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GLOBAL ANALYSIS FOR A DELAYED SIV MODEL WITH DIRECT AND ENVIRONMENTAL TRANSMISSIONS*

Tongqian Zhang[1,2,†], Xinzhu Meng[1,2] and Tonghua Zhang[3]

Abstract In this paper, we propose a new SIV epidemic model with time delay, which also involves both direct and environmental transmissions. For such model, we first introduce the basic reproduction number $\mathscr{R}$ by using the next generation matrix. And then global stability of the equilibria is discussed by means of Lyapunov functionals and LaSalle's invariance principle for delay differential equations, which shows that the infection-free equilibrium of the system is globally asymptotically stable if $\mathscr{R} < 1$ and the epidemic equilibrium of the system is globally asymptotically stable for $\mathscr{R} > 1$.

Keywords Global stability, environmental transmission, time delay, Lyapunov functional.

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1. Introduction and model formulation

It is well known that many diseases are transmitted by close contact with the source of infection. But recently empirical studies suggest that environmental transmission is also playing an important role in the spread of some diseases including many

(a) human diseases such as gastroenteritis [7], cholera [15, 17], chronic wasting disease [22], tetanus [27];

(b) animal diseases - Hepatitis E virus (HEV) in pigs [2], rabbit haemorrhagic disease [10], avian cholera [4], epizootics of plague [30]), to name a few; and

(c) zoonoses: salmonella [32], Nipah and Hendra viral diseases [8], bovine spongiform encephalopathy [1], highly pathogenic avian influenza [23, 31], brucellosis [6, 14], for instance.

[†]the corresponding author. Email address:zhangtongqian@sudst.edu.cn (T. Zhang)
[1]College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao 266590, P. R. China
[2]State Key Laboratory of Mining Disaster Prevention and Control Co-founded by Shandong Province and the Ministry of Science and Technology, Shandong University of Science and Technology, Qingdao 266590, P.R.China
[3]Department of Mathematics, Swinburne University of Technology, Hawthorn, VIC 3122, Australia
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Complexity of transmission dynamics of certain diseases between the host and the pathogen in the environment has led to both direct host to host and indirect environment to host transmission pathways [11, 26, 33]. However, the common mathematical models for describing the spread of infectious diseases only considered one of the hand [3, 5, 12, 13, 18, 19, 21]. So it is desired to establish epidemic models involving both direct and environmental transmissions. Few works have been done, for example Breban et al. [24] proposed a general multi-strain model with direct and environmental transmission to analyze the transpiration dynamics of avian influenza viruses as follows,

$$\begin{cases} \dfrac{dS}{dt} = \pi - \mu S - S\displaystyle\sum_{i=1}^{n}\beta_i I_i - \rho S f(V), \\ \dfrac{dI_j}{dt} = \beta_j S I_j - (\mu+\gamma_j)I_j + \rho S f(V)\varepsilon_j(V), \\ \dfrac{dV_j}{dt} = \omega_j I_j - \eta_j V_j, \end{cases} \tag{1.1}$$

where $j = 1, 2, \cdots, n$. S represents the number of susceptible individuals, I_j represents the number of individuals infected with strain j, and V_j represents the number of virion of strain j contaminating the environment; the mixed viral population is denoted by $V = \{V_1, V_2, \cdots, V_n\}$; Roche et al. [25] expanded model (1.1) by considering the mortality due to diseases into the following one

$$\begin{cases} \dfrac{dS}{dt} = \nu - \left(\sigma\beta I(t) + \rho\dfrac{V(t)}{V(t)+K}\right)S(t) - \mu S(t), \\ \dfrac{dI}{dt} = \left(\sigma\beta I(t) + \rho\dfrac{V(t)}{V(t)+K}\right)S(t) - (\mu+\gamma+\alpha)I(t), \\ \dfrac{dV}{dt} = \omega I(t) - \eta V(t). \end{cases} \tag{1.2}$$

Nowadays, time delays are commonly used in biological models to reflect some biological facts, for example, a gestation period or reaction time of a population [34]. Motivated by the biological meaning, we introduce a time delay, τ in the process of an individual becoming infectious, which results in

$$\begin{cases} \dfrac{dS}{dt} = \nu - \left(\sigma\beta I(t) + \rho\dfrac{V(t)}{V(t)+K}\right)S(t) - \mu S(t), \\ \dfrac{dI}{dt} = e^{-\mu\tau}\left(\sigma\beta I(t-\tau) + \rho\dfrac{V(t-\tau)}{V(t-\tau)+K}\right)S(t-\tau) - (\mu+\gamma+\alpha)I(t), \\ \dfrac{dV}{dt} = \omega I(t) - \eta V(t), \end{cases} \tag{1.3}$$

where $S(t)$ represents the number of susceptible individuals, $I(t)$ represents the number of individuals infected and $V(t)$ represents the number of virion in the environment. ν represents the input of susceptible individuals. σ and β represent the host contact rate and β denotes infectiousness, respectively. ρ represents the uptake rate of the environmental reservoir.

The rest of paper is to investigate the globally asymptotic stability of the equilibria of model (1.3), and is organised as follows. We first derive the basic reproduction number and show the existence of infection-free equilibrium and the endemic equilibrium in Section 2. Then Section 3 dedicates to the investigation of local stability of the equilibria. In Section 4, we focus on the globally asymptotic stability of the two equilibria. Finally, we briefly conclude the paper in Section 5.

2. Basic reproduction number and the existence of the equilibria

Let $C = C([-\tau, 0], R^3)$ denote the Banach space of continuous functions from $[-\tau, 0]$ to R_3 and the initial condition of (1.3) be

$$S(\varsigma) = \phi_1(\varsigma) \geq 0, I(\varsigma) = \phi_2(\varsigma) \geq 0, V(\varsigma) = \phi_3(\varsigma) \geq 0, \varsigma \in [-\tau, 0], \tag{2.1}$$

where $\phi = (\phi_1, \phi_2, \phi_3)^T \in C$. Then a standard procedure from [9,16] will show that the solution of the initial value problem (1.3) and (2.1) exists for all $t > 0$, and furthermore it is unique, non-negative and ultimately bounded.

Obviously (1.3) has a infection-free equilibria $E_0 = (\nu/\mu, 0, 0)$. Denote $m = \mu+\gamma+\alpha$ for the sake of simplicity and let $z = (I, V, S)^T$. Then we have from model (1.3)

$$z' = F(z) - V(z),$$

where

$$F(z) = \begin{pmatrix} e^{-\mu\tau}\left(\sigma\beta I(t-\tau) + \rho\frac{V(t-\tau)}{V(t-\tau)+K}\right)S(t-\tau) \\ 0 \\ 0 \end{pmatrix},$$

$$V(z) = \begin{pmatrix} mI \\ -\omega I + \eta V \\ -\nu + \left(\sigma\beta I(t) + \rho\frac{V(t)}{V(t)+K}\right)S(t) + \mu S(t) \end{pmatrix}.$$

The Jacobian matrices of $F(z)$ and $V(z)$ at E_0 are

$$DF(E_0) = \begin{pmatrix} F_{2\times 2} & 0 \\ & 0 \\ 0 \quad 0 & 0 \end{pmatrix} \text{ and } DV(E_0) = \begin{pmatrix} V_{2\times 2} & 0 \\ & 0 \\ \frac{\sigma\beta\nu}{\mu} \quad \frac{\rho\nu}{K\mu} & \mu \end{pmatrix},$$

respectively, where

$$F_{2\times 2} = \begin{pmatrix} e^{-\mu\tau}\dfrac{\sigma\beta\nu}{\mu} & e^{-\mu\tau}\frac{\rho\nu}{K\mu} \\ 0 & 0 \end{pmatrix}, V_{2\times 2} = \begin{pmatrix} m & 0 \\ -\omega & \eta \end{pmatrix}.$$

Then from [29] the next generation matrix for model (1.3) is FV^{-1} and the spectral radius of matrix FV^{-1} is

$$\rho(FV^{-1}) = \frac{e^{-\mu\tau}\nu(\sigma\beta\eta K + \rho\omega)}{m\eta\mu K} \equiv \mathscr{R}, \tag{2.2}$$

which is the basic reproduction number of system (1.3).

It is easy to check that (1.3) has an equilibrium $E_1 = (S^*, I^*, V^*)$ with

$$S^* = \frac{v - e^{\mu\tau}mI^*}{\mu} = \frac{e^{\mu\tau}m(\omega I^* + \eta K)}{\sigma\beta\omega I^* + \sigma\beta\eta K + \rho\omega}, V^* = \frac{\omega I^*}{\eta},$$

substituting which into the second equation of (1.3) and equating to zero yield

$$a(I^*)^2 + bI^* + c = 0, \tag{2.3}$$

where

$$\begin{aligned} a =& \sigma\beta m\omega > 0, \\ b =& \sigma\beta mK\eta + \rho\omega m + m\omega\mu - e^{-\mu\tau}\sigma\beta\omega\nu, \\ c =& m\eta\mu K - e^{-\mu\tau}\nu(\sigma\beta\eta K + \rho\omega). \end{aligned}$$

Then we know that if $\mathscr{R} > 1$, i.e. $c < 0$, equation (2.3) has a unique positive root. Then we reach

Theorem 2.1. *If $\mathscr{R} > 1$, model (1.3) has an endemic equilibrium $E_1(S^*, I^*, V^*)$, where*

$$S^* = \frac{e^{\mu\tau}m(\omega I^* + \eta K)}{\sigma\beta\omega I^* + \sigma\beta\eta K + \rho\omega}, I^* = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, V^* = \frac{\omega I^*}{\eta}.$$

3. Local stability of the equilibria

Regarding the local stability, we have the following theorem.

Theorem 3.1. *For model (1.3), we have*

(i) E_0 is locally stable if $\mathscr{R} < 1$ and unstable if $\mathscr{R} > 1$; and

(ii) E_1 is locally stable when it exists.

Proof. We firstly prove (i). Letting $x = S - S_0, y = I, z = V$ in (1.3) yields

$$\begin{cases} x'(t) = -(\sigma\beta y(t) + \rho z(t))xS(t) - (\sigma\beta y(t) + \rho z(t))S_0 - \mu x(t), \\ y'(t) = e^{-\mu\tau}(\sigma\beta y(t-\tau) + \rho z(t-\tau))x(t-\tau) \\ \qquad\quad + e^{-\mu\tau}(\sigma\beta y(t-\tau) + \rho z(t-\tau))S_0 - my(t), \\ z'(t) = \omega y(t) - \eta z(t). \end{cases} \tag{3.1}$$

Notice the Jacobian of model (3.1) evaluated at E_0 is

$$J(E_0) = \begin{pmatrix} -\mu & -\sigma\beta S_0 & -\rho S_0 \\ 0 & e^{-\mu\tau}\sigma\beta S_0 - m & e^{-\mu\tau}\rho S_0 \\ 0 & \omega & -\eta \end{pmatrix},$$

from which we have the characteristic equation

$$(-\mu - \lambda)((e^{-\mu\tau}\sigma\beta S_0 e^{-\lambda\tau} - m - \lambda)(-\eta - \lambda) - e^{-\mu\tau}\rho\omega S_0) = 0, \tag{3.2}$$

Clearly, it has a root $\lambda = -\mu < 0$. Then we only need to analyze the distribution of roots, which determines the stability of solution of system (3.1), of equation

$$P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0, \tag{3.3}$$

where

$$P(\lambda) = \lambda^2 + A\lambda + B, Q(\lambda) = C\lambda + D,$$

and

$$A = m + \eta,\ B = m\eta - e^{-\mu\tau}\rho\omega S_0,\ C = -e^{-\mu\tau}\sigma\beta S_0,\ D = -e^{-\mu\tau}\eta\sigma\beta S_0.$$

When $\tau = 0$, (3.3) reduces to

$$\lambda^2 + (m + \eta - \sigma\beta S_0)\lambda + [m\eta - S_0(\rho\omega + \sigma\beta\eta)] = 0. \tag{3.4}$$

Since that $\mathscr{R}_{\tau=0} < 1$ implies $m+\eta-\sigma\beta S_0 > 0$ and $m\eta - S_0(\rho\omega+\sigma\beta\eta) > 0$, we know the two roots of (3.4) have always negative real part. Next, we assume equation (3.3) with $\tau \neq 0$ has a pair of pure imaginary roots $\pm i\varpi(\varpi > 0)$, which implies equation

$$F(\varpi) = \varpi^4 + (A^2 - C^2 - 2B)\varpi^2 + B^2 - D^2 = 0 \tag{3.5}$$

has at least one positive solution. Since $\mathscr{R} = \frac{e^{-\mu\tau}S_0(\sigma\beta\eta+\rho\omega)}{m\eta} < 1$, then $B > 0, D < 0$ and $m > e^{-\mu\tau}\sigma\beta S_0$. Then it is to see that

$$\begin{aligned} B^2 - D^2 &= (B - D)(B + D) \\ &= (B - D)(m\eta - e^{-\mu\tau}\rho\omega S_0 - e^{-\mu\tau}\sigma\beta S_0) \\ &= (B - D)m\eta(1 - e^{-\mu\tau}S_0(\sigma\beta\eta + \rho\omega)) \\ &= (B - D)m\eta(1 - \mathscr{R}) \\ &> 0, \end{aligned}$$

and

$$\begin{aligned} A^2 - C^2 - 2B &= (m + \eta)^2 - (e^{-\mu\tau}\sigma\beta S_0)^2 - 2(m\eta - e^{-\mu\tau}\rho\omega S_0) \\ &= \eta^2 + 2e^{-\mu\tau}\rho\omega S_0 + m^2 - (e^{-\mu\tau}\sigma\beta S_0)^2 \\ &> \eta^2 + 2e^{-\mu\tau}\rho\omega S_0 \\ &> 0. \end{aligned}$$

Then we can conclude that when $\mathscr{R} < 1$, equation (3.5) has no positive real root, which leads to that equation (3.4) does not have pure imaginary root. It then implies that all the roots of (3.3) have always negative real parts. Thus the infection-free equilibrium E_0 of system is locally stable.

Similarly we can discuss the local stability of E_1. To this end, we let $x = S - S^*, y = I - I^*, z = V - V^*$, to shift the equilibrium to the original. Then linearisation at the original results in a characteristic equation

$$P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0, \tag{3.6}$$

where

$$P(\lambda) = b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0,$$

and

$$Q(\lambda) = c_2\lambda^2 + c_1\lambda + c_0,$$

with

$$\begin{aligned}
b_0 &= \Lambda m\eta,\\
b_1 &= \Lambda\eta + S^* m\eta + \Lambda m,\\
b_2 &= S^*\eta + \Lambda + S^* m,\\
b_3 &= S^*,\\
c_0 &= -\mu {S^*}^2(\sigma\beta\eta + \omega\rho)e^{-\mu\tau},\\
c_1 &= -{S^*}^2(\sigma\beta\eta + \mu\sigma\beta + \rho\omega)e^{-\mu\tau},\\
c_2 &= -{S^*}^2 e^{-\mu\tau}\sigma\beta.
\end{aligned}$$

When $\tau = 0$, we claim that the roots of (3.6) always have negative real parts. In fact, if $\tau = 0$, (3.6) reduces

$$f(\lambda) = a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0,$$

where

$$\begin{aligned}
a_3 &= S^*\eta, & a_2 &= S^*\eta^2 + \Lambda\eta + {S^*}^2\rho\omega,\\
a_1 &= \Lambda\rho\omega S^* + \Lambda\eta^2 + mI^*S^*\eta\sigma\beta, & a_0 &= mI^*S^*\eta\rho\omega + mI^*S^*\eta^2\sigma\beta.
\end{aligned}$$

Obviously, $a_3, a_2, a_1, a_0 > 0$ and

$$\begin{aligned}
& a_2a_1 - a_0a_3\\
=& 2\Lambda\rho\omega {S^*}^2\eta^2 + \Lambda^2\rho\omega S^*\eta + \Lambda\rho^2\omega^2 {S^*}^3\\
& + \Lambda\eta^4 S^* + \Lambda^2\eta^3 + mI^*{S^*}^2\eta^3\sigma\beta + mI^*S^*\eta^2\sigma\beta\Lambda\\
& + mI^*{S^*}^3\eta\sigma\beta\rho\omega - mI^*{S^*}^2\eta^2\rho\omega - mI^*{S^*}^2\eta^3\sigma\beta\\
=& \Lambda\rho {S^*}^2\omega\eta^2 + \Lambda^2\rho S^*\omega\eta + \Lambda\rho^2{S^*}^3\omega^2 + \Lambda\eta^4 S^* + \Lambda^2\eta^3 + {S^*}^3\eta^2\mu\rho\omega\\
& + \Lambda\sigma\beta {S^*}^3\eta * \rho\omega - \mu\sigma\beta {S^*}^4\eta\rho\omega + \Lambda^2\sigma\beta S^*\eta^2 - \mu\sigma\beta {S^*}^2\eta^2\Lambda\\
=& \Lambda\rho {S^*}^2\omega\eta^2 + \Lambda^2\rho S^*\omega\eta + \Lambda\rho^2{S^*}^3\omega^2 + \Lambda\eta^4 S^* + \Lambda^2\eta^3 + {S^*}^3\eta^2\mu\rho\omega\\
& + (\sigma\beta\rho\omega\eta {S^*}^3 + \Lambda\sigma\beta S^*\eta^2)(\Lambda - \mu S^*)\\
=& \Lambda\rho {S^*}^2\omega\eta^2 + \Lambda^2\rho S^*\omega\eta + \Lambda\rho^2{S^*}^3\omega^2 + \Lambda\eta^4 S^* + \Lambda^2\eta^3 + {S^*}^3\eta^2\mu\rho\omega\\
& + (\sigma\beta\rho\omega\eta {S^*}^3 + \Lambda\sigma\beta S^*\eta^2)\mu S^*(\mathscr{R} - 1) > 0.
\end{aligned}$$

Then by the Rouche-Hurwitz Criterion [28], all the roots of $f(\lambda) = 0$ always have negative real parts. We then assume (3.6) has a pair of pure imaginary roots $i\varpi(\varpi > 0)$, which results in

$$G(\varpi) = d_3\varpi^6 + d_2\varpi^4 + d_1\varpi^2 + d_0 = 0, \tag{3.7}$$

here

$$\begin{aligned}
d_0 &= b_0^2 - c_0^2, & d_1 &= b_1^2 + 2c_0c_2 - 2b_0b_2 - c_1^2,\\
d_2 &= b_2^2 - 2b_1b_3 - c_2^2, & d_3 &= b_3^2 > 0.
\end{aligned}$$

Notice $\mathscr{R} = \dfrac{e^{-\mu\tau} S_0(\sigma\beta\eta + \rho\omega)}{m\eta} > 1$, then

$$\begin{aligned} d_0 &= (b_0 + |c_0|)(b_0 - |c_0|) = (b_0 + |c_0|)(\Lambda m\eta - \mu {S^*}^2(\sigma\beta\eta + \omega\rho)e^{-\mu\tau}) \\ &= (b_0 + |c_0|)\Lambda m\eta(1 - \frac{1}{\mathscr{R}}) > 0, \end{aligned}$$

$$\begin{aligned} d_1 &= (S^*\eta + \Lambda + S^*m)^2 - 2(\Lambda\eta + S^*m\eta + \Lambda m)S^* - (-{S^*}^2 e^{-\mu\tau}\sigma\beta)^2 \\ &= (S^*\eta)^2 + \Lambda^2 + (S^*m)^2 - ({S^*}^2 e^{-\mu\tau}\sigma\beta)^2 \\ &= (S^*\eta)^2 + \Lambda^2 + (S^*m)^2 - {S^*}^2(S^* e^{-\mu\tau}\sigma\beta)^2 \\ &= (S^*\eta)^2 + \Lambda^2 + (S^*m)^2 - {S^*}^2(\frac{\Lambda}{\mu\mathscr{R}}\frac{\sigma\beta\mu m\eta\mathscr{R}}{\Lambda(\sigma\beta\eta + \rho\omega)})^2 \\ &= (S^*\eta)^2 + \Lambda^2 + (S^*m)^2 - m{S^*}^2\left(\frac{\sigma\beta\eta}{\sigma\beta\eta + \rho\omega}\right)^2 \\ &> (S^*\eta)^2 + \Lambda^2 > 0 \end{aligned}$$

and

$$\begin{aligned} d_2 =&(\Lambda\eta + S^*m\eta + \Lambda m)^2 + 2\mu {S^*}^4(\sigma\beta\eta + \omega\rho)(e^{-\mu\tau})^2\sigma\beta - 2\Lambda m\eta(S^*\eta + \Lambda + S^*m) \\ &- ({S^*}^2(\sigma\beta\eta + \mu\sigma\beta + \rho\omega)e^{-\mu\tau})^2 \\ =&(\Lambda\eta)^2 + (S^*m\eta)^2 + (\Lambda m)^2 - 2(e^{-\mu\tau})^2{S^*}^2\sigma\beta\eta\rho\omega \\ =&(\Lambda\eta)^2 + (S^*m\eta)^2 + (\Lambda m)^2 - 2{S^*}^2\left(\frac{\Lambda}{\mu\mathscr{R}}\right)^2\left(\frac{\mu m\eta\mathscr{R}}{\Lambda(\sigma\beta\eta + \rho\omega)}\right)^2\sigma\beta\eta\rho\omega \\ =&(\Lambda\eta)^2 + (\Lambda m)^2 + (S^*m\eta)^2 - (S^*m\eta)^2\frac{2\sigma\beta\eta\rho\omega}{(\sigma\beta\eta + \rho\omega)^2} \\ >&(\Lambda\eta)^2 + (\Lambda m)^2 > 0. \end{aligned}$$

Then equation (3.6) have no pure imaginary roots $\pm i\varpi$. Thus all roots of (3.6) have always negative real parts. This completes the proof. □

4. Global stability of the equilibria

Following the idea of [20], we will prove the global stability of the two equilibria of model (1.3) in this section. Let $(S_t, I_t, V_t)^T = (S(t+\theta), I(t+\theta), V(t+\theta))^T, (-\tau \leq \theta \leq 0)$ be any solution of (1.3) with the initial condition (1.3). Then we can prove

Theorem 4.1. *For model* (1.3) *and* $\tau \geq 0$, *we have*

(i) if $\mathscr{R} < 1$, *the infection-free equilibrium* E_0 *is globally asymptotically stable;*

(ii) if $\mathscr{R} = 1$, *the infection-free equilibrium* E_0 *is globally attractive; and*

(iii) if $\mathscr{R} > 1$, *the epidemic equilibrium* E_1 *is globally asymptotically stable.*

Proof. Define

$$G = \{\phi = (\phi_1, \phi_2, \phi_3)^T \in C \mid S_0 \geq \phi_1 \geq 0, \phi_2 \geq 0, \phi_3 \geq 0\}.$$

Then it is a positively invariant with respect to (1.3), i.e., for any $t \geq 0$, $S(t) \leq S_0$. In fact, from the first equation of (1.3), we get $\frac{dS}{dt} \leq \nu - \mu S(t)$, then we have $\limsup_{t\to+\infty} S(t) \leq S_0$. For any $\phi = (\phi_1, \phi_2, \phi_3)^T \in G$, let $(S(t), I(t), V(t))^T$ be the solution of (1.3) with the initial function ϕ. If there is $t' > 0$ such that $S(t') > S_0$ and $\frac{dS}{dt}\mid_{t=t'} > 0$, then from the first equation of (1.3), we have

$$\begin{aligned}\left.\frac{dS}{dt}\right|_{t=t'} &= \nu - \left(\sigma\beta I(t') + \rho\frac{V(t')}{V(t')+K}\right)S(t_1) - \mu S(t')\\ &\leq -\left(\sigma\beta I(t') + \frac{V(t')}{V(t')+K}\right)S(t') \leq 0,\end{aligned}$$

which is a contradiction. Thus, for any $t \geq 0$, we have $S(t) \leq S_0$, i.e., G is a positively invariant with respect to (1.3).

When $\mathscr{R} < 1$, we define on G

$$L_1(\phi) = W_1(0) + U_1(\phi) + U_2(\phi), \tag{4.1}$$

where

$$\begin{aligned}W_1(0) &= \phi_1(0) - S_0 - S_0 \ln\frac{\phi_1(0)}{S_0} + e^{\mu\tau}\phi_2(0) + k\phi_3(0),\\ U_1(\phi) &= \int_{-\tau}^{0} \sigma\beta\phi_1(\xi)\phi_2(\xi)d\xi, \quad U_2(\phi) = \int_{-\tau}^{0} \rho\frac{\phi_1(\xi)\phi_3(\xi)}{\phi_3(\xi)+k}d\xi,\end{aligned}$$

and here $k > 0$ is a constant to be determined. Obviously, $L_1(\phi)$ is continuous on the subset $\overline{G} \in C$. It follows from (1.3) and (4.1) that

$$\begin{aligned}\left.\frac{dL_1(\phi)}{dt}\right|_{(1.3)} \leq &-\frac{\mu}{\phi_1(0)}(\phi_1(0)-S_0)^2 + \omega\left(k - \frac{me^{\mu\tau} - \sigma\beta S_0}{\omega}\right)\phi_2(0)\\ &+ \eta\left(\frac{\rho S_0}{\eta K} - k\right)\phi_3(0).\end{aligned} \tag{4.2}$$

Since $\mathscr{R} < 1$, we have $\frac{\rho S_0}{\eta K} < \frac{me^{\mu\tau} - \sigma\beta S_0}{\omega}$. So we can choose $k > 0$ such that $\frac{\rho S_0}{\eta K} < k < \frac{me^{\mu\tau} - \sigma\beta S_0}{\omega}$. Hence, we have that $\frac{dL_1(\phi)}{dt}\mid_{(1.3)} \leq 0$ for any $\phi \in G$. Then $L_1(\phi)$ is a Lyapunov functional on the subset $G \in C$.

Furthermore, define

$$\Omega_1 = \{\phi \in G | \frac{dL_1(\phi)}{dt}\mid_{(1.3)} = 0\}.$$

Obviously, $\frac{dL_1(\phi)}{dt}\mid_{(1.3)} = 0$ holds if and only if $S_t = S_0, I_t = 0, V_t = 0$. Let $\mathscr{M}$ be the largest invariant set in Ω_1 with respect to (1.3). Then $\mathscr{M} = \{E_0\}$. In fact, $\mathscr{M}$ is not empty since $E_0 \in \mathscr{M}$. For $\forall\phi \in \mathscr{M}$, let $(S(t), I(t), V(t))^T$ be the solution of (1.3) under the initial function ϕ. According to the invariance of $\mathscr{M}$, for any $t \in R$, we get $(S_t, I_t, V_t) \in \mathscr{M} \subset \Omega_1$. Then $S(t) \equiv S_0, I(t) \equiv 0, V(t) \equiv 0$, for any $t \in R$. Thus, $\mathscr{M} = \{E_0\}$. By using Lyapunov-LaSalle invariance principal [9,12,13,16], we have that E_0 is global asymptotic stability for any time delay $\tau \geq 0$. This proves the conclusion (i).

When $\mathscr{R} = 1$. Define a Lyapunov functional $L_2(\phi)$ on G as follows,

$$L_2(\phi) = W_2(0) + U_1(\phi) + U_2(\phi), \tag{4.3}$$

where

$$W_2(0) = \phi_1(0) - S_0 - S_0 \ln \frac{\phi_1(0)}{S_0} + e^{\mu\tau}\phi_2(0) + \frac{\rho S_0}{\eta K}\phi_3(0),$$

and $\Omega_2 = \{\phi \in G | \frac{dL_2(\phi)}{dt} |_{(1.3)} = 0\}$. Then a similar discussion as for $\mathscr{R} < 1$ leads to the conclusion (ii).

For the case of $\mathscr{R} > 1$, we define function $L_3(\phi)$ on G as follows,

$$L_3(\phi) = W_3(0) + \sigma\beta S^* I^* U_3(\phi) + \frac{\rho S^* V^*}{V^* + K} U_4(\phi), \tag{4.4}$$

here

$$\begin{aligned} W_3(0) =& \phi_1(0) - S^* - S^* \ln \frac{\phi_1(0)}{S^*} + e^{\mu\tau}\left(\phi_2(0) - I^* - I^* \ln \frac{\phi_2(0)}{I^*}\right) \\ &+ \frac{\rho S^*}{(V^* + K)\eta}\left(\phi_3(0) - V^* - V^* \ln \frac{\phi_3(0)}{V^*}\right), \\ U_3(t) =& \int_{-\tau}^{0} \left[\frac{\phi_1(\xi)\phi_2(\xi)}{S^* I^*} - 1 + \ln \frac{\phi_1(\xi)\phi_2(\xi)}{S^* I^*}\right] d\xi, \\ U_4(t) =& \int_{-\tau}^{0} \left[\frac{\frac{\phi_1(\xi)\phi_3(\xi)}{\phi_3(\xi)+K}}{\frac{S^* V^*}{V^*+K}} - 1 + \ln \frac{\phi_1(\xi)\phi_3(\xi)}{\frac{S^* V^*}{V^*+K}}\right] d\xi. \end{aligned}$$

We claim that $L_3(\phi)$ is a Lyapunov functional. First, it follows from (1.3) and (4.4) that

$$\begin{aligned} \frac{dL_3(\phi)}{dt} =& -\mu\frac{(\phi_1(0) - S^*)^2}{\phi_1(0)} + 2\sigma\beta S^* I^* - \frac{\sigma\beta {S^*}^2 I^*}{\phi_1(0)} - \frac{\sigma\beta I^* \phi_1(-\tau)\phi_2(-\tau)}{\phi_2(0)} \\ &+ \sigma\beta S^* I^* \ln \frac{\phi_1(-\tau)\phi_2(-\tau)}{\phi_1(0)\phi_2(0)} + \frac{3\rho S^* V^*}{V^* + K} - \frac{\rho {S^*}^2 V^*}{(V^* + K)\phi_1(0)} \\ &- \frac{\rho S^* {V^*}^2 \phi_2(0)}{\phi_3(0) I^* (V^* + K)} - \frac{\rho\phi_1(-\tau)\phi_3(-\tau) I^*}{\phi_2(0)(\phi_3(-\tau) + K)} + \frac{\rho S^* V^*}{V^* + K} \ln \frac{\frac{\phi_1(-\tau)\phi_3(-\tau)}{\phi_3(-\tau)+K}}{\frac{\phi_1(0)\phi_3(0)}{\phi_3(0)+K}} \\ &+ \frac{S^*}{\phi_1(0)}\left(\sigma\beta\phi_2(0) + \rho\frac{\phi_3(0)}{\phi_3(0) + K}\right)\phi_1(0) + \frac{\rho S^* V^* \phi_2(0)}{(V^* + K) I^*} - e^{\mu\tau} m\phi_2(0) \\ &- \frac{\rho S^*}{V^* + K}\phi_3(0) \end{aligned}$$

factoring the last term of which gives

$$\begin{aligned} \frac{dL_3(\phi)}{dt} =& -\mu\frac{(\phi_1(0) - S^*)^2}{\phi_1(0)} && (4.5) \\ &+ \sigma\beta S^* I^* \left(2 - \frac{S^*}{\phi_1(0)} - \frac{\phi_1(-\tau)\phi_2(-\tau)}{S^* \phi_2(0)} + \ln \frac{\phi_1(-\tau)\phi_2(-\tau)}{\phi_1(0)\phi_2(0)}\right) && (4.6) \\ &+ \frac{\rho S^* V^*}{V^* + K}\left(4 - \frac{S^*}{\phi_1(0)} - \frac{V^* \phi_2(0)}{\phi_3(0) I^*} - \frac{\phi_3(0) + K}{V^* + K}\right. \end{aligned}$$

$$- \frac{V^*+K}{S^*V^*}\frac{\phi_1(-\tau)\phi_3(-\tau)I^*}{\phi_2(0)(\phi_3(-\tau)+K)} + \ln \frac{\frac{\phi_1(-\tau)\phi_3(-\tau)}{\phi_3(-\tau)+K}}{\frac{\phi_1(0)\phi_3(0)}{\phi_3(0)+K}}\Bigg) \tag{4.7}$$

$$+ \frac{\rho S^*V^*}{V^*+K}\left(-1 - \frac{\phi_3(0)}{V^*} + \frac{\phi_3(0)+K}{V^*+K} + \frac{V^*+K}{\phi_3(0)+K}\frac{\phi_3(0)}{V^*}\right) \tag{4.8}$$

Noticing

$$2 - \frac{S^*}{\phi_1(0)} - \frac{\phi_1(-\tau)\phi_2(-\tau)}{S^*\phi_2(0)} + \ln\frac{\phi_1(-\tau)\phi_2(-\tau)}{\phi_1(0)\phi_2(0)} \tag{4.9}$$

$$= \left(1 - \frac{S^*}{\phi_1(0)} + \ln\frac{S^*}{\phi_1(0)}\right) \tag{4.10}$$

$$+ \left(1 - \frac{\phi_1(-\tau)\phi_2(-\tau)}{S^*\phi_2(0)} + \ln\frac{\phi_1(-\tau)\phi_2(-\tau)}{S^*\phi_2(0)}\right), \tag{4.11}$$

$$4 - \frac{S^*}{\phi_1(0)} - \frac{V^*\phi_2(0)}{\phi_3(0)I^*} - \frac{\phi_3(0)+K}{V^*+K} - \frac{V^*+K}{S^*V^*}\frac{\phi_1(-\tau)\phi_3(-\tau)I^*}{\phi_2(0)(\phi_3(-\tau)+K)}$$

$$+ \ln\frac{\phi_1(-\tau)\phi_3(-\tau)/(\phi_3(-\tau)+K)}{\phi_1(0)\phi_3(0)/(\phi_3(0)+K)} \tag{4.12}$$

$$= \left(1 - \frac{S^*}{\phi_1(0)} - \ln\frac{S^*}{\phi_1(0)}\right) \tag{4.13}$$

$$+ \left(1 - \frac{V^*\phi_2(0)}{I^*\phi_3(0)} - \ln\frac{V^*\phi_2(0)}{I^*\phi_3(0)}\right) \tag{4.14}$$

$$+ \left(1 - \frac{\phi_3(0)+K}{V^*+K} + \ln\frac{\phi_3(0)+K}{V^*+K}\right) \tag{4.15}$$

$$+ \left(1 - \frac{V^*+K}{S^*V^*}\frac{\phi_1(-\tau)\phi_3(-\tau)I^*}{\phi_2(0)(\phi_3(-\tau)+K)} + \ln\frac{V^*+K}{S^*V^*}\frac{\phi_1(-\tau)\phi_3(-\tau)I^*}{\phi_2(0)(\phi_3(-\tau)+K)}\right), \tag{4.16}$$

and

$$-1 - \frac{\phi_3(0)}{V^*} + \frac{\phi_3(0)+K}{V^*+K} + \frac{V^*+K}{\phi_3(0)+K}\frac{\phi_3(0)}{V^*} = -\frac{K(\phi_3(0)-V^*)^2}{(\phi_3(0)+K)(V^*+K)V^*} \le 0.$$

And further noticing the function $\mathscr{H}(t) = 1 - s(t) + \ln s(t)$ is always non-positive for any function $s(t) > 0$, and $\mathscr{H}(t) = 0$ if and only if $s(t) = 1$, we have that the terms in (4.10)-(4.11) and (4.13)-(4.16) are always non-positive. Hence, we have that $\frac{dL_3(\phi)}{dt} \le 0$. Therefore, $L_3(\phi)$ is a Lyapunov functional on the subset $G \in C$.

Next, in order to employ the Lyapunov-LaSalle invariance principal, we define $\Omega_3 = \{\phi \in G | \frac{dL_3(\phi)}{dt} |_{(1.3)} = 0\}$. It is clear that $\frac{dL_3(\phi)}{dt} |_{(1.3)} = 0$ holds if and only if

$$\frac{S_t(0)}{S^*} = \frac{S_t(-\tau)I_t(-\tau)}{S^*I_t(0)} = \frac{S_t(-\tau)V_t(-\tau)I^*}{S^*V^*I_t(0)} = \frac{V^*I_t(0)}{I^*V_t(0)} = 1. \tag{4.17}$$

Let $\mathscr{U}$ be the largest invariant set in Ω_3 with respect to (1.3). It is easy to show $\mathscr{U} = \{E_1\}$. Finally, the Lyapunov-LaSalle invariance principal implies the conclusion (iii). □

5. Conclusion

In this paper, we first proposed a new SIV epidemic model with both direct and environmental transmission, which is a delayed version of the model in [25]. By constructing suitable Lyapunov functionals and using Lyapunov-LaSalle invariance principal, the completed global stabilities for model (1.3) were discussed. It is suggested that the global stabilities of the equilibria of model (1.3) strongly depend on the basic reproductive number $\mathscr{R}$: the infection-free equilibrium of the system is globally asymptotically stable if $\mathscr{R} < 1$ and the epidemic equilibrium of the system is globally asymptotically stable for $\mathscr{R} > 1$.

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