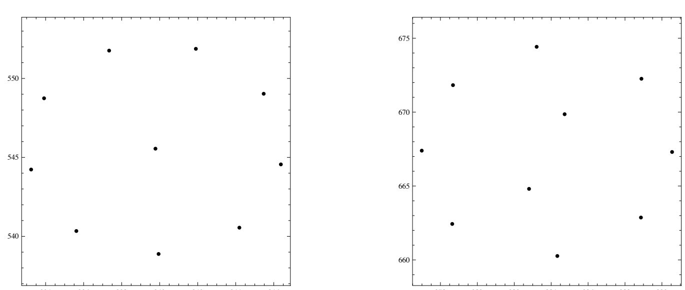


Figure 3. Trajectories of the velocities of the 10 particles with $\alpha = 0.1,\ R = 5$.



(a) (b)

Figure 4. The pictures show the equilibrium configurations may have two cases: (a): nine agents at the vertices of the regular hexagon and one particle at its center; (b): eight agents are distributed at the vertices of the regular octagon to form an outer ring; the remaining ones are stably and evenly located inside the outer ring eventually.

5. Conclusions

A flock model with the singular communication rate and extra interaction items is considered in this paper. Some sufficient conditions have been established to the realization of obstacle avoidance and asymptotic flocking. Particularly, we present a rigorous theoretical proof of collision avoidance between any two agents.

Since the speed of information transmission is limited and each agent requires a certain amount of time for information processing, time delay is unavoidable and should be considered in modeling, which is more practical. Therefore, the future work is to concentrate on the impact of these delay on the results of this paper.

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