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SYMMETRIC POSITIVE SOLUTIONS FOR SECOND ORDER BOUNDARY VALUE PROBLEMS WITH INTEGRAL BOUNDARY CONDITIONS ON TIME SCALES

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Abstract This paper investigates the existence of symmetric positive solutions for a class of nonlinear boundary value problem of second order dynamic equations with integral boundary conditions on time scales. Under suitable conditions, the existence of symmetric positive solution is established by using monotone iterative technique.

Keywords Positive solution, symmetric solution, time scales.

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1. Introduction

The theory of boundary value problems is experiencing a rapid development. Many methods are used to study this kind of problems such as fixed point theorems, shooting method, iterative method with upper and lower solutions, etc. We refer the readers to the papers [13, 18, 23, 24]. Among them, the method of upper and lower solutions and the monotone iterative technique play extremely important roles in proving the existence of solutions to boundary value problems. Boundary value problems with positive solutions describe many phenomena in the applied mathematical sciences found in the theory of nonlinear diffusion generated by nonlinear sources, thermal ignition of gases and concentration in chemical or biological problems. For details and references see [8–10, 17]. Boundary value problems with integral boundary conditions also constitute a very interesting and important class of problems. They include two, three, multi point and nonlocal boundary value problems as special cases. The theory of boundary value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to the nonlinear problems with integral boundary conditions; we refer readers to [11,16,20] for examples and references. The calculus on time scales was introduced by Stefan Hilger in this Ph.D thesis in 1988 [14] in order to create a theory that can unify discrete and continuous analysis. A time scale is an arbitrary nonempty closed subset of real numbers. Thus the real numbers, the integers, the natural numbers, the closed intervals, the Cantor set, i.e. are examples of time scales. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale, which is an arbitrary closed subset of the reals. By choosing the time scale to be the set of real numbers, the general

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result yields a result concerning an ordinary differential equation, and by choosing the time scale to be the set of integers, the same general result yields a result for difference equations. However, since there are many other time scales than just the set of real numbers or the set of integers, one has a much more general result. We may summarize the above and state that "Unification and Extension" are the two main features of the time scales calculus. By using the theory of time scales we can also study biological, heat transfer, economic, stock market and epidemic models [3, 15, 22]. Hence, the study of dynamic equations on time scales is worthwhile and has theoretical and practical values [5,6]. Recently, for the existence problems of positive solutions of boundary value problems on time scales, some authors have obtained many results; for details, see [1, 2, 21] and the references therein. However, they did not further provide characteristic of positive solutions, such as symmetry that not only has its theoretical value, such as in studying chemical structures [12].

In 2009, Boucherif [5] investigated the existence of positive solutions of the following boundary value problem:

$$
\begin{aligned}
&-y''(t) = f(t, y(t)), \qquad t \in (0,1),\\
&y(0) - ay'(0) = \int_0^1 g_0(s)y(s)ds,\\
&y(1) - by'(1) = \int_0^1 g_1(s)y(s)ds,
\end{aligned}
$$

using Krasnoselskiis fixed point theorem. In 2011, Benchohra, Nieto and Ouahab [4] studied the existence of solutions for following problem:

$$
\begin{aligned}
&-y''(t) = f(t, y(t)), \qquad t \in (0,1),\\
&y(0) = 0, \quad y(1) = \int_0^1 g(s)y(s)ds,
\end{aligned}
$$

using nonlinear alternative of the Leray Schauder type and the Banach contraction principle. In [19], Li and Zhang considered the second order p-Laplacian dynamic equations with integral boundary conditions on time scales

$$
\begin{aligned}
&(\phi_p(x^{\triangle}(t)))^{\nabla} + \lambda f(t, x(t), x^{\triangle}(t)) = 0, \qquad t \in (0,T),\\
&x^{\triangle}(0) = 0, \quad \alpha x(T) - \beta x(0) = \int_0^T g(s)x(s)\nabla s.
\end{aligned}
$$

By using Legget-Williams fixed point theorem, they obtained the existence criteria of at least three positive solutions.

Inspired by the work mentioned above, we concentrate on proving the existence of at least one symmetric positive solution to the second order nonlinear boundary value problem on a time scale T given by

$$
u^{\triangle\nabla}(t) + f(t, u(t), u^{\triangle}(t)) = 0, \qquad t \in (a,b), \tag{1.1}
$$

$$
\alpha u(a) - \beta \lim_{t \to a^+} u^{\triangle}(t) = \int_a^b h_1(s)u(s)\nabla s, \tag{1.2}
$$

$$
\alpha u(b) + \beta \lim_{t \to b^-} u^{\triangle}(t) = \int_a^b h_2(s)u(s)\nabla s, \tag{1.3}
$$

where T is a symmetric time scale, $\alpha, \beta > 0$, the functions $h_1, h_2 \in L^1([a,b])$ are nonnegative, symmetric on $[a,b]$ and the function $f : [a,b] \times [0,\infty) \times R \to [0,\infty)$ is continuous.

In this paper, we study more general problem and some new results are obtained for the existence of at least one positive symmetric solution for the above problem by using monotone iterative technique. The results are even new for the special cases of differential equations and difference equations, as well as in the general time scale setting.

Now, we present some symmetric definition.

Definition 1.1. A time scale T is said to be symmetric if for any given $t \in \mathrm{T}$, we have $b + a - t \in \mathrm{T}$.

Definition 1.2. A function $u : \mathrm{T} \to R$ is said to be symmetric on T if for any given $t \in \mathrm{T}$, $u(t) = u(b + a - t)$.

Throughout this paper T is a symmetric time scale with a, b are points in T. By an interval (a,b), we always mean the intersection of the real interval (a,b) with the given time scale, that is $(a,b) \cap \mathrm{T}$. Other types of intervals are defined similarly. For the details of basic notions connected to time scales we refer to $[1,2]$.

2. The Preliminary Lemmas

In this section we collect some preliminary results that will be used in subsequent section. Throughout the paper we will assume that the following conditions are satisfied:

(H1) $\alpha, \beta > 0$,

(H2) $h_1, h_2 \in L^1([a,b])$ are nonnegative, symmetric on $[a,b]$, $1 - v_1 - v_2 > 0$, where $\mu = 2\alpha\beta + \alpha^2(b-a)$, $v_1 = \int_a^b h_1(t)k_1(t)\nabla t$, $v_2 = \int_a^b h_2(t)k_2(t)\nabla t$, $k_2(t) = \frac{\beta + \alpha(t-a)}{\mu}$ and $k_1(t) = \frac{\beta + \alpha(b-t)}{\mu}$,

(H3) $f : [a,b] \times [0,\infty) \times R \to [0,\infty)$ is continuous, $f(.,u,v)$ is symmetric on $[a,b]$ and $f(t,u,v) = f(t,u,-v)$ for all $(t,u,v) \in [a,b] \times [0,\infty) \times R$,

(H4) $f(t,.,v), f(t,u,.)$ are nondecreasing for each $(t,u,v) \in [a,b] \times [0,\infty) \times R$.

The lemmas in this section are based on the boundary value problem

$$-u^{\triangle\nabla}(t) = p(t), t \in (a,b) \tag{2.1}$$

with boundary conditions $(1.2) - (1.3)$.

To prove the main result, we will employ following lemmas.

Lemma 2.1. *Let (H_1), (H_2) hold and $\mu \neq 0$. Then for any $p \in C([a,b])$, the boundary value problem $(2.1) - (1.2) - (1.3)$ has a unique solution u given by*

$$u(t) = \int_a^b H(t,s)p(s)\nabla s,$$

where

$$H(t,s) = G(t,s) + B_1(t)\int_a^b G(s,\tau)h_1(\tau)\nabla\tau + B_2(t)\int_a^b G(s,\tau)h_2(\tau)\nabla\tau, \tag{2.2}$$

$$G(t,s)=\frac{1}{\mu}\begin{cases}(\beta+\alpha(s-a))(\beta+\alpha(b-t)),\ a\le s\le t\le b,\\ (\beta+\alpha(t-a))(\beta+\alpha(b-s)),\ a\le t\le s\le b,\end{cases}\tag{2.3}$$

where

$$B_1(t)=\frac{(1-v_2)k_1(t)+v_2k_2(t)}{1-v_1-v_2},\quad B_2(t)=\frac{(1-v_1)k_2(t)+v_1k_1(t)}{1-v_1-v_2}.$$

Proof. Suppose that u is a solution of the problem (2.1). It is easy to see by integration of (2.1) that

$$-u^{\triangle}(t)+\lim_{t\to a^+}u^{\triangle}(t)=\int_a^t p(s)\nabla s.$$

Letting $A=\lim_{t\to a^+}u^{\triangle}(t)$, then we have

$$u^{\triangle}(t)=A-\int_a^t p(s)\nabla s.$$

Integrating again, we get

$$u(t)=u(a)+A(t-a)-\int_a^t\int_a^s p(r)\nabla r\Delta s.$$

By the boundary condition, we obtain

$$\begin{cases}\alpha u(a)-\beta A=\displaystyle\int_a^b h_1(s)u(s)\nabla s,\\ \alpha u(a)+[\alpha(b-a)+\beta]A=\displaystyle\int_a^b h_2(s)u(s)\nabla s+\alpha\int_a^b\int_a^s p(r)\nabla r\Delta s+\beta\int_a^b p(s)\nabla s.\end{cases}$$

Then, we find

$$A=\frac{1}{\alpha(b-a)+2\beta}\Big[\int_a^b(h_2(s)-h_1(s))u(s)\nabla s+\alpha\int_a^b\int_a^s p(r)\nabla r\Delta s+\beta\int_a^b p(s)\nabla s\Big]$$

and

$$\begin{aligned}u(a)=&\frac{\beta}{\alpha(\alpha(b-a)+2\beta)}\int_a^b(h_2(s)-h_1(s))u(s)\nabla s+\frac{1}{\alpha}\int_a^b h_1(s)u(s)\nabla s\\&+\frac{\beta}{\alpha(\alpha(b-a)+2\beta)}\Big[\alpha\int_a^b\int_a^s p(r)\nabla r\Delta s+\beta\int_a^b p(s)\nabla s\Big].\end{aligned}$$

Thus, we have

$$\begin{aligned}u(t)=&\frac{\beta}{\alpha(\alpha(b-a)+2\beta)}\int_a^b(h_2(s)-h_1(s))u(s)\nabla s+\frac{1}{\alpha}\int_a^b h_1(s)u(s)\nabla s\\&+\frac{\beta}{\alpha(\alpha(b-a)+2\beta)}\Big[\alpha\int_a^b\int_a^s p(r)\nabla r\Delta s+\beta\int_a^b p(s)\nabla s\Big]\\&+\frac{t-a}{\alpha(b-a)+2\beta}\int_a^b(h_2(s)-h_1(s))u(s)\nabla s\end{aligned}$$

$$
\begin{aligned}
&+\frac{t-a}{\alpha(b-a)+2\beta}\Big[\alpha\int_a^b\int_a^s p(r)\nabla r\Delta s+\beta\int_a^b p(s)\nabla s\Big]-\int_a^t\int_a^s p(r)\nabla r\Delta s\\
&=\frac{\beta+\alpha(t-a)}{\mu}\int_a^b h_2(s)u(s)\nabla s+\frac{\beta+\alpha(b-t)}{\mu}\int_a^b h_1(s)u(s)\nabla s\\
&+\int_a^b G(t,s)p(s)\nabla s,
\end{aligned}
$$

from which, we obtain

$$
u(t)=k_2(t)\int_a^b h_2(s)u(s)\nabla s+k_1(t)\int_a^b h_1(s)u(s)\nabla s+\int_a^b G(t,s)p(s)\nabla s, \tag{2.4}
$$

where $G(t,s)$ is defined by (2.3).

Considering

$$
\int_a^b h_1(t)k_1(t)\nabla t=\int_a^b h_1(t)k_2(t)\nabla t,
$$

multiplying (2.4) with $h_1(t)$ and integrating it, we get

$$
(1-v_1)\int_a^b h_1(s)u(s)\nabla s-v_1\int_a^b h_2(s)u(s)\nabla s=A_1, \tag{2.5}
$$

where

$$
A_1=\int_a^b h_1(t)\int_a^b G(t,s)p(s)\nabla s\nabla t.
$$

Similarly, multiplying (2.4) with $h_2(t)$ and integrating it again, we have

$$
(1-v_2)\int_a^b h_2(s)u(s)\nabla s-v_2\int_a^b h_1(s)u(s)\nabla s=A_2, \tag{2.6}
$$

where

$$
A_2=\int_a^b h_2(t)\int_a^b G(t,s)p(s)\nabla s\nabla t.
$$

From (2.5) and (2.6), we find

$$
\begin{aligned}
\int_a^b h_1(s)u(s)\nabla s&=\frac{1-v_2}{1-v_1-v_2}A_1+\frac{v_1}{1-v_1-v_2}A_2,\\
\int_a^b h_2(s)u(s)\nabla s&=\frac{1-v_1}{1-v_1-v_2}A_2+\frac{v_2}{1-v_1-v_2}A_1.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
u(t)&=\int_a^b G(t,s)p(s)\nabla s+B_1(t)\int_a^b h_1(t)\int_a^b G(t,s)p(s)\nabla s\nabla t\\
&\quad+B_2(t)\int_a^b h_2(t)\int_a^b G(t,s)p(s)\nabla s\nabla t\\
&=\int_a^b H(t,s)p(s)\nabla s,
\end{aligned}
$$

where $H(t,s)$ is defined in (2.2). The proof is complete. $\square$

For studying the existence of symmetric positive solutions to the boundary value problem (1.1)−(1.2)−(1.3), we have to take the functions h_1 and h_2 in the boundary conditions (1.2), (1.3) are same, i.e. $h_1(t) = h_2(t) = h(t)$ on $[a,b]$. Otherwise the solution of the problem can not be symmetric. In this case, by elementary calculations, we get

$$v_1(t) = v_2(t) = \frac{1}{2\alpha}\int_a^b h(s)\nabla s, t \in [a,b].$$

Then, the assumption (H_2) becomes the following expression:
(H_2) $h \in L^1([a,b])$ is nonnegative, symmetric on $[a,b]$, $\alpha - v > 0$, where $v = \int_a^b h(s)\nabla s$.

Throughout this paper we'll take $h_1 = h_2 = h$. Thus, a unique solution u can be written as follows:

$$u(t) = \int_a^b H(t,s)p(s)\nabla s,$$

where

$$H(t,s) = G(t,s) + B\int_a^b G(s,\tau)h(\tau)\nabla\tau,$$

where $G(t,s)$ is defined by (2.3), $\mu = 2\alpha\beta + \alpha^2(b-a)$ and $B = \dfrac{1}{\alpha - v}$.

Lemma 2.2. *Assume that (H_1) and (H_2) hold. Then we have*

(i) $H(t,s) > 0, \quad G(t,s) > 0$, *for* $t,s \in [a,b]$,

(ii) $H(b+a-t, b+a-s) = H(t,s), \quad G(b+a-t,b+a-s) = G(t,s)$, *for* $t,s \in [a,b]$,

(iii) $\frac{1}{\mu}\beta^2\gamma \le H(t,s) \le H(s,s) \le \frac{1}{\mu}\gamma D$ *and* $\frac{1}{\mu}\beta^2 \le G(t,s) \le G(s,s) \le \frac{1}{\mu}D$, *for* $t,s \in [a,b]$,

where $D = (\beta + \alpha(b-a))^2$, $\gamma = 1 + Bv$.

Proof. It is clear that (i) hold. Now we prove that (ii) and (iii) hold. First, we consider (ii). If $t \le s$, then $b+a-t \ge b+a-s$. Using (2.3) , we get

$$\begin{aligned} G(b+a-t,b+a-s) &= \frac{1}{\mu}(\beta + \alpha(b+a-s-a))(\beta + \alpha(b-(b+a-t))) \\ &= \frac{1}{\mu}(\beta + \alpha(b-s))(\beta + \alpha(t-a)) = G(t,s). \end{aligned}$$

Similarly, we can prove that $G(b+a-t,b+a-s) = G(t,s)$, for $s \le t$. Thus we have $G(b+a-t,b+a-s) = G(t,s)$, for $t,s \in [a,b]$. Now by (2.2), for $t,s \in [a,b]$, we have

$$\begin{aligned}
&H(b+a-t,b+a-s)\\
&=G(b+a-t,b+a-s)+B\int_a^b G(b+a-s,\tau)h(\tau)\nabla\tau\\
&=G(t,s)+B\int_b^a G(b+a-s,b+a-\tau)h(b+a-\tau)\nabla(b+a-\tau)\\
&=G(t,s)+B\int_a^b G(s,\tau)h(\tau)\nabla\tau=H(t,s).
\end{aligned}$$

So (ii) is established. Now we show that (iii) holds. In fact, if $t\le s$, from (2.3) and the assumption (H_2), then we get

$$\begin{aligned}
G(t,s)&=\frac{1}{\mu}(\beta+\alpha(t-a))(\beta+\alpha(b-s))\le\frac{1}{\mu}(\beta+\alpha(s-a))(\beta+\alpha(b-t))=G(s,s)\\
&\le\frac{1}{\mu}(\beta+\alpha(b-a))(\beta+\alpha(b-a))=\frac{1}{\mu}(\beta+\alpha(b-a))^2=\frac{1}{\mu}D.
\end{aligned}$$

Similarly, we can prove that $G(t,s)\le G(s,s)\le\frac{1}{\mu}D$ for $s\le t$. Therefore $G(t,s)\le G(s,s)\le\frac{1}{\mu}D$, for $t,s\in[a,b]$. And then, by (2.2), we have

$$\begin{aligned}
H(t,s)&=G(t,s)+B\int_a^b G(s,\tau)h(\tau)\nabla\tau\le G(s,s)+B\int_a^b G(\tau,\tau)h(\tau)\nabla\tau\\
&\le\frac{1}{\mu}D+\frac{1}{\mu}DB\int_a^b h(\tau)\nabla\tau=\frac{1}{\mu}D(1+Bv)=\frac{1}{\mu}D\gamma.
\end{aligned}$$

On the other hand, for $t,s\in[a,b]$, we have

$$G(t,s)\ge\frac{1}{\mu}\beta\beta=\frac{1}{\mu}\beta^2.$$

And then, we get

$$\begin{aligned}
H(t,s)&=G(t,s)+B\int_a^b G(s,\tau)h(\tau)\nabla\tau\\
&\ge\frac{1}{\mu}\beta^2+\frac{1}{\mu}\beta^2B\int_a^b h(\tau)\nabla\tau=\frac{1}{\mu}\beta^2\gamma.
\end{aligned}$$

Thus for $t,s\in[a,b]$, we have $\frac{1}{\mu}\beta^2\gamma\le H(t,s)\le H(s,s)\le\frac{1}{\mu}\gamma D$ and $\frac{1}{\mu}\beta^2\le G(t,s)\le G(s,s)\le\frac{1}{\mu}D$. This completes the proof. □

Let E denote the Banach space $C[a,b]$ with the norm $\|u\|=\max\{\|u\|_\infty,\|u^\triangle\|_\infty\}$, where $\|u\|_\infty=\max_{t\in[a,b]}|u(t)|$. Define the cone $P\subset E$ by $P=\{u\in E: u(t)\ge 0$ is symmetric and concave, $t\in[a,b]\}$.

For any $u\in P$, $S:P\to E$ is defined by

$$Su(t)=\int_a^b H(t,s)f(s,u(s),u^\triangle(s))\nabla s,t\in[a,b].$$

Lemma 2.3. *Let $(H_1)-(H_4)$ hold. Then $S : P \to P$ is completely continuous and nondecreasing.*

Proof. First, it is clear that S is continuous and completely continuous. Now we shall prove that $S(P) \subseteq P$. Let $u \in P$. Obviously, Su is concave. From the expression of Su and the positivity of the function f, combining with Lemma 2.2, we know that Su is nonnegative on $[a, b]$. We now prove that Su is symmetric on $[a, b]$.

If we take the $\triangle$ derivative on both sides of $u(b + a - s) = u(s)$, by using the chain rule on time scale we get $u^{\triangle}(b + a - s)(-1) = u^{\triangle}(s)$.
Considering $u \in P$, the assumption (H_3) and Lemma 2.2, we have

$$\begin{aligned}
&Su(b + a - t)\\
&= \int_a^b H(b + a - t, s)f(s, u(s), u^{\triangle}(s))\nabla s\\
&= \int_b^a H(b + a - t, b + a - s)f(b + a - s, u(b + a - s), u^{\triangle}(b + a - s))\nabla(b + a - s)\\
&= \int_a^b H(b + a - t, b + a - s)f(b + a - s, u(s), u^{\triangle}(b + a - s))\nabla s\\
&= \int_a^b H(t, s)f(s, u(s), -u^{\triangle}(s))\nabla s = \int_a^b H(t, s)f(s, u(s), u^{\triangle}(s))\nabla s = Su(t).
\end{aligned}$$

Therefore Su is symmetric. So, we get $S(P) \subseteq P$.

Finally, we shall show that Su is nondecreasing about u. Let $u_1(t), u_2(t) \in P$ with $u_1(t) \le u_2(t)$, for all $t \in [a, b]$. Then, by the properties of a cone, we have $u_2(t) - u_1(t) \in P$. Since $u_2 - u_1$ is concave and symmetric, there exists an η that $u_2 - u_1$ is nondecreasing on $[a, \eta]$ and non increasing on $[\eta, b]$. Thus, we get

$$\begin{cases} u_1^{\triangle}(t) \le u_2^{\triangle}(t),\ t \in [a, \eta],\\ u_1^{\triangle}(t) \ge u_2^{\triangle}(t),\ t \in [\eta, b], \end{cases} \tag{2.7}$$

where $\eta := \inf\{\delta \in [a, b] : \sup_{t\in[a,b]} (u_2 - u_1)(t) = (u_2 - u_1)(\delta)\}$. Hence, in view of (2.7), the assumptions (H_3) and (H_4) , for all $t \in [a, b]$, we obtain

$$\begin{aligned}
Su_1(t) &= \int_a^b H(t, s)f(s, u_1(s), u_1^{\triangle}(s))\nabla s\\
&= \int_a^{\eta} H(t, s)f(s, u_1(s), u_1^{\triangle}(s))\nabla s + \int_{\eta}^b H(t, s)f(s, u_1(s), u_1^{\triangle}(s))\nabla s\\
&\le \int_a^{\eta} H(t, s)f(s, u_2(s), u_2^{\triangle}(s))\nabla s + \int_{\eta}^b H(t, s)f(s, u_2(s), -u_1^{\triangle}(s))\nabla s\\
&\le \int_a^{\eta} H(t, s)f(s, u_2(s), u_2^{\triangle}(s))\nabla s + \int_{\eta}^b H(t, s)f(s, u_2(s), -u_2^{\triangle}(s))\nabla s\\
&= \int_a^{\eta} H(t, s)f(s, u_2(s), u_2^{\triangle}(s))\nabla s + \int_{\eta}^b H(t, s)f(s, u_2(s), u_2^{\triangle}(s))\nabla s\\
&= \int_a^b H(t, s)f(s, u_2(s), u_2^{\triangle}(s))\nabla s = Su_2(t).
\end{aligned}$$

So, Su is nondecreasing. The proof is complete. $\square$

3. Main Results

In this section we discuss the existence of at least one positive solution for the problem $(1.1)-(1.3)$. We obtain the following existence result.

Theorem 3.1. *Assume that $(H_1)-(H_4)$ hold. If there exist two positive numbers $a_1 < c$ such that*

$$\sup_{t\in[a,b]} f(t,c,c) \le a_1 \tag{3.1}$$

and

$$c \ge \max\{\frac{2\gamma(b-a)D}{\mu}, \frac{\mu-\alpha\beta}{\mu}, \frac{(b-a)(\mu-\alpha\beta)}{\mu}\}a_1 \tag{3.2}$$

then the problem $(1.1)-(1.3)$ has a positive solution $w^* \in P$ with

$$\|w^*\|_\infty \le c \quad \textit{and} \quad \lim_{n\to\infty} S^n w_0 = w^*,$$

where $w_0(t) = (b-a)a_1G(t,t) + \dfrac{\gamma(b-a)a_1D}{\mu}$ and $G(t,t)$ is given in (2.3).

Proof. Let $P_c = \{w \in P : \|w\| \le c\}$. First, we shall show that $SP_c \subseteq P_c$. Let $w \in P_c$, then we have

$$0 \le w(t) \le \|w\|_\infty \le c \quad \text{and} \quad w^\triangle(t) \le \|w^\triangle\|_\infty \le c, \text{for all} \quad t \in [a,b], \tag{3.3}$$

Thus, by $(3.1), (3.2)$, (3.3) and Lemma 2.2, for all $t \in [a,b]$, we get

$$Sw(t) = \int_a^b H(t,s)f(s,w(s),w^\triangle(s))\nabla s \le \int_a^b \frac{\gamma D}{\mu} f(s,c,c)\nabla s \le \frac{\gamma D(b-a)a_1}{\mu} \le c.$$

On the other hand, from $(2.2), (2.3), (3.1), (3.2)$ and (3.3), for all $t \in [a,b]$, we obtain

$$\begin{aligned}(Sw)^\triangle(t) &= \int_a^b H^\triangle(t,s)f(s,w(s),w^\triangle(s))\nabla s \le \int_a^b G^\triangle(t,s)a_1\nabla s \\ &\le \int_a^b \frac{(\alpha\beta+\alpha^2(b-a))}{\mu}a_1\nabla s = \frac{(\mu-\alpha\beta)}{\mu}a_1\int_a^b \nabla s \\ &= \frac{(b-a)(\mu-\alpha\beta)}{\mu}a_1 \le c.\end{aligned}$$

Hence, we get $\|Sw\| \le c$ and therefore, we have $SP_c \subseteq P_c$.

Let $w_0(t) = (b-a)a_1G(t,t) + \dfrac{\gamma(b-a)a_1D}{\mu}$, for $t \in [a,b]$, then using (3.2) and Lemma (2.2), it can be easily seen that $\|w_0\|_\infty \le c$ and using the the expression of the Green function, for all $t \in [a,b]$, we have

$$\begin{aligned}w_0^\triangle(t) &= (b-a)a_1G^\triangle(t,t) = \frac{(b-a)a_1}{\mu}(\alpha^2(b+a) - \alpha^2(t+\sigma(t))) \\ &\le \frac{(b-a)^2a_1\alpha^2}{\mu},\end{aligned} \tag{3.4}$$

If $(b-a) \leq 1$, then from (3.2), we obtain

$$\frac{(b-a)^2 a_1 \alpha^2}{\mu} \leq \frac{(b-a) a_1 \alpha^2}{\mu} = \frac{(\mu - 2\alpha\beta)}{\mu} a_1 \leq \frac{(\mu - \alpha\beta)}{\mu} a_1 \leq c. \tag{3.5}$$

If $(b-a) \geq 1$, then from (3.2), we get

$$\frac{(b-a)^2 a_1 \alpha^2}{\mu} \leq \frac{(b-a)(b-a)^2 a_1 \alpha^2}{\mu} \leq \frac{D(b-a)}{\mu} a_1 \leq c. \tag{3.6}$$

Noticing (3.4), (3.5) and (3.6) , we have $\|(w_0)^{\triangle}\|_\infty \leq c$ and $w_0(t) \in P_c$.

Let $w_1(t) = Sw_0(t)$, then $w_1(t) \in P_c$. We denote $w_{n+1} = Sw_n = S^{n+1} w_0$, $(n = 0, 1, 2, ...)$.

Since $SP_c \subseteq P_c$, we have $w_n \in P_c$, $(n = 0, 1, 2, ...)$. From Lemma 2.3, S is compact, we assert that $\{w_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{w_{n_k}\}_{k=1}^{\infty}$ and there exists $w^* \in P_c$ such that $w_{n_k} \to w^*$. Now, using Lemma 2.2, the definition of S and (3.1), we get

$$\begin{aligned}
w_1(t) =& Sw_0(t) = \int_a^b H(t,s) f(s, w_0(s), w_0^{\triangle}(s)) \nabla s \\
=& \int_a^b G(t,s) f(s, w_0(s), w_0^{\triangle}(s)) \nabla s \\
& + B \int_a^b \int_a^b G(s,\tau) h(\tau) f(s, w_0(s), w_0^{\triangle}(s)) \nabla\tau \nabla s \\
\leq& \int_a^b G(t,t) a_1 \nabla s + B \int_a^b \int_a^b G(s,s) h(\tau) a_1 \nabla\tau \nabla s \\
=& (b-a) G(t,t) a_1 + B v a_1 \int_a^b G(s,s) \nabla s \\
\leq& (b-a) a_1 G(t,t) + \frac{\gamma (b-a) a_1 D}{\mu} = w_0(t).
\end{aligned}$$

Thus, we have $w_1(t) \leq w_0(t)$, for all $t \in [a,b]$. Then, from Lemma 2.3, we know that $Sw_1(t) \leq Sw_0(t)$, for all $t \in [a,b]$, which means $w_2(t) \leq w_1(t)$. By induction, $w_{n+1} \leq w_n$, $(n = 0, 1, 2, ...)$. Hence, we obtain that $w_n \to w^*$. Since S is continuous, we assert that $Sw^* = w^*$. It is well known that the fixed point of the operator S is the solution of the boundary value problem $(1.1) - (1.3)$. This completes the proof. □

Example 3.1. Consider the following second-order boundary value problem with integral conditions on $T = Z$;

$$u^{\triangle\nabla}(t) + t^2[(u^{\triangle}(t))^2 + (u(t))^4 + 10^{-4}] = 0, t \in (-3, 3), \tag{3.7}$$

$$10u(-3) - 10 \lim_{t \to -3^+} u^{\triangle}(t) = \int_{-3}^{3} \frac{1}{s^4 + 4} u(s) \nabla s, \tag{3.8}$$

$$10u(3) + 10 \lim_{t \to 3^-} u^{\triangle}(t) = \int_{-3}^{3} \frac{1}{s^4 + 4} u(s) \nabla s. \tag{3.9}$$

It is easy to check that the assumption $(H_1) - (H_4)$ hold and $\mu = 800$, $v \cong 0,762$, $B \cong 0,108$, $\gamma \cong 1,082$, $D = 4900$. Set $c = 10^{-4}$ and $a_1 = 10^{-6}$. Thus we can verify that conditions (3.1) − (3.2) are satisfied. Then applying Theorem 3.1, the problem (3.7) − (3.9) has a positive solution $w^* \in P$ with $\|w^*\|_\infty \leq 10^{-4}$ and $\lim\limits_{n\to\infty} S^n w_0 = w^*$, where $w_0(t) \cong \dfrac{3}{4}10^{-6}(69,04 - t^2)$.

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