Journal of Applied Analysis and Computation
Volume 6, Number 2, May 2016, 515–530
Website:http://jaac-online.com/
DOI:10.11948/2016037

EVANS FUNCTIONS AND BIFURCATIONS OF STANDING WAVE FRONTS OF A NONLINEAR SYSTEM OF REACTION DIFFUSION EQUATIONS*

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Abstract Consider the following nonlinear system of reaction diffusion equations arising from mathematical neuroscience $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\alpha[\beta H(u-\theta)-u]-w$, $\frac{\partial w}{\partial t}=\varepsilon(u-\gamma w)$. Also consider the nonlinear scalar reaction diffusion equation $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\alpha[\beta H(u-\theta)-u]$. In these model equations, $\alpha>0$, $\beta>0$, $\gamma>0$, $\varepsilon>0$ and $\theta>0$ are positive constants, such that $0<2\theta<\beta$. In the model equations, $u=u(x,t)$ represents the membrane potential of a neuron at position x and time t, $w=w(x,t)$ represents the leaking current, a slow process that controls the excitation.

The main purpose of this paper is to couple together linearized stability criterion (the equivalence of the nonlinear stability, the linear stability and the spectral stability of the standing wave fronts) and Evans functions (complex analytic functions) to establish the existence, stability, instability and bifurcations of standing wave fronts of the nonlinear system of reaction diffusion equations and to establish the existence and stability of the standing wave fronts of the nonlinear scalar reaction diffusion equation.

Keywords Nonlinear system of reaction diffusion equations, standing wave fronts, existence, stability, instability, bifurcation, linearized stability criterion, Evans functions.

MSC(2010) 35Q20.

1. Introduction

1.1. The Mathematical Model Equations

Consider the following nonlinear system of reaction diffusion equations arising from mathematical neuroscience

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\alpha[\beta H(u-\theta)-u]-w, \tag{1.1}$$
$$\frac{\partial w}{\partial t}=\varepsilon(u-\gamma w). \tag{1.2}$$

Also consider the nonlinear scalar reaction diffusion equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\alpha[\beta H(u-\theta)-u]. \tag{1.3}$$

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*Project partly supported by a Faculty Research Grant of Lehigh University.

In these model equations, $u = u(x,t)$ represents the membrane potential of a neuron at position x and time t, $w = w(x,t)$ represents the leaking current, a slow process that controls the excitation of neuron membrane. The positive constants $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\varepsilon > 0$ and $\theta > 0$ represent neurobiological mechanisms. The positive constant $\theta > 0$ represents the threshold for excitation. The function $H = H(u-\theta)$ represents the Heaviside step function, which is defined by $H(u-\theta) = 0$ for all $u < \theta$, $H(0) = \frac{1}{2}$ and $H(u-\theta) = 1$ for all $u > \theta$. When an action potential is generated across a neuron membrane, Na^+ ion activation is considerably faster than K^+ ion activation. The positive constant ε represents the ratio of the activation of Na^+ ion channels over the activation of K^+ ion channels. Letting $\varepsilon = 0$ and $w = 0$, equation (1.3) may be obtained from system (1.1)-(1.2). See Feroe [5–7], McKean [8–10], McKean and Moll [11], Rinzel and Keller [12], Rinzel and Terman [13], Terman [14], Wang [15] and [16] for more neurobiological backgrounds of the model system.

1.2. The Main Difficulty, the Main Purposes and the Main Strategy

The main purpose of this paper is to accomplish the existence and stability of standing wave fronts of the nonlinear system of reaction diffusion equations and the nonlinear scalar reaction diffusion equation. The existence of the standing wave fronts follow from standard ideas, methods and techniques in dynamical systems. The stability of the standing wave fronts will be accomplished by coupling together linearized stability criterion and Evans functions. The interesting and difficult points are that the eigenvalue problems obtained by using linearization technique and the method of separation of variables involve the Dirac delta impulse functions. This makes it very difficult to establish the equivalence of the nonlinear stability, the linear stability and the spectral stability of the standing wave fronts. The main strategy to overcome the difficulty is to use the special structure of the model equations to reduce the eigenvalue problems to simplified differential equations and to use the fundamental matrix and the method of variation of parameters to construct general solutions of the eigenvalue problems. Another very interesting point is that the parameter ε plays no role in the existence of the standing wave fronts, but it does play a very important role in the stability of the standing wave fronts. It is worth of pointing out that the nonlinear system is not a singular perturbation problem for the existence and stability of the standing wave fronts.

The construction and application of Evans functions to stability analysis of the standing wave fronts of the nonlinear system of reaction diffusion equations (1.1)-(1.2) have been open for a long time. This paper aims to provide positive solutions to the open problems. Mathematically and biologically, these are very important/interesting problems. We believe that the same ideas also work for the existence and stability of multiple traveling pulse solutions of system (1.1)-(1.2).

1.3. Previous Related Results

There have been some results on the existence and stability of the standing wave fronts of the nonlinear system of reaction diffusion equations (1.1)-(1.2). However, previous proofs on the existence, stability, instability or bifurcations missed a few key points and that is why their mathematical analysis are not rigorously correct. Maybe the claimed results are based on numerical simulations for some specific parameters.

1.4. The Main Results

Theorem 1.1. *(I) Let the positive constants* $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\varepsilon > 0$ *and* $\theta > 0$ *satisfy the condition* $2(1+\alpha\gamma)\theta = \alpha\beta\gamma$. *Then there exist two monotone standing wave fronts* $(U_1, W_1) \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} - \{0\})$ *and* $(U_2, W_2) \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} - \{0\})$ *to the nonlinear system of reaction diffusion equations (1.1)-(1.2). The monotone standing wave fronts are given explicitly by*

$$\begin{aligned}
U_1(x) &= \theta \exp\left(\sqrt{\alpha + \frac{1}{\gamma}}x\right), && \text{on} \quad (-\infty, 0),\\
U_1(x) &= \frac{\alpha\beta\gamma}{1+\alpha\gamma} - \theta \exp\left(-\sqrt{\alpha + \frac{1}{\gamma}}x\right), && \text{on} \quad (0, \infty);\\
U_2(x) &= \frac{\alpha\beta\gamma}{1+\alpha\gamma} - \theta \exp\left(\sqrt{\alpha + \frac{1}{\gamma}}x\right), && \text{on} \quad (-\infty, 0),\\
U_2(x) &= \theta \exp\left(-\sqrt{\alpha + \frac{1}{\gamma}}x\right), && \text{on} \quad (0, \infty).
\end{aligned}$$

(II) The monotone fronts are stable if $\gamma^2\varepsilon > 1$ *and they are unstable if* $0 < \gamma^2\varepsilon < 1$. *The value* $\gamma^2\varepsilon = 1$ *is a bifurcation point for the monotone standing wave fronts of system (1.1)-(1.2).*
(III) Let the positive constants $\alpha > 0$, $\beta > 0$ *and* $\theta > 0$ *satisfy the condition* $2\theta = \beta$. *Then there exist two stable monotone standing wave fronts* $U_1 \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} - \{0\})$ *and* $U_2 \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} - \{0\})$ *to the nonlinear scalar reaction diffusion equation* (1.3). *The standing wave fronts are given explicitly by*

$$\begin{aligned}
U_1(x) &= \theta \exp(\sqrt{\alpha}x), && \text{on} \quad (-\infty, 0),\\
U_1(x) &= \beta - \theta \exp(-\sqrt{\alpha}x), && \text{on} \quad (0, \infty);\\
U_2(x) &= \beta - \theta \exp(\sqrt{\alpha}x), && \text{on} \quad (-\infty, 0),\\
U_2(x) &= \theta \exp(-\sqrt{\alpha}x), && \text{on} \quad (0, \infty).
\end{aligned}$$

2. The Mathematical Analysis and the Proofs of the Main Results

The main purpose of this section is to accomplish the existence and bifurcations of the standing wave fronts. We will couple together linearized stability criterion, Evans functions (complex analytic functions) to accomplish the stability, instability and bifurcation of the standing wave fronts. The proof of Theorem 1.1 consists of several steps.

Let $0 < (1+\alpha\gamma)\theta < \alpha\beta\gamma$. There exist two stable constant solutions $(U, W) = (0, 0)$ and $(U, W) = (\frac{\alpha\beta\gamma}{1+\alpha\gamma}, \frac{\alpha\beta}{1+\alpha\gamma})$.

The proof of Theorem 1.1 consists of several steps.

1. The existence. A standing wave front of (1.1)-(1.2) satisfies the following system of differential equations

$$U'' + \alpha[\beta H(U-\theta) - U] - W = 0,$$
$$\varepsilon(U - \gamma W) = 0.$$

That is

$$U'' - \left(\alpha + \frac{1}{\gamma}\right) U + \alpha\beta H(U-\theta) = 0.$$

Suppose that the increasing standing wave front satisfies the conditions: $U < \theta$ on $(-\infty, 0)$, $U(0) = \theta$, $U'(0) > 0$ and $U > \theta$ on $(0, \infty)$. Then the differential equation reduces to the boundary value problems

$$U'' - \left(\alpha + \frac{1}{\gamma}\right) U = 0, \qquad \text{on} \quad (-\infty, 0), \qquad U(0) = \theta;$$
$$U'' - \left(\alpha + \frac{1}{\gamma}\right) U + \alpha\beta = 0, \qquad \text{on} \quad (0, \infty), \qquad U(0) = \theta.$$

It is easy to solve these differential equations to find the following explicit solutions which also satisfy the boundary conditions

$$U(x) = \theta \exp\left(\sqrt{\alpha + \frac{1}{\gamma}}\, x\right), \qquad \text{on } (-\infty, 0);$$
$$U(x) = \frac{\alpha\beta\gamma}{1+\alpha\gamma} + \left(\theta - \frac{\alpha\beta\gamma}{1+\alpha\gamma}\right) \exp\left(-\sqrt{\alpha + \frac{1}{\gamma}}\, x\right), \qquad \text{on } (0, \infty).$$

Moreover, the solution is continuously differentiable everywhere. The explicit decreasing standing wave front may be obtained very similarly. The proof of the existence of the monotone standing wave fronts of the nonlinear system of reaction diffusion equations (1.1)-(1.2) is finished. A standing wave front of equation (1.3) satisfies the differential equation

$$U'' + \alpha[\beta H(U-\theta) - U] = 0.$$

The proof of the existence of the monotone standing wave fronts of the nonlinear scalar reaction diffusion equation (1.3) is very similar and is omitted.

Let us study the stability of the standing wave fronts of the nonlinear system of reaction diffusion equations (1.1)-(1.2). To keep the mathematical analysis clear, let us focus on the stability analysis of the increasing front. The stability analysis of the decreasing front is very similar.

2. The eigenvalue problems. Let $(P(x,t), Q(x,t)) = (u(x,t), w(x,t))$. Then system (1.1)-(1.2) becomes

$$\frac{\partial P}{\partial t} = \frac{\partial^2 P}{\partial x^2} + \alpha[\beta H(P-\theta) - P] - Q,$$
$$\frac{\partial Q}{\partial t} = \varepsilon(P - \gamma Q).$$

The standing wave front $(U(x), W(x))$ is a stationary solution of this system. Linearizing the nonlinear system about the standing wave front yields

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} + \alpha[\beta\delta(U-\theta)p - p] - q,$$
$$\frac{\partial q}{\partial t} = \varepsilon(p - \gamma q),$$

where $\delta = \delta(x)$ represents the Dirac delta impulse function, defined by

$$\int_{\mathbb{R}} \delta(x-y)\psi(y)\mathrm{d}y = \psi(x),$$
$$\delta(x) = 0, \text{ for all } x \neq 0, \ \delta(x) = \infty, \quad \text{for } x = 0,$$

for all real valued functions $\psi \in C_0^\infty(\mathbb{R})$.

Suppose that $(p(x,t), q(x,t)) = \exp(\lambda t)(\psi_1(x), \psi_2(x))$ is a complex solution of this linear system, where λ is a complex number, ψ_1 and ψ_2 are complex, bounded, continuous functions defined on $\mathbb{R}$. This leads to the following eigenvalue problem

$$\lambda\psi_1 = \psi_1'' - \alpha\psi_1 - \psi_2 + \alpha\beta\delta(U-\theta)\psi_1,$$
$$\lambda\psi_2 = \varepsilon(\psi_1 - \gamma\psi_2).$$

Define the linear differential operator $\mathcal{L}$ by

$$\mathcal{L}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}\psi_1'' - \alpha\psi_1 - \psi_2 + \alpha\beta\delta(U-\theta)\psi_1\\ \varepsilon(\psi_1 - \gamma\psi_2)\end{pmatrix}.$$

Definition 2.1. If there exists a complex number λ_0 and there exists a complex, vector valued, bounded, continuous function $\psi_0(\lambda_0, x) = \begin{pmatrix}\psi_{01}(\lambda_0, x)\\ \psi_{02}(\lambda_0, x)\end{pmatrix}$ defined on $\mathbb{R}$, such that $\mathcal{L}\psi_0 = \lambda_0\psi_0$, then λ_0 is called an eigenvalue and $\psi_0 = \begin{pmatrix}\psi_{01}\\ \psi_{02}\end{pmatrix}$ is called an eigenfunction of the eigenvalue problem.

To see that $\lambda_0 = 0$ is an eigenvalue and $(\psi_1(x), \psi_2(x)) = (U'(x), W'(x))$ is an eigenfunction of the eigenvalue problem, let us differentiate the standing wave equations

$$U'' + \alpha[\beta H(U-\theta) - U] - W = 0,$$
$$\varepsilon(U - \gamma W) = 0,$$

with respect to x to get

$$U''' - \alpha U' - W' + \alpha\beta\delta(U-\theta)U' = 0$$
$$\varepsilon(U' - \gamma W') = 0.$$

Definition 2.2. (I) The standing wave front of the nonlinear system of reaction diffusion equations (1.1)-(1.2) is stable, if $\max\{\mathrm{Re}\lambda : \lambda \in \sigma(\mathcal{L}), \lambda \neq 0\} \leq -C_0$ and if $\lambda_0 = 0$ is an algebraically simple eigenvalue, where $\sigma(\mathcal{L})$ represents the spectrum of the linear differential operator $\mathcal{L}$ and $C_0 > 0$ is a positive constant.
(II) The standing wave front of the nonlinear system of reaction diffusion equations (1.1)-(1.2) is unstable, if there exists an eigenvalue λ_0 with positive real part or if the neutral eigenvalue $\lambda_0 = 0$ is not simple.

Following John Evans' idea in Evans [1–4], the essential spectrum of the linear differential operator $\mathcal{L}$ is easy to find and it is given by

$$\sigma_{\text{essential}}(\mathcal{L}) = \{\lambda \in \mathbb{C} : \lambda = \lambda_1(r) \text{ or } \lambda = \lambda_2(r), r \in \mathbb{R}\},$$

where

$$\lambda_1(r) = -\frac{1}{2}\left[\alpha + r^2 + \gamma\varepsilon + \sqrt{(\alpha + r^2 - \gamma\varepsilon)^2 - 4\varepsilon}\right],$$
$$\lambda_2(r) = -\frac{1}{2}\left[\alpha + r^2 + \gamma\varepsilon - \sqrt{(\alpha + r^2 - \gamma\varepsilon)^2 - 4\varepsilon}\right].$$

It is easy to find that the essential spectrum of $\mathcal{L}$ causes no problem to the stability of the monotone standing wave front of the nonlinear system of reaction diffusion equations (1.1)-(1.2).

Note that in the eigenvalue problem, we can solve the second equation to get $\psi_2 = \frac{\varepsilon}{\lambda+\gamma\varepsilon}\psi_1$. For convenience, let $\psi = \psi_1$. Now the eigenvalue problem becomes

$$\lambda\psi = \psi'' - \alpha\psi - \frac{\varepsilon}{\lambda + \gamma\varepsilon}\psi + \alpha\beta\delta(U - \theta)\psi,$$

which is equivalent to

$$\lambda\psi = \psi'' - \alpha\psi - \frac{\varepsilon}{\lambda + \gamma\varepsilon}\psi + \alpha\beta\frac{\psi(0)}{U'(0)}\delta(x).$$

3. The solutions of the eigenvalue problems.

Now let us study the eigenvalues and eigenfunctions of the eigenvalue problem. The eigenvalue problem may be written as a first order linear system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\psi \\ \psi'\end{pmatrix} = \begin{pmatrix}0 & 1 \\ \alpha + \lambda + \frac{\varepsilon}{\lambda+\gamma\varepsilon} & 0\end{pmatrix}\begin{pmatrix}\psi \\ \psi'\end{pmatrix} - \alpha\beta\frac{\psi(\lambda, 0)}{U'(0)}\delta(x)\begin{pmatrix}0 \\ 1\end{pmatrix}.$$

The solution of the homogeneous system

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\psi \\ \psi'\end{pmatrix} = \begin{pmatrix}0 & 1 \\ \alpha + \lambda + \frac{\varepsilon}{\lambda+\gamma\varepsilon} & 0\end{pmatrix}\begin{pmatrix}\psi \\ \psi'\end{pmatrix}$$

is given by

$$\begin{pmatrix}\psi(\lambda, \varepsilon, x) \\ \psi_x(\lambda, \varepsilon, x)\end{pmatrix} = C_1 \exp\left(\sqrt{\alpha + \lambda + \frac{\varepsilon}{\lambda + \gamma\varepsilon}}x\right)\begin{pmatrix}1 \\ \sqrt{\alpha + \lambda + \frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix}$$
$$+ C_2 \exp\left(-\sqrt{\alpha + \lambda + \frac{\varepsilon}{\lambda + \gamma\varepsilon}}x\right)\begin{pmatrix}1 \\ -\sqrt{\alpha + \lambda + \frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix},$$

where C_1 and C_2 are constants.

Let us diagonalize the coefficient matrix. Define

$$T(\lambda, \varepsilon) = \begin{pmatrix}1 & 1 \\ \sqrt{\alpha + \lambda + \frac{\varepsilon}{\lambda+\gamma\varepsilon}} & -\sqrt{\alpha + \lambda + \frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix}.$$

Then the inverse matrix exists and it is given by

$$[T(\lambda,\varepsilon)]^{-1} = \frac{1}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}} \begin{pmatrix} \sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}} & 1 \\ \sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}} & -1 \end{pmatrix}.$$

Now

$$\begin{aligned} &[T(\lambda,\varepsilon)]^{-1} \begin{pmatrix} 0 & 1 \\ \alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon} & 0 \end{pmatrix} T(\lambda,\varepsilon) \\ &= \begin{pmatrix} \sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}} & 0 \\ 0 & -\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}} \end{pmatrix}. \end{aligned}$$

Clearly

$$X(\lambda,\varepsilon,x) = \begin{pmatrix} \exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right) & \exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right) \\ \sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right) & -\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right) \end{pmatrix}$$

is a fundamental matrix of the homogeneous system.

By using the method of variation of parameters and the fundamental matrix, we find a bounded particular solution of the eigenvalue problem. The particular solution is given by

$$\begin{aligned} &\frac{\alpha\beta\psi(\lambda,\varepsilon,0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)} \exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right) \\ &\cdot \begin{pmatrix} 1 \\ \sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}} \end{pmatrix} [1-H(x)] \\ &+ \frac{\alpha\beta\psi(\lambda,\varepsilon,0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)} \exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right) \\ &\cdot \begin{pmatrix} 1 \\ -\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}} \end{pmatrix} H(x). \end{aligned}$$

Therefore, the general solution of the eigenvalue problem is given by

$$\begin{aligned} \begin{pmatrix} \psi(\lambda,\varepsilon,x) \\ \psi_x(\lambda,\varepsilon,x) \end{pmatrix} &= C_1(\lambda,\varepsilon)\exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right) \begin{pmatrix} 1 \\ \sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}} \end{pmatrix} \\ &+C_2(\lambda,\varepsilon)\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right) \begin{pmatrix} 1 \\ -\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}} \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha\beta\psi(\lambda,\varepsilon,0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)\\
&\cdot\begin{pmatrix}1\\ \sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix}[1-H(x)]\\
&+\frac{\alpha\beta\psi(\lambda,\varepsilon,0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)\\
&\cdot\begin{pmatrix}1\\ -\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix}H(x),
\end{aligned}
$$

where $C_1(\lambda,\varepsilon)$ and $C_2(\lambda,\varepsilon)$ are independent of x, but they depend on the parameters λ and ε. The general solution is bounded on $\mathbb{R}$ if and only if $\begin{pmatrix}C_1(\lambda,\varepsilon)\\ C_2(\lambda,\varepsilon)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$.

The first component of the general solution of the eigenvalue problem is given by

$$
\begin{aligned}
\psi(\lambda,\varepsilon,x)=\;&C_1(\lambda,\varepsilon)\exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)\\
&+C_2(\lambda,\varepsilon)\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)\\
&+\frac{\alpha\beta\psi(\lambda,\varepsilon,0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)[1-H(x)]\\
&+\frac{\alpha\beta\psi(\lambda,\varepsilon,0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)H(x).
\end{aligned}
$$

Let us find the relationship between $C_1(\lambda,\varepsilon)$, $C_2(\lambda,\varepsilon)$ and $\psi(\lambda,\varepsilon,0)$. Letting $x=0$ in the first component of the general solution leads to

$$
\psi(\lambda,\varepsilon,0)=C_1(\lambda,\varepsilon)+C_2(\lambda,\varepsilon)+\frac{\alpha\beta\psi(\lambda,\varepsilon,0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}.
$$

Hence

$$
C_1(\lambda,\varepsilon)+C_2(\lambda,\varepsilon)=\left[1-\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\right]\psi(\lambda,\varepsilon,0).
$$

To be an eigenfunction of the eigenvalue problem, the solution must be bounded as $x\to-\infty$. Let $C_2(\lambda,\varepsilon)=0$ to ensure that asymptotic behaviour. Then

$$
C_1(\lambda,\varepsilon)=\left[1-\frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\right]\psi(\lambda,\varepsilon,0).
$$

4. The Evans function. Let $\Omega = \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda > -\gamma\varepsilon\}$. Define the Evans function for the monotone standing wave front of the nonlinear system of reaction diffusion equations (1.1)-(1.2) by

$$\mathcal{E}_{\text{front}}(\lambda,\varepsilon) = 1 - \frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)} = 1 - \sqrt{\frac{\alpha+\frac{1}{\gamma}}{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}}. \tag{2.1}$$

Define the Evans function of the decreasing standing wave front by

$$\mathcal{E}_{\text{back}}(\lambda,\varepsilon) = 1 - \frac{\alpha\beta}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)} = 1 - \sqrt{\frac{\alpha+\frac{1}{\gamma}}{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}}. \tag{2.2}$$

Note that

$$\begin{aligned}
&\lim_{|\lambda|\to\infty} \mathcal{E}_{\text{front}}(\lambda,\varepsilon) = 1,\\
&\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{front}}(\lambda,\varepsilon) = \frac{1}{2}\frac{1}{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\left[1-\frac{\varepsilon}{(\lambda+\gamma\varepsilon)^2}\right]\sqrt{\frac{\alpha+\frac{1}{\gamma}}{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}},\\
&\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{front}}(0,\varepsilon) = \frac{1}{2}\left(1-\frac{1}{\gamma^2\varepsilon}\right)\frac{1}{\alpha+\frac{1}{\gamma}} > 0, \qquad \text{if } \gamma^2\varepsilon > 1,\\
&\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{front}}(0,\varepsilon) = \frac{1}{2}\left(1-\frac{1}{\gamma^2\varepsilon}\right)\frac{1}{\alpha+\frac{1}{\gamma}} < 0, \qquad \text{if } \gamma^2\varepsilon < 1.
\end{aligned}$$

Now the compatible solution of the eigenvalue problem is given by

$$\begin{aligned}
\begin{pmatrix}\psi(\lambda,\varepsilon,x)\\ \psi_x(\lambda,\varepsilon,x)\end{pmatrix} &= C_1(\lambda,\varepsilon)\exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)\begin{pmatrix}1\\ \sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix}\\
&+C_2(\lambda,\varepsilon)\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)\begin{pmatrix}1\\ -\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix}\\
&+\frac{\alpha\beta\psi(\lambda,\varepsilon,0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)\\
&\cdot\begin{pmatrix}1\\ \sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix}[1-H(x)]\\
&+\frac{\alpha\beta\psi(\lambda,\varepsilon,0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)\\
&\cdot\begin{pmatrix}1\\ -\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}\end{pmatrix}H(x),
\end{aligned}$$

where $C_1(\lambda,\varepsilon)+C_2(\lambda,\varepsilon) = \mathcal{E}_{\text{front}}(\lambda,\varepsilon)\psi(\lambda,\varepsilon,0)$. The first component of the compatible solution is given by

$$\begin{aligned}\psi(\lambda,\varepsilon,x) = {} & C_1(\lambda,\varepsilon)\exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right) \\ & + C_2(\lambda,\varepsilon)\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right) \\ & + \frac{\alpha\beta\psi(\lambda,\varepsilon,0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)[1-H(x)] \\ & + \frac{\alpha\beta\psi(\lambda,\varepsilon,0)}{2\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}U'(0)}\exp\left(-\sqrt{\alpha+\lambda+\frac{\varepsilon}{\lambda+\gamma\varepsilon}}x\right)H(x).\end{aligned}$$

The compatible solution of the eigenvalue problem is bounded on $\mathbb{R}$ if and only if $\mathcal{E}_{\text{front}}(\lambda,\varepsilon)=0$. Let $\mathcal{E}_{\text{front}}(\lambda,\varepsilon)=0$, there exist two eigenvalues

$$\lambda_0 = 0, \qquad \lambda = \frac{1}{\gamma} - \gamma\varepsilon.$$

5. The stability/instability of the increasing standing wave front.

Let us review the linearized stability criterion. The nonlinear stability of the standing wave front of the nonlinear system of reaction diffusion equations (1.1)-(1.2) is equivalent to its linear stability, which is equivalent to the spectral stability.

By using the definitions of the stability and instability of the standing wave front of the nonlinear system of reaction diffusion equations and also by using the linearized stability criterion, we find that the increasing standing wave front is stable if $\gamma^2\varepsilon > 1$ and it is unstable if $\gamma^2\varepsilon < 1$.

6. The bifurcations of the increasing standing wave front.

Obviously, the value $\gamma^2\varepsilon = 1$ is the bifurcation point.

7. The stability of the standing wave fronts of the nonlinear scalar reaction diffusion equation (1.3)**.** For the increasing standing wave front of the nonlinear scalar reaction diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha[\beta H(u-\theta) - u],$$

the eigenvalue problem is

$$\lambda\psi = \psi'' - \alpha\psi + \alpha\beta\frac{\psi(0)}{U'(0)}\delta(x).$$

The linear differential operator is defined by

$$\mathcal{L}_0\psi = \psi'' - \alpha\psi + \alpha\beta\frac{\psi(0)}{U'(0)}\delta(x).$$

The essential spectrum of the operator $\mathcal{L}_0$ is given by

$$\sigma_{\text{essential}}(\mathcal{L}_0) = \{\lambda\in\mathbb{C} : \lambda = -\alpha - r^2, r\in\mathbb{R}\}.$$

The eigenvalue problem $\lambda\psi = \psi'' - \alpha\psi + \alpha\beta\frac{\psi(0)}{U'(0)}\delta(x)$ may be written as a first order linear system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\psi\\ \psi'\end{pmatrix} = \begin{pmatrix}0 & 1\\ \alpha+\lambda & 0\end{pmatrix}\begin{pmatrix}\psi\\ \psi'\end{pmatrix} - \alpha\beta\frac{\psi(0)}{U'(0)}\delta(x)\begin{pmatrix}0\\ 1\end{pmatrix}.$$

The general solution of the eigenvalue problem is given by

$$\begin{aligned}\begin{pmatrix}\psi(\lambda,x)\\ \psi_x(\lambda,x)\end{pmatrix} &= C_1(\lambda)\exp(\sqrt{\alpha+\lambda}x)\begin{pmatrix}1\\ \sqrt{\alpha+\lambda}\end{pmatrix}\\ &+C_2(\lambda)\exp(-\sqrt{\alpha+\lambda}x)\begin{pmatrix}1\\ -\sqrt{\alpha+\lambda}\end{pmatrix}\\ &+\frac{\alpha\beta\psi(\lambda,0)}{2\sqrt{\alpha+\lambda}U'(0)}\exp(\sqrt{\alpha+\lambda}x)\begin{pmatrix}1\\ \sqrt{\alpha+\lambda}\end{pmatrix}[1-H(x)]\\ &+\frac{\alpha\beta\psi(\lambda,0)}{2\sqrt{\alpha+\lambda}U'(0)}\exp(-\sqrt{\alpha+\lambda}x)\begin{pmatrix}1\\ -\sqrt{\alpha+\lambda}\end{pmatrix}H(x).\end{aligned}$$

The first component of the general solution is given by

$$\begin{aligned}\psi(\lambda,x) &= C_1(\lambda)\exp(\sqrt{\alpha+\lambda}x) + C_2(\lambda)\exp(-\sqrt{\alpha+\lambda}x)\\ &+\frac{\alpha\beta\psi(\lambda,0)}{2\sqrt{\alpha+\lambda}U'(0)}\exp(\sqrt{\alpha+\lambda}x)[1-H(x)]\\ &+\frac{\alpha\beta\psi(\lambda,0)}{2\sqrt{\alpha+\lambda}U'(0)}\exp(-\sqrt{\alpha+\lambda}x)H(x).\end{aligned}$$

The Evans function is defined by

$$\mathcal{E}_{\text{front}}(\lambda) = 1 - \sqrt{\frac{\alpha}{\alpha+\lambda}},$$

for all complex numbers $\lambda \in \Omega_0 = \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda > -\alpha\}$. There exists no nonzero eigenvalue in the right half complex plane. The neutral eigenvalue $\lambda_0 = 0$ is algebraically simple. The proof of Theorem 1.1 is completed. □

3. Conclusion and Remarks

3.1. Summary

Consider the following nonlinear system of reaction diffusion equations arising from mathematical neuroscience

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \alpha[\beta H(u-\theta) - u] - w,\\ \frac{\partial w}{\partial t} &= \varepsilon(u - \gamma w).\end{aligned}$$

Also consider the nonlinear scalar reaction diffusion equations

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha[\beta H(u-\theta) - u].$$

There exist two standing wave fronts to both the nonlinear system of reaction diffusion equations and the nonlinear scalar reaction diffusion equation. For the system, if $\gamma^2\varepsilon > 1$, then the standing wave fronts are stable. If $0 < \gamma^2\varepsilon < 1$. then the standing wave fronts are unstable. The value $\gamma^2\varepsilon = 1$ is the bifurcation point for both monotone standing wave fronts. For the nonlinear scalar reaction diffusion equations, the standing wave fronts are stable.

Summary of the eigenvalue problem

$$\lambda\psi = \psi'' - \alpha\psi - \frac{\varepsilon}{\lambda + \gamma\varepsilon}\psi + \alpha\beta\frac{\psi(0)}{U'(0)}\delta(x),$$

and the Evans function

$$\mathcal{E}_{\text{front}}(\lambda,\varepsilon) = 1 - \sqrt{\frac{\alpha + \frac{1}{\gamma}}{\alpha + \lambda + \frac{\varepsilon}{\lambda+\gamma\varepsilon}}},$$

for the nonlinear system of reaction diffusion equations

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha[\beta H(u-\theta) - u] - w,$$
$$\frac{\partial w}{\partial t} = \varepsilon(u - \gamma w).$$

(I) The Evans function $\mathcal{E} = \mathcal{E}_{\text{front}}(\lambda,\varepsilon)$ is a complex analytic function of λ and ε, it is real-valued if λ is real.

(II) The complex number $\lambda_0 \in \Omega$ is an eigenvalue of the eigenvalue problem, if and only if λ_0 is a zero of the Evans function, that is, $\mathcal{E}_{\text{front}}(\lambda_0,\varepsilon) = 0$. In particular, $\mathcal{E}_{\text{front}}(0,\varepsilon) = 0$.

(III) The imaginary part of the Evans function $\mathcal{E}_{\text{front}}(\lambda,\varepsilon)$ is equal to zero if and only if the imaginary part of λ is equal to zero. In another word, all eigenvalues of the eigenvalue problems are real.

(IV) The algebraic multiplicity of any eigenvalue λ_0 of the eigenvalue problem is equal to the order of the zero λ_0 of the Evans function $\mathcal{E} = \mathcal{E}_{\text{front}}(\lambda,\varepsilon)$.

(V) The Evans function enjoys the following limit

$$\lim_{|\lambda|\to\infty} |\mathcal{E}_{\text{front}}(\lambda,\varepsilon)| = 1,$$

in the right half plane $\{\lambda \in \mathbb{C}\colon \text{Re}\lambda > -\gamma\varepsilon\}$.

(VI) Let $\gamma^2\varepsilon > 1$. There hold the following results on the imaginary axis $\mathrm{i}\mathbb{R}$

$$\sup_{\lambda\in\mathrm{i}\mathbb{R}} |\mathcal{E}_{\text{front}}(\lambda,\varepsilon)| = 1, \qquad \sup_{\lambda\in\mathrm{i}\mathbb{R}} |1 - \mathcal{E}_{\text{front}}(\lambda,\varepsilon)| = 1.$$

Moreover, there holds the following estimate on the imaginary axis

$$|\mathcal{E}_{\text{front}}(\lambda,\varepsilon)| > 0,$$

for all $\lambda \in \mathrm{i}\mathbb{R}$ but $\lambda \neq 0$.

(VII) Let $\gamma^2\varepsilon > 1$. There hold the following uniform estimates

$$0 < |\mathcal{E}_{\text{front}}(\lambda, \varepsilon)| < 1,$$

for all $\lambda \in \mathbb{C}$, with Re$\lambda \geq 0$ but $\lambda \neq 0$.

(VIII) The derivative of the Evans function at $\lambda_0 = 0$ is given by

$$\begin{aligned}
&\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{front}}(0, \varepsilon) = \frac{1}{2}\frac{1}{\alpha + \frac{1}{\gamma}}\left(1 - \frac{1}{\gamma^2\varepsilon}\right), \\
&\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{front}}(0, \varepsilon) > 0, \text{ if } \gamma^2\varepsilon > 1, \\
&\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{front}}(0, \varepsilon) = 0, \text{ if } \gamma^2\varepsilon = 1, \\
&\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{front}}(0, \varepsilon) < 0, \text{ if } \gamma^2\varepsilon < 1.
\end{aligned}$$

(IX) There exist exactly two solutions (counting multiplicities) to the equation $\mathcal{E}_{\text{front}}(\lambda, \varepsilon) = 0$: one is the neutral eigenvalue $\lambda = 0$. If $0 < \gamma^2\varepsilon < 1$, then the other eigenvalue is a positive eigenvalue, given by

$$\lambda_0 = \frac{1}{\gamma} - \gamma\varepsilon > 0.$$

If $\gamma^2\varepsilon > 1$, then the other eigenvalue is a negative eigenvalue, given by

$$\lambda_0 = \frac{1}{\gamma} - \gamma\varepsilon < 0.$$

If $\gamma^2\varepsilon = 1$, then the other eigenvalue is $\lambda = 0$. For this case, $\lambda = 0$ is a repeated eigenvalue.

(X) Let $\gamma^2\varepsilon > 1$. On the right half real axis $\{\lambda \in \mathbb{R} : \lambda > 0\}$, there hold the following results

$$\begin{aligned}
&\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{front}}(\lambda, \varepsilon) = \frac{1}{2}\left[1 - \frac{\varepsilon}{(\lambda+\varepsilon)^2}\right]\frac{1}{\alpha + \lambda + \frac{\varepsilon}{\lambda+\gamma\varepsilon}}\sqrt{\frac{\alpha + \frac{1}{\gamma}}{\alpha + \lambda + \frac{\varepsilon}{\lambda+\gamma\varepsilon}}} > 0. \\
&\mathcal{E}_{\text{front}}(0, \varepsilon) = 0, \qquad \lim_{\lambda\to\infty}\mathcal{E}_{\text{front}}(\lambda, \varepsilon) = 1,
\end{aligned}$$

3.2. Further Directions and Open Problems

Consider the following nonlinear singularly perturbed system of reaction diffusion equations arising from mathematical neuroscience

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \alpha[\beta H(u-\theta) - u] - w, \\
\frac{\partial w}{\partial t} &= \varepsilon(u - \gamma w).
\end{aligned}$$

The existence and stability of fast multiple traveling pulse solutions $(U, W) = (U(x+\nu_{\text{fast}}(\varepsilon)t), W(x+\nu_{\text{fast}}(\varepsilon)t))$ with the fast moving coordinates $z = x+\nu_{\text{fast}}(\varepsilon)t$

and the fast wave speeds $\nu_{\text{fast}}(\varepsilon)$, the existence and instability of slow multiple traveling pulse solutions $(U, W) = (U(x + \nu_{\text{slow}}(\varepsilon)t), W(x + \nu_{\text{slow}}(\varepsilon)t))$ with the slow moving coordinates $z = x + \nu_{\text{slow}}(\varepsilon)t$ and the slow wave speeds $\nu_{\text{slow}}(\varepsilon)$ have been open for a long time. In the future, we wish to accomplish these results.

We will construct Evans functions to study the stability/instability of the multiple traveling pulse solutions. For the fast and slow multiple traveling pulse solutions of the nonlinear singularly perturbed system of reaction diffusion equations (1.1)-(1.2), there may hold the following representations for the Evans functions

$$\begin{aligned}
&\mathcal{E}_{\text{fast}-\text{single}-\text{pulse}}(\lambda, \varepsilon) \\
&= \mathcal{E}_{\text{front}}(\lambda)\mathcal{E}_{\text{back}}(\lambda) + \mathcal{E}_{\text{singular}-\text{perturbation}-1}(\lambda, \varepsilon), \\
&\mathcal{E}_{\text{fast}-\text{multiple}-\text{pulse}}(\lambda, \varepsilon) \\
&= [\mathcal{E}_{\text{fast}-\text{single}-\text{pulse}}(\lambda, \varepsilon)]^m + \mathcal{E}_{\text{singular}-\text{perturbation}-2}(\lambda, \varepsilon), \\
&\mathcal{E}_{\text{slow}-\text{single}-\text{pulse}}(\lambda, \varepsilon) \\
&= \mathcal{E}_{\text{standing}-\text{single}-\text{pulse}}(\lambda, \varepsilon) + \mathcal{E}_{\text{singular}-\text{perturbation}-3}(\lambda, \varepsilon), \\
&\mathcal{E}_{\text{slow}-\text{multiple}-\text{pulse}}(\lambda, \varepsilon) \\
&= [\mathcal{E}_{\text{slow}-\text{single}-\text{pulse}}(\lambda, \varepsilon)]^m + \mathcal{E}_{\text{singular}-\text{perturbation}-4}(\lambda, \varepsilon),
\end{aligned}$$

for all $\lambda \in \mathbb{C}$ with $\lambda > -\gamma\varepsilon$ and for all $\varepsilon > 0$, where $\mathcal{E}_{\text{front}}(\lambda)$ stands for the Evans functions of the traveling wave front of

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha[\beta H(u - \theta) - u],$$

and $\mathcal{E}_{\text{back}}(\lambda)$ stands for the traveling wave back of

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha[\beta H(u - \theta) - u] - w_0,$$

for $w_0 = \alpha(\beta - 2\theta)$. Moreover

$$\begin{aligned}
\mathcal{E}_{\text{singular}-\text{perturbation}-1}(\lambda, \varepsilon) &= \lambda \exp\big[-\frac{1}{\varepsilon}O(1)\big]O(1), \\
\mathcal{E}_{\text{singular}-\text{perturbation}-2}(\lambda, \varepsilon) &= \lambda \exp\big[-\frac{1}{\varepsilon}O(1)\big]O(1), \\
\mathcal{E}_{\text{singular}-\text{perturbation}-3}(\lambda, \varepsilon) &= \lambda \exp\big[-\frac{1}{\varepsilon}O(1)\big]O(1), \\
\mathcal{E}_{\text{singular}-\text{perturbation}-4}(\lambda, \varepsilon) &= \lambda \exp\big[-\frac{1}{\varepsilon}O(1)\big]O(1).
\end{aligned}$$

Let us compute the derivatives with respect to λ. We have

$$\begin{aligned}
\frac{\partial}{\partial \lambda}\mathcal{E}_{\text{fast}-\text{single}-\text{pulse}}(\lambda, \varepsilon) = \mathcal{E}'_{\text{front}}(\lambda)\mathcal{E}_{\text{back}}(\lambda) + \mathcal{E}_{\text{front}}(\lambda)\mathcal{E}'_{\text{back}}(\lambda) \\
+ \frac{\partial}{\partial \lambda}\mathcal{E}_{\text{singular}-\text{perturbation}-1}(\lambda, \varepsilon),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{fast}-\text{multiple}-\text{pulse}}(\lambda,\varepsilon)\\
&= m[\mathcal{E}_{\text{front}}(\lambda)\mathcal{E}_{\text{back}}(\lambda)+\mathcal{E}_{\text{singular}-\text{perturbation}-1}(\lambda,\varepsilon)]^{m-1}\\
&\quad\cdot\left\{\mathcal{E}'_{\text{front}}(\lambda)\mathcal{E}_{\text{back}}(\lambda)+\mathcal{E}_{\text{front}}(\lambda)\mathcal{E}'_{\text{back}}(\lambda)+\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{singular}-\text{perturbation}-1}(\lambda,\varepsilon)\right\}\\
&\quad+\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{singular}-\text{perturbation}-2}(\lambda,\varepsilon),
\end{aligned}$$

for all $\lambda\in\mathbb{C}$ with $\lambda>-\gamma\varepsilon$ and for all $\varepsilon>0$.

Let $\lambda=0$, note that $\mathcal{E}_{\text{front}}(0)=0$ and $\mathcal{E}_{\text{back}}(0)=0$. Therefore, we see that

$$\begin{aligned}
&\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{fast}-\text{single}-\text{pulse}}(0,\varepsilon)\\
&=\mathcal{E}'_{\text{front}}(0)\mathcal{E}_{\text{back}}(0)+\mathcal{E}_{\text{front}}(0)\mathcal{E}'_{\text{back}}(0)+\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{singular}-\text{perturbation}-1}(0,\varepsilon)\\
&=\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{singular}-\text{perturbation}-1}(0,\varepsilon)>0,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{fast}-\text{multiple}-\text{pulse}}(0,\varepsilon)\\
&=m[\mathcal{E}_{\text{front}}(0)\mathcal{E}_{\text{back}}(0)+\mathcal{E}_{\text{singular}-\text{perturbation}-1}(0,\varepsilon)]^{m-1}\\
&\quad\cdot[\mathcal{E}'_{\text{front}}(0)\mathcal{E}_{\text{back}}(0)+\mathcal{E}_{\text{front}}(0)\mathcal{E}'_{\text{back}}(0)+\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{singular}-\text{perturbation}-}(0,\varepsilon)]\\
&\quad+\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{singular}-\text{perturbation}-2}(0,\varepsilon)\\
&=\frac{\partial}{\partial\lambda}\mathcal{E}_{\text{singular}-\text{perturbation}-2}(0,\varepsilon)>0.
\end{aligned}$$

References

[1] John W. Evans, *Nerve axon equations, I Linear approximations*, Indiana University Mathematics Journal, 21(1971), 877–885.

[2] John W. Evans, *Nerve axon equations, II Stability at rest*, Indiana University Mathematics Journal, 22(1972), 75–90.

[3] John W. Evans, *Nerve axon equations, III Stability of the nerve impulse,* Indiana University Mathematics Journal, 22(1972), 577–593.

[4] John W. Evans, *Nerve axon equations, IV The stable and the unstable impulse,* Indiana University Mathematics Journal, 24(1975), 1169–1190.

[5] John A. Feroe, *Traveling waves of infinitely many pulses in nerve equations,* Mathematical Biosciences, 55(1981), 189–203.

[6] John A. Feroe, *Existence and stability of multiple impulse solutions of a nerve equation,* SIAM Journal on Applied Mathematics, 42(1982), 235–246.

[7] John A. Feroe, *Existence of traveling wave trains in nerve axon equations,* SIAM Journal on Applied Mathematics, 46(1986), 1079–1097.

[8] Henry P. McKean, *Nagumo's equation,* Advances in Mathematics, 4(1970), 209–223.

[9] Henry P. McKean, *Stabilization of solutions of a caricature of the Fitzhugh-Nagumo equation,* Communications in Pure and Applied Mathematics, 36(1983), 291–324.

[10] Henry P. McKean, *Stabilization of solutions of a caricature of the Fitzhugh-Nagumo equation. II,* Communications in Pure and Applied Mathematics, 37(1984), 299–301.

[11] Henry P. McKean and Victor Moll, *Stabilization to the standing wave in a simple caricature of the nerve equation,* Communications in Pure and Applied Mathematics, 39(1986), 485–529.

[12] John Rinzel and Joseph B. Keller, *Traveling wave solutions of a nerve conduction equation,* Biophysical Journal, 13(1973), 1313–1337.

[13] John Rinzel and David Terman, *Propagation phenomena in a bistable reaction-diffusion system,* SIAM Journal on Applied Mathematics, 42(1982), 1111–1137.

[14] David Terman, *Threshold phenomena for a reaction-diffusion system,* Journal of Differential Equations, 47(1983), 406–443.

[15] Wei-Ping Wang, *Multiple impulse solutions to McKean's caricature of the nerve equation. I. Existence,* Communications on Pure and Applied Mathematics, 41(1988), 71–103.

[16] Wei-Ping Wang, *Multiple impulse solutions to McKean's caricature of the nerve equation. II. Stability,* Communications on Pure and Applied Mathematics, 41(1988), 997–1025.