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# NEW CONSTRUCTION OF HIGHER-ORDER LOCAL CONTINUOUS PLATFORMS FOR ERROR CORRECTION METHODS*

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**Abstract** Error correction method (ECM) [6,7] which has been recently developed, is based on the construction of a local approximation to the solution on each time step, and has the excellent convergence order $O(h^{2p+2})$, provided the local approximation has a local residual error $O(h^p)$. In this paper, we construct a higher-order continuous local platform to develop higher-order semi-explicit one-step ECM for solving initial value time dependent differential equations. It is shown that special choices of parameters for the local platform can lead to the improvement of the well-known explicit fourth and fifth order Runge-Kutta methods. Numerical experiments demonstrate the theoretical results.

**Keywords** Error correction method, local platform, stiff initial value problem, Runge Kutta methods.

**MSC(2010)** 34A45, 65L04, 65L20, 65L70.

## 1. Introduction

We discuss the numerical solution of the initial value problem (IVP) given by

$$\frac{d\phi}{dt} = f(t, \phi(t)), \qquad t \in [t_0, t_f], \qquad y(t_0) = \phi_0. \tag{1.1}$$

Many numerical techniques [2,3,10] have been developed for the accurate and efficient solution of the IVPs, including linear multi-step methods, Runge-Kutta methods and operating splitting techniques, etc. In this paper, we focus on the performance of the error correction method (ECM), recently developed by P. Kim et al. [6,7,9]. The basic idea of ECM is on the solution of the perturbed problem for a given local approximation to the true solution on each time step. Note that the $p$-stage implicit Runge-Kutta (RK) method achieves the order of accuracy $2p$ [5] and it needs to solve a simultaneous system of equations at each time step by a costly Newton-type iteration. Unlike the traditional RK methods, ECM has the excellent convergence $O(h^{2p+2})$ if one can make a local approximation $y(t)$ to the true solution on each time step such that the local residual error $f(t_n, \phi(t_n)) - \phi'(t_n)$ has the asymptotic behavior $O(h^p)$, where $p$ is any positive integer. In this paper, we focus on the construction of higher order continuous local polynomial approximations on each time step to build up much higher order ECM.

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Usually, to construct a continuous one-step method [4], one can extend an existing discrete method by including some additional function evaluations.

$$K_i = f\Big(t_0 + c_i h, \phi_0 + h\sum_{j=1}^{i-1} a_{ij}K_j\Big), \qquad i = 1, \cdots, \nu, \tag{1.2}$$

$$\phi(t_0 + \theta h) = \phi_0 + h\sum_{i=1}^{\nu} b_i(\theta)K_i, \qquad \theta \in [0, 1], \tag{1.3}$$

where $\phi(t_0 + \theta h)$ is a continuous approximation to $\phi(t)$ in $[t_0, t_0 + h]$ and $b_i(\theta)$, $i = 1, \cdots, \nu$ are polynomials of degree $\le d$, where $d$ is a positive integer. Up to the 4th degree local approximations, the number of unknowns to calculate $\phi(t)$ at each level is the same as the system size induced by (1.2). However, when the degree is bigger than 4, the number of unknowns at some levels exceeds the given information derived from (1.2). That is, it is difficult to construct the local approximations over the 5th degree.

In the proposed scheme, we add the following function evaluations

$$K_i(-h) = f\left(t_0 - c_i h, \phi_0 - h\sum_{j=1}^{i-1} a_{ij}K_j\right), \qquad i = 1, \cdots, \nu, \tag{1.4}$$

on the traditional scheme (1.2). By combining the additional function evaluations (1.4) with (1.2), the number of unknowns can be sufficiently minimized at each level. Moreover, the system size at each level is automatically reduced, so that the proposed scheme can minimize the overall computational costs. That is, the proposed scheme can be more efficient than the traditional way in the sense of computational costs. Also, it can stably control the number of unknowns within the given information derived from (1.4).

The aim of this paper is to construct quartic and quintic local polynomial platforms to the solution on each time step, which generalize the classical fourth and fifth order RK methods. The regime of the implicit RK methods is the basic tool to construct the local approximation. We show that each local polynomial has several free parameters, and the special choices of the parameters lead to the fourth and fifth order explicit RK methods. That is, the corresponding ECMs which are specified by the parameters based on the fourth and fifth degree local platform, have the orders of accuracy up to 10 and 12 having almost L-stability [11], respectively.

This paper is organized as follows. In Sec. 2, we briefly describe the explicit one-step ECM. In Sec. 3, we construct quartic and quintic local approximations generalizing most classical RK methods. Preliminary numerical results are presented in Sec. 4 to give numerical evidences for the theoretical analysis. Finally in Sec. 5, we summarize our results and discuss some possibilities to improve the efficiency of the new scheme.

## 2. Error Correction Methods (ECM)

In this section, we briefly review the ECM algorithm. Let $y_m$ be an approximation to the true solution $\phi(t)$ at time $t_m$ by a numerical method. Let $y(t)$ be a local approximation for the true solution $\phi(t)$ on the local time step $[t_m, t_{m+1}]$ satisfying

$$y(t_m) = y_m, \tag{2.1}$$

and

$$F(t) := f(t, y(t)) - y'(t) = O(h^p), \quad t \in (t_m, t_{m+1}], \tag{2.2}$$

for some positive number $p$. By using the Taylor's expansion,

$$f(t, \phi(t)) = f(t, y(t)) + f_\phi(t, y(t))(\phi(t) - y(t)) + O((\phi(t) - y(t))^2), \tag{2.3}$$

where the generic constant is depending only on the bound $f_{\phi\phi}$, one may have the following asymptotic linear ODE

$$\psi'(t) = \varphi(t)\psi(t) + F(t) + O(\psi(t)^2), \quad t \in (t_m, t_{m+1}), \tag{2.4}$$

where $\psi(t)$ is a perturbation of $\phi(t)$ on $[t_m, t_{m+1}]$ defined by

$$\psi(t) := \phi(t) - y(t), \quad t \in [t_m, t_{m+1}] \tag{2.5}$$

and $\varphi(t)$ is defined by

$$\varphi(t) = f_\phi(t, y(t)). \tag{2.6}$$

Now, by applying the Chebyshev collocation method (CCM) to (2.4) and using the relation in (2.5), one can have a discrete system as follows:

$$\mathcal{A}_n\Phi_n = \left(\mathcal{L}_n - \frac{h}{2}\mathcal{J}_n\right)\Phi_n = \mathcal{A}_n y_n + \frac{h}{2} f_n - (\phi(t_m) - y_m)b_n + r_n, \tag{2.7}$$

where $b_n := [\dot{l}_0^n(s_1^n), \cdots, \dot{l}_0^n(s_n^n)]^T$, the matrices $\mathcal{L}_n, \mathcal{A}_n$ and $\mathcal{J}_n$ are defined by

$$\mathcal{A}_n = (a_{jk}), \quad \mathcal{J}_n = (J_{jk}), \quad \mathcal{L}_n = (L_{jk}), \quad 1 \le j, k \le n, \tag{2.8}$$

whose entries are defined by

$$L_{jk} := \dot{l}_k^n(s_j^n), \quad J_{jk} := \varphi(t(s_j^n))\delta_{jk}, \quad a_{jk} := L_{jk}(s_j^n) - \frac{h}{2}J_{jk}, \tag{2.9}$$

and $n \times 1$ vectors $\Phi_n, y_n, f_n$ and $r_n$ are defined by

$$\begin{aligned}
\Phi_n &= [\phi(t(s_1^n)), \cdots, \phi(t(s_n^n))]^T, \quad y_n = [y(t(s_1^n)), \cdots, y(t(s_n^n))]^T \\
f_n &= [F(t(s_1^n)), \cdots, F(t(s_n^n))]^T, \quad r_n = [r_1, \cdots, c_n]^T,
\end{aligned} \tag{2.10}$$

where $r_j$ consist of the asymptotic term in (2.4) and the interpolation errors. Here, $l_k^n(s)$ is the interpolation polynomial of degree $n$ induced by Chebyshev polynomials, $t = t(s)$ is the change of variables transforming the computational region $[t_m, t_{m+1}]$ to the reference domain $[-1, 1]$ such that

$$t = t(s) = t_m + \frac{h}{2}(1 + s), \quad s \in [-1, 1], \tag{2.11}$$

and $s_j^n$ are the Chebyshev-Gauss-Lobatto (CGL) points such that

$$s_j^n = \cos\frac{(n-j)\pi}{n}, \quad j = 0, 1, \cdots, n.$$

Solving the discrete system (2.7) gives

$$\phi(t(s_k^n)) = y(t(s_k^n)) + \mu_k^m - (\phi(t_m) - y_m)\nu_k^m + \omega_k^m, \quad k = 1, \cdots, n, \tag{2.12}$$

where $\mu_k^m, \nu_k^m$ and $\omega_k^m$ are the $k^{th}$ components of the solutions for the linear systems

$$\mathcal{A}_n x = \frac{h}{2} f_n, \quad \mathcal{A}_n x = b_n, \quad \text{and} \quad \mathcal{A}_n x = r_n, \tag{2.13}$$

respectively, $k = 1, \cdots, n$. Hence, by using the fact $t(s_n^n) = t_{m+1}$ and truncating the last two unknown terms in the right hand side of (2.12), the ECM is defined as follows:

$$y_{m+1} = y(t_{m+1}) + \mu_n^m, \quad m \geq 0; \quad y_0 = \phi_0. \tag{2.14}$$

For more details on ECM, we refer to [6,7].

## 3. Construction of the higher order local continuous platform

In this section, we construct the local continuous approximation based on polynomials up to degree 5 which serves as a basis of the ECM. Prior to the construction, it is convenient to introduce and simplify notations to handle derivatives. We denote $f^0 = 1$ and $f^1 = f(t, \phi)$ which are derivatives of $t$ and $\phi(t)$ with respect to $t$, respectively. Also, we write the second derivative of $\phi$ with respect to $t$ as

$$\phi'' = (f^1)' = \frac{\partial f^1}{\partial t}\frac{dt}{dt} + \frac{\partial f^1}{\partial \phi}\frac{d\phi}{dt} = f_t^1 f^0 + f_\phi^1 f^1. \tag{3.1}$$

If we write $z_0$ for $\frac{\partial z}{\partial t}$ and $z_1$ for $\frac{\partial z}{\partial y}$, then we get

$$\phi'' = f_t^1 f^0 + f_\phi^1 f^1 = \sum_{i=1}^{1} f_i^1 f^i := f_j^1 f^j, \tag{3.2}$$

where the last equality means the Einstein summation symbol, which says that any repeated subscript or superscript in a multiplication term is to be summed over its range (0 to 1). Interested readers are referred to [4] for further details on the Einstein summation symbol.

Based on these notations, we will consider Taylor's expansion of $\phi(t)$ at time $t = t_m$ as follows:

$$\begin{aligned}
\phi(t) =& \phi(t_m) + (t - t_m) f^1 + \frac{(t - t_m)^2}{2} f_j^1 f^j + \frac{(t - t_m)^3}{6} \left( f_{jk}^1 f^j f^k + f_j^1 f_k^j f^k \right) \\
&+ \frac{(t - t_m)^4}{24} \left( f_{jkl}^1 f^j f^k f^l + 3 f_{jk}^1 f_l^j f^k f^l + f_j^1 f_{kl}^j f^k f^l + f_j^1 f_k^j f_l^k f^l \right) \\
&+ \frac{(t - t_m)^5}{120} \Big( f_{jklm}^1 f^j f^k f^l f^m + f_j^1 f_{klm}^j f^k f^l f^m + 6 f_{jkl}^1 f^j f^k f_m^l f^m \\
&+ 4 f_{jk}^1 f^j f_{lm}^k f^l f^m + f_j^1 f_k^j f_{lm}^k f^l f^m + 7 f_{jk}^1 f^j f_l^k f_m^l f^m + 3 f_{jl}^1 f_k^j f^k f_m^l f^m \\
&+ f_j^1 f_k^j f_l^k f_m^l f^m \Big) + O(h^6),
\end{aligned} \tag{3.3}$$

where coefficients are evaluated at $t = t_m$. Also, the Taylor's expansion of $f(t +$

$\alpha, y+\beta)$ at the point $(t, y)$ is given by

$$
\begin{aligned}
f(t+\alpha, y+\beta) = & f^1 + \alpha f_0^1 + \beta f_1^1 + \frac{1}{2}\left(\alpha^2 f_{00}^1 + 2\alpha\beta f_{01}^1 + \beta^2 f_{11}^1\right) \\
& + \frac{1}{6}\left(\alpha^3 f_{000}^1 + 3\alpha^2\beta f_{001}^1 + 3\alpha\beta^2 f_{011}^1 + \beta^3 f_{111}^1\right) \\
& + \frac{1}{24}\left(\alpha^4 f_{0000}^1 + 4\alpha^3\beta f_{0001}^1 + 6\alpha^2\beta^2 f_{0011}^1 + 4\alpha\beta^3 f_{0111}^1\right. \\
& \left. + \beta^4 f_{1111}^1\right) + \cdots . \qquad (3.4)
\end{aligned}
$$

## 3.1. Local polynomial approximation of degree p=4

In this subsection, we give the technique to construct a local polynomial approximation of degree 4 for which the seven unknown coefficients $f_j^1 f^j$, $f_{jk}^1 f^j f^k$, $f_j^1 f_k^j f^k$, $f_{jkl}^1 f^j f^k f^l$,$f_{jk}^1 f_l^j f^k f^l$, $f_j^1 f_{kl}^j f^k f^l$ and $f_j^1 f_k^j f_l^k f^l$ in (3.3) must be determined. We begin with the determination of three unknowns $f_j^1 f^j$,$f_{jk}^1 f^j f^k$ and $f_{jkl}^1 f^j f^k f^l$.

### 3.1.1. 1st level

For this, we consider

$$
K_{1,i} = f(t_m + \alpha_{1,i} h, y_m + \alpha_{1,i} h f(t_m, y_m)), \qquad i = 0, 1, 2, \qquad (3.5)
$$

where $\alpha_{1,i}$ are arbitrary parameters to be determined. Applying the Taylor's expansion (3.4) to (3.5) leads to the following three equations for $f_j^1 f^j$, $f_{jk}^1 f^j f^k$ and $f_{jkl}^1 f^j f^k f^l$.

$$
K_{1,i} = f^1 + \alpha_{1,i} h f_j^1 f^j + \frac{\alpha_{1,i}^2 h^2}{2} f_{jk}^1 f^j f^k + \frac{\alpha_{1,i}^3 h^3}{6} f_{jkl}^1 f^j f^k f^l + O(h^4), \qquad i = 0, 1, 2. \qquad (3.6)
$$

Hence, by solving these simultaneous equations, a 3 by 3 system is needed to solve for $f_j^1 f^j$, $f_{jk}^1 f^j f^k$ and $f_{jkl}^1 f^j f^k f^l$. Notice that to generally construct a local polynomial approximation of degree $p$, we need to solve a $p$ by $p$ matrix. However, it is more expensive and complicated to solve when $p$ becomes large. In this paper, unlike the traditional way to solve the 3 by 3 system, we introduce a new notation $\widehat{K_{1,i}}$ to reduce the system size to a 2 by 2 system using $K_{1,i}(h)$ and $K_{1,i}(-h)$ as follows:

$$
\widehat{K_{1,i}} := K_{1,i}(h) - K_{1,i}(-h), \quad i = 0, 1, \qquad (3.7)
$$

where $K_{1,i}(h) = K_{1,i}$ and

$$
\begin{aligned}
K_{1,i}(-h) &:= f(t_m - \alpha_{1,i} h, y_m - \alpha_{1,i} h f(t_m, y_m)) \\
&= f^1 - (\alpha_{1,i} h) f_j^1 f^j + \frac{(\alpha_{1,i} h)^2}{2} f_{jk}^1 f^j f^k - \frac{(\alpha_{1,i} h)^3}{6} f_{jkl}^1 f^j f^k f^l + O(h^4). \qquad (3.8)
\end{aligned}
$$

Then,

$$
\widehat{K_{1,i}} = 2(\alpha_{1,i} h) f_j^1 f^j + h^3 \frac{\alpha_{1,i}^3}{3} f_{jkl}^1 f^j f^k f^l + O(h^4). \qquad (3.9)
$$

If we define $\gamma_1$ and $\gamma_3$ as follows:

$$
\begin{pmatrix} \gamma_1 \\ \gamma_3 \end{pmatrix} = A_1^{-1} \begin{pmatrix} \widehat{K_{1,0}} \\ \widehat{K_{1,1}} \end{pmatrix}, \qquad (3.10)
$$

where $A_1 = \begin{pmatrix} 2\alpha_{1,0}h & \frac{\alpha_{1,0}^3 h^3}{3} \\ 2\alpha_{1,1}h & \frac{\alpha_{1,1}^3 h^3}{3} \end{pmatrix}$, then $f_j^1 f^j$ and $f_{jkl}^1 f^j f^k f^l$ can be obtained

$$\begin{pmatrix} f_j^1 f^j \\ f_{jkl}^1 f^j f^k f^l \end{pmatrix} = \begin{pmatrix} \gamma_1 + O(h^3) \\ \gamma_3 + O(h) \end{pmatrix}, \tag{3.11}$$

Note that for determination of the coefficients, the matrix $A_1$ should be non-singular or $det(A_1) = \frac{2}{3}\alpha_{1,0}\alpha_{1,1}h^4(\alpha_{1,1}^2 - \alpha_{1,0}^2) \neq 0$, so $\alpha_{1,0}$ and $\alpha_{1,1}$ should satisfy the following conditions:

$$\begin{cases} \alpha_{1,i} \neq 0, & i = 0, 1 \\ \alpha_{1,1} \neq \pm\, \alpha_{1,0}. \end{cases} \tag{3.12}$$

Based on the calculation for (3.10), $K_{1,i}$ can be rewritten as

$$K_{1,i} = f^1 + \alpha_{1,i}h\gamma_1 + \frac{\alpha_{1,i}^2 h^2}{2} f_{jk}^1 f^j f^k + \frac{\alpha_{1,i}^3 h^3}{6}\gamma_3 + O(h^4). \tag{3.13}$$

Since $\gamma_1$ and $\gamma_3$ are now known values, $f_{jk}^1 f^j f^k$ is simply determined by

$$\begin{aligned} f_{jk}^1 f^j f^k &= \gamma_2 + O(h^2) \\ &= \Big(2K_{1,0} - 2f^1 - 2\alpha_{1,0}h\gamma_1 - \frac{\alpha_{1,0}^3 h^3}{3}\gamma_3\Big)/\alpha_{1,0}^2 h^2 + O(h^2), \end{aligned} \tag{3.14}$$

provided $\alpha_{1,0} \neq 0$.

### 3.1.2. 2nd level

Next, for the determination of the unknown coefficients $f_j^1 f_k^j f^k$, $f_{jk}^1 f_l^j f^k f^l$ and $f_j^1 f_{kl}^j f^k f^l$, we consider

$$K_{2,i} := f(t_m + \alpha_{2,i}h, y_m + h\delta_{2,i}), \qquad \delta_{2,i} = \beta_{2,i}f^1 + (\alpha_{2,i} - \beta_{2,i})K_{1,i} \tag{3.15}$$

where $\alpha_{2,i}$ and $\beta_{2,i}$ are parameters to be determined. Applying Taylor's expansion and $K_{1,i}$ to Eq. (3.15) leads to the following equations

$$\begin{aligned} K_{2,i} = f^1 &+ (\alpha_{2,i}h)\gamma_1 + \frac{1}{2}(\alpha_{2,i}h)^2\gamma_2 + \mu_i\alpha_{1,i}h^2 f_j^1 f_k^j f^k + \frac{1}{6}(\alpha_{2,i}h)^3\gamma_3 \\ &+ h^3\mu_i\frac{\alpha_{1,i}^2}{2} f_j^1 f_{kl}^j f^k f^l + h^3\mu_i\alpha_{1,i}\alpha_{2,i} f_{jk}^1 f^j f_l^k f^l + O(h^4), \end{aligned} \tag{3.16}$$

where $\mu_i = \alpha_{2,i} - \beta_{2,i}$ for $i = 0, 1, 2$. Similarly above, instead of solving a 3 by 3 system, we introduce a new notation $\widehat{K_{2,i}}$ by defining $\widehat{K_{2,i}} = K_{2,i}(h) - K_{2,i}(-h)$. Then

$$\begin{aligned} \widehat{K_{2,i}} = \hat{D}_i &+ h^3\mu_i(\alpha_{1,i})^2 f_j^1 f_{kl}^j f^k f^l + 2h^3\mu_i\alpha_{1,i}\alpha_{2,i} f_{jk}^1 f^j f_l^k f^l \\ &+ O(h^4), \quad i = 0, 1, \end{aligned} \tag{3.17}$$

where $\hat{D}_i = 2(\alpha_{2,i}h)\gamma_1 + \frac{1}{3}(\alpha_{2,i}h)^3\gamma_3$. By solving the system (3.17), one may determine $f^1_j f^j_{kl} f^k f^l$ and $f^1_{jk} f^j f^k_l f^l$ with the formula

$$\begin{pmatrix} f^1_j f^j_{kl} f^k f^l \\ f^1_{jk} f^j f^k_l f^l \end{pmatrix} = \begin{pmatrix} \gamma_5 + O(h) \\ \gamma_6 + O(h) \end{pmatrix}, \tag{3.18}$$

where

$$\begin{pmatrix} \gamma_5 \\ \gamma_6 \end{pmatrix} = B_1^{-1} \begin{pmatrix} \widehat{K_{2,0}} - \hat{D}_0 \\ \widehat{K_{2,1}} - \hat{D}_1 \end{pmatrix}, \quad B_1 := \begin{pmatrix} \mu_0\alpha_{1,0}^2 h^3 & 2\mu_0\alpha_{1,0}\alpha_{2,0}h^3 \\ \mu_1\alpha_{1,1}^2 h^3 & 2\mu_1\alpha_{1,1}\alpha_{2,1}h^3 \end{pmatrix}. \tag{3.19}$$

As mentioned above, for the determination of coefficients $f^1_j f^j_{kl} f^k f^l$ and $f^1_{jk} f^j f^k_l f^l$, the matrix $B_1$ is non-singular. Since $det(B_1) = 2h^6\mu_0\mu_1\alpha_{1,0}\alpha_{1,1}(\alpha_{1,0}\alpha_{2,1} - \alpha_{1,1}\alpha_{2,0}) \neq 0$, the conditions to be satisfied are as follows:

$$\begin{cases} \alpha_{1,i} \neq 0 \quad & i = 0, 1, \\ \alpha_{2,i} \neq \beta_{2,i} \quad & i = 0, 1, \\ \alpha_{1,0}\alpha_{2,1} - \alpha_{1,1}\alpha_{2,0} \neq 0. \end{cases} \tag{3.20}$$

Based on the previous calculation, $K_{2,i}$ can be simplified as follows:

$$\begin{aligned} K_{2,i} &= f^1 + (\alpha_{2,i}h)\gamma_1 + \frac{1}{2}(\alpha_{2,i}h)^2\gamma_2 + \mu_i\alpha_{1,i}h^2 f^1_j f^j_k f^k \\ &+ \frac{1}{6}(\alpha_{2,i}h)^3\gamma_3 + h^3\mu_i\frac{\alpha_{1,i}^2}{2}\gamma_5 + h^3\mu_i\alpha_{1,i}\alpha_{2,i}\gamma_6, \end{aligned} \tag{3.21}$$

Hence, $f^1_j f^j_k f^k$ is easily obtained by

$$\begin{aligned} f^1_j f^j_k f^k &= \gamma_4 + O(h^2) \\ &= \Big(K_{2,0} - f^1 - (\alpha_{2,0}h)\gamma_1 + \frac{1}{2}(\alpha_{2,0}h)^2\gamma_2 - \frac{1}{6}(\alpha_{2,0}h)^3\gamma_3 \\ &+ h^3\mu_0\frac{\alpha_{1,0}^2}{2}\gamma_5 + h^3\mu_0\alpha_{1,0}\alpha_{2,0}\gamma_6\Big)/\mu_0\alpha_{1,0}h^2 + O(h^2), \end{aligned} \tag{3.22}$$

with $\mu_0\alpha_{1,0} \neq 0$.

### 3.1.3. 3rd level

Finally, for the determination of the remaining unknown coefficient $f^1_j f^j_k f^k_l f^l$ in the expression (3.3), we consider

$$K_3 = f(t_m + \alpha_3 h, y_m + h\rho_3) \tag{3.23}$$

where

$$\rho_3 = \beta_{3,0}f^1 + \beta_{3,1}K_{1,0} + (\alpha_3 - \beta_{3,0} - \beta_{3,1})K_{2,0}, \tag{3.24}$$

where $\beta_{3,i}, i = 0, 1$ and $\alpha_3$ are arbitrary parameters to be determined.

$$\begin{aligned} f^1_j f^j_k f^k_l f^l &= \gamma_7 + O(h) \\ &= \frac{1}{h^3\nu_3}\Big(K_3 - f^1 - \gamma_1 h\alpha_3 - h^2\Big(\nu_1\gamma_4 + \frac{\alpha_3^2}{2}\gamma_2\Big) \\ &- h^3\Big(\nu_2\gamma_5 + \alpha_3\nu_1\gamma_6 + \frac{\alpha_3^3}{6}\gamma_3\Big)\Big) + O(h), \end{aligned} \tag{3.25}$$

where

$$\begin{aligned}\nu_1 &:= \beta_{3,1}(\alpha_{1,0}-\alpha_{2,0}) + (\alpha_3-\beta_{3,0})\alpha_{2,0},\\ \nu_2 &:= \frac{\alpha_{1,0}^2}{2}\beta_{3,1} + \frac{\alpha_{2,0}^2}{2}(\alpha_3-\beta_{3,0}-\beta_{3,1}),\\ \nu_3 &:= (\alpha_3-\beta_{3,0}-\beta_{3,1})(\alpha_{2,0}-\beta_{2,0})\alpha_{1,0}.\end{aligned} \tag{3.26}$$

Finally, by substituting (3.10), (3.14), (3.19), (3.22) and (3.25) into (3.3) and approximating $\phi(t_m)$ with $y_m$, one may define the local approximation $y(t)$ of degree 4 by

$$\begin{aligned}y(t) = y_m + (t-t_m)f(t_m,y_m) &+ \frac{(t-t_m)^2}{2}\gamma_1 + \frac{(t-t_m)^3}{6}\Big(\gamma_2+\gamma_4\Big)\\ &+ \frac{(t-t_m)^4}{24}\Big(\gamma_3+3\gamma_6+\gamma_5+\gamma_7\Big),\end{aligned} \tag{3.27}$$

where $\gamma_i,\ i=1,\cdots,7$, are defined in (3.10), (3.14), (3.19), (3.22) and (3.25).

## 3.2. Local polynomial approximation of degree p=5

Similarly to the previous subsection, in this subsection, we construct a local polynomial approximation of degree 5 by finding approximations for 15 unknown coefficients in the equation (3.3), which are the total derivatives of $\phi$ with order larger than 2.

### 3.2.1. 1st level

First, we start with finding the leading coefficients of each order term in (3.4), $f_j^1 f^j$, $f_{jk}^1 f^j f^k$, $f_{jkl}^1 f^j f^k f^l$ and $f_{jklm}^1 f^j f^k f^l f^m$ using

$$K_{1,i} := f(t_m+\alpha_{1,i}h, y_m+\alpha_{1,i}hf(t_m,y_m)),\quad i=0,1,2,3 \tag{3.28}$$

where $\alpha_{1,i}$ are arbitrary parameters to be determined. The Taylor expansion can be applied to Eq. (3.28), and then following equations for $f_j^1 f^j$, $f_{jk}^1 f^j f^k$, $f_{jkl}^1 f^j f^k f^l$ and $f_{jklm}^1 f^j f^k f^l f^m$ can be obtained.

$$\begin{aligned}K_{1,i} = f^1 + (\alpha_{1,i}h)f_j^1 f^j &+ \frac{(\alpha_{1,i}h)^2}{2}f_{jk}^1 f^j f^k + \frac{(\alpha_{1,i}h)^3}{6}f_{jkl}^1 f^j f^k f^l\\ &+ \frac{(\alpha_{1,i}h)^4}{24}f_{jklm}^1 f^j f^k f^l f^m + O(h^5) \qquad i=0,1,2,3.\end{aligned} \tag{3.29}$$

As mentioned above, instead of solving a 4 by 4 system, we introduce a new notation $\widehat{K_{1,i}}$ to reduce the system size using $K_{1,i}(h)$ and $K_{1,i}(-h)$,

$$\widehat{K_{1,i}} = K_{1,i}(h) - K_{1,i}(-h), \tag{3.30}$$

where $K_{1,i}(h) = K_{1,i}$ and

$$\begin{aligned}K_{1,i}(-h) &:= f(t_m-\alpha_{1,i}h, y_m-\alpha_{1,i}hf(t_m,y_m))\\ &= f^1 - (\alpha_{1,i}h)f_j^1 f^j + \frac{(\alpha_{1,i}h)^2}{2}f_{jk}^1 f^j f^k - \frac{(\alpha_{1,i}h)^3}{6}f_{jkl}^1 f^j f^k f^l\\ &\quad + \frac{(\alpha_{1,i}h)^4}{24}f_{jklm}^1 f^j f^k f^l f^m + O(h^5).\end{aligned} \tag{3.31}$$

Then,

$$\widehat{K_{1,i}} = 2\alpha_{1,i}hf^1_j f^j + h^3\frac{\alpha^3_{1,i}}{3} f^1_{jkl} f^j f^k f^l. \tag{3.32}$$

Since

$$\begin{pmatrix} f^1_j f^j \\ f^1_{jkl} f^j f^k f^l \end{pmatrix} = \begin{pmatrix} \gamma_1 + O(h^4) \\ \gamma_3 + O(h^2) \end{pmatrix}, \tag{3.33}$$

the coefficients can be determined by

$$\begin{pmatrix} \gamma_1 \\ \gamma_3 \end{pmatrix} = A_1^{-1} \begin{pmatrix} \widehat{K_{1,0}} \\ \widehat{K_{1,1}} \end{pmatrix}, \tag{3.34}$$

where $A_1 = \begin{pmatrix} 2\alpha_{1,0}h & \frac{\alpha^3_{1,0}h^3}{3} \\ 2\alpha_{1,1}h & \frac{\alpha^3_{1,1}h^3}{3} \end{pmatrix}$. Note that for determination of the coefficients, the matrix $A_1$ should be non-singular or $det(A_1) = \frac{2}{3}\alpha_{1,0}\alpha_{1,1}h^4(\alpha^2_{1,1} - \alpha^2_{1,0}) \neq 0$. Therefore,$\alpha_{1,0}$ and $\alpha_{1,1}$ should satisfy the following conditions:

$$\begin{cases} \alpha_{1,i} \neq 0, \quad i = 0, 1 \\ \alpha_{1,1} \neq \pm\alpha_{1,0}. \end{cases} \tag{3.35}$$

Similarly above, we introduce a new notation $\widetilde{K_{1,i}}$ as follows:

$$\widetilde{K_{1,i}} = K_{1,i}(h) + K_{1,i}(-h), \tag{3.36}$$

then we get

$$\widetilde{K_{1,i}} = 2f^1 + h^2\alpha^2_{1,i} f^1_{jk} f^j f^k + h^4\frac{\alpha^4_{1,i}}{12} f^1_{jklm} f^j f^k f^l f^m. \tag{3.37}$$

Since

$$\begin{pmatrix} f^1_{jk} f^j f^k \\ f^1_{jklm} f^j f^k f^l f^m \end{pmatrix} = \begin{pmatrix} \gamma_2 + O(h^3) \\ \gamma_4 + O(h) \end{pmatrix}, \tag{3.38}$$

the coefficients can be determined by solving these equations

$$\begin{pmatrix} \gamma_2 \\ \gamma_4 \end{pmatrix} = A_2^{-1} \begin{pmatrix} \widetilde{K_{1,0}} - 2f^1 \\ \widetilde{K_{1,1}} - 2f^1 \end{pmatrix}, \tag{3.39}$$

where $A_2 = \begin{pmatrix} \alpha^2_{1,0}h^2 & \frac{\alpha^4_{1,0}h^4}{12} \\ \alpha^2_{1,1}h^2 & \frac{\alpha^4_{1,1}h^4}{12} \end{pmatrix}$ and $A_2$ should be non-singular. Since the $det(A_2) = \frac{1}{12}\alpha^2_{1,0}\alpha^2_{1,1}h^6(\alpha^2_{1,1} - \alpha^2_{1,0}) \neq 0$, the condition to be satisfied for determination is the same as (3.35). Therefore, the conditions for the parameters are

$$\begin{cases} \alpha_{1,i} \neq 0, \quad i = 0, 1 \\ \alpha_{1,1} \neq \pm\alpha_{1,0}. \end{cases} \tag{3.40}$$

and $\alpha_{1,2}$ and $\alpha_{1,3}$ become free parameters to be assigned arbitrarily without any constraints, unlike the traditional way.

**Remark 3.1.** Notice that the traditional scheme generates a 4 by 4 system to solve (3.28), while the proposed scheme needs only two 2 by 2 systems ((3.34) and (3.39)), so that it can reduce the total computational costs. This is one of remarkable improvements of the proposed scheme.

### 3.2.2. 2nd level

Secondly, we consider

$$K_{2,i} := f(t_m + \alpha_{2,i}h, y_m + h\delta_{2,i}), \qquad \delta_{2,i} = \beta_{2,i}f^1 + (\alpha_{2,i} - \beta_{2,i})K_{1,i} \tag{3.41}$$

where $\alpha_{2,i}$ and $\beta_{2,i}$ are parameters to be determined. Applying Taylor's expansion and $K_{1,i}$ to Eq. (3.41) leads to the following equations

$$\begin{aligned} K_{2,i} = {} & f^1 + (\alpha_{2,i}h)\gamma_1 + \frac{1}{2}(\alpha_{2,i}h)^2\gamma_2 + \mu_i\alpha_{1,i}h^2 f^1_j f^j_k f^k + \frac{1}{6}(\alpha_{2,i}h)^3\gamma_3 \\ & + h^3\mu_i\frac{(\alpha_{1,i})^2}{2}f^1_j f^j_{kl}f^k f^l + h^3\mu_i\alpha_{1,i}\alpha_{2,i}f^1_{jk}f^j f^k_l f^l + \frac{1}{24}(\alpha_{2,i}h)^4\gamma_4 \\ & + h^4\mu_i\frac{\alpha^3_{1,i}}{6}f^1_j f^j_{klm}f^k f^l f^m + h^4\mu_i\frac{\alpha^2_{1,i}}{2}\alpha_{2,i}f^1_{jk}f^j f^k_{lm}f^l f^m \\ & + h^4\mu_i^2\frac{\alpha^2_{1,i}}{2}f^1_{jk}f^j_l f^l f^k_m f^m + h^4\mu_i\frac{\alpha^2_{2,i}}{2}\alpha_{1,i}f^1_{jkl}f^j f^k f^l_m f^m \\ & + O(h^5) \qquad\qquad i = 0, 1, 2, 3, \end{aligned} \tag{3.42}$$

where $\mu_i = \alpha_{2,i} - \beta_{2,i}$. Since there are 7 unknowns, we need a 7 by 7 matrix to solve (3.41) in the traditional way. However, unnecessary values $K_{1,i}$ in (3.41) should be calculated to generate the relevant conditions, so it leads expensive computational costs and unnecessary storage costs. To hurdle this drawback, similarly above, we introduce a new notation $\widehat{K_{2,i}}$ by $\widehat{K_{2,i}} = K_{2,i}(\alpha_{2,i}, \beta_{2,i}, h) - K_{2,i}(\alpha_{2,i}, \beta_{2,i}, -h)$, then

$$\widehat{K_{2,i}} = \hat{D}_i + h^3\mu_i\alpha^2_{1,i}f^1_j f^j_{kl}f^k f^l + 2h^3\mu_i\alpha_{1,i}\alpha_{2,i}f^1_{jk}f^j f^k_l f^l, \tag{3.43}$$

where $\hat{D}_i = 2(\alpha_{2,i}h)\gamma_1 + \frac{1}{3}(\alpha_{2,i}h)^3\gamma_3$. Therefore, by solving these equations (3.43), one may determine $f^1_j f^j_{kl}f^k f^l$ and $f^1_{jk}f^j f^k_l f^l$ with the formula

$$\begin{pmatrix} f^1_j f^j_{kl}f^k f^l \\ f^1_{jk}f^j f^k_l f^l \end{pmatrix} = \begin{pmatrix} \gamma_5 + O(h^2) \\ \gamma_6 + O(h^2) \end{pmatrix}, \tag{3.44}$$

where

$$\begin{pmatrix} \gamma_5 \\ \gamma_6 \end{pmatrix} = B_1^{-1}\begin{pmatrix} \widehat{K_{2,0}} - \hat{D}_0 \\ \widehat{K_{2,1}} - \hat{D}_1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} \mu_0\alpha^2_{1,0}h^3 & 2\mu_0\alpha_{1,0}\alpha_{2,0}h^3 \\ \mu_1\alpha^2_{1,1}h^3 & 2\mu_1\alpha_{1,1}\alpha_{2,1}h^3 \end{pmatrix}. \tag{3.45}$$

For the determination of coefficients $f^1_j f^j_{kl}f^k f^l$ and $f^1_{jk}f^j f^k_l f^l$, the matrix $B_1$ is nonsingular. Since $det(B_1) = 2h^6\mu_0\mu_1\alpha_{1,0}\alpha_{1,1}(\alpha_{1,0}\alpha_{2,1} - \alpha_{1,1}\alpha_{2,0}) \neq 0$, the condition to be satisfied is as follows:

$$\begin{cases} \alpha_{1,i} \neq 0, \qquad i = 0, 1, \\ \alpha_{2,i} \neq \beta_{2,i}, \qquad i = 0, 1, \\ \alpha_{1,0}\alpha_{2,1} - \alpha_{1,1}\alpha_{2,0} \neq 0. \end{cases} \tag{3.46}$$

Similarly, we again introduce another notation $\widetilde{K_{2,i}} = K_{2,i}(\alpha_{2,i}, \beta_{2,i}, h) + K_{2,i}(-\alpha_{2,i}, -\beta_{2,i}, h)$. Then, one may check that

$$\widetilde{K_{2,i}} = \widetilde{D_i} + h^4\mu_i(\alpha_{1,i})^2\alpha_{2,i}f^1_{jk}f^jf^k_{lm}f^lf^m + h^4\mu_i^2\alpha_{1,i}^2f^1_{jk}f^j_lf^lf^k_mf^m, \tag{3.47}$$

where $\widetilde{D_i} = 2f^1 + (\alpha_{2,i}h)^2\gamma_2 + 2h^3\mu_i\alpha_{1,i}\alpha_{2,i}\gamma_6 + \frac{1}{12}(\alpha_{2,i}h)^4\gamma_4$. Therefore, by solving these equations (3.47) to determine $f^1_{jk}f^jf^k_{lm}f^lf^m$ and $f^1_{jk}f^j_lf^lf^k_mf^m$, we can get

$$\begin{pmatrix} f^1_{jk}f^jf^k_{lm}f^lf^m \\ f^1_{jk}f^j_lf^lf^k_mf^m \end{pmatrix} = \begin{pmatrix} \gamma_7 + O(h) \\ \gamma_8 + O(h) \end{pmatrix}, \tag{3.48}$$

where

$$\begin{pmatrix} \gamma_7 \\ \gamma_8 \end{pmatrix} = B_2^{-1}\begin{pmatrix} \widetilde{K_{2,0}} - \widetilde{D_0} \\ \widetilde{K_{2,1}} - \widetilde{D_1} \end{pmatrix}, \quad B_2 = \begin{pmatrix} \mu_0\alpha_{1,0}^2\alpha_{2,0}h^4 & \mu_0^2\alpha_{1,0}^2h^4 \\ \mu_1\alpha_{1,1}^2\alpha_{2,1}h^4 & \mu_1^2\alpha_{1,1}^2h^4 \end{pmatrix}. \tag{3.49}$$

To satisfy the non-singularity of $B_2$, the following conditions should be satisfied

$$\begin{cases} \alpha_{1,0} \neq 0, \quad i = 0,1, \\ \alpha_{2,0} \neq \beta_{2,0}, \quad i = 0,1, \\ \alpha_{2,0}\mu_1 - \mu_0\alpha_{2,1} \neq 0, \quad \alpha_{2,1}\beta_{2,0} - \beta_{2,1}\alpha_{2,0} \neq 0. \end{cases} \tag{3.50}$$

Based on the calculation for $\widehat{K_{2,i}}$ and $\widetilde{K_{2,i}}$, we lastly define $\overline{K_{2,i}} = K_{2,i}(\alpha_{2,i}, \beta_{2,i}, h) - K_{2,i}(-\alpha_{2,i}, -\beta_{2,i}, -h)$. Then one may check that

$$\begin{aligned} \overline{K_{2,i}} &= \overline{D_i} + 2h^2\mu_i\alpha_{1,i}f^1_jf^j_kf^k + \frac{h^4}{3}\mu_i\alpha_{1,i}^3f^1_jf^j_{klm}f^kf^lf^m \\ &+ h^4\mu_i\alpha_{2,i}^2\alpha_{1,i}f^1_{jkl}f^jf^kf^l_mf^m \end{aligned} \tag{3.51}$$

where $\overline{D_i} = 2\alpha_{2,i}h\gamma_1 + \frac{1}{3}(\alpha_{2,i}h)^3\gamma_3 + h^3\mu_i\alpha_{1,i}^2\gamma_5$. Therefore, by solving these equations (3.51), one may determine $f^1_jf^j_kf^k$, $f^1_jf^j_{klm}f^kf^lf^m$, and $f^1_{jkl}f^jf^kf^l_mf^m$ as follows:

$$\begin{pmatrix} f^1_jf^j_kf^k \\ f^1_jf^j_{klm}f^kf^lf^m \\ f^1_{jkl}f^jf^kf^l_mf^m \end{pmatrix} = \begin{pmatrix} \gamma_9 + O(h^3) \\ \gamma_{10} + O(h) \\ \gamma_{11} + O(h) \end{pmatrix} = B_3^{-1}\begin{pmatrix} \overline{K_{2,0}} - \overline{D_0} + O(h^3) \\ \overline{K_{2,1}} - \overline{D_1} + O(h) \\ \overline{K_{2,2}} - \overline{D_2} + O(h) \end{pmatrix}, \tag{3.52}$$

where $B_3 = \begin{pmatrix} 2\mu_0\alpha_{1,0}h^2 & \mu_0\frac{(\alpha_{1,0})^3}{3}h^4 & \mu_0(\alpha_{2,0})^2\alpha_{1,0}h^4 \\ 2\mu_1\alpha_{1,1}h^2 & \mu_1\frac{(\alpha_{1,1})^3}{3}h^4 & \mu_1(\alpha_{2,1})^2\alpha_{1,1}h^4 \\ 2\mu_2\alpha_{1,2}h^2 & \mu_2\frac{(\alpha_{1,2})^3}{3}h^4 & \mu_2(\alpha_{2,2})^2\alpha_{1,2}h^4 \end{pmatrix}$ with the conditions

$$\begin{cases} \alpha_{1,0} \neq 0, \quad i = 0,1,2, \\ \alpha_{2,0} \neq \beta_{2,0}, \quad i = 0,1,2, \\ \alpha_{1,1}^2\alpha_{2,2}^2 - \alpha_{1,2}^2\alpha_{2,1}^2 - \alpha_{1,0}^2\alpha_{2,2}^2 + \alpha_{1,0}^2\alpha_{2,1}^2 + \alpha_{2,0}^2\alpha_{1,2}^2 - \alpha_{2,0}^2\alpha_{1,1}^2 \neq 0. \end{cases}$$

**Remark 3.2.** Note that instead of solving a 7 by 7 system directly with generating unnecessary function values done in traditional way, the proposed scheme can reduce the system to a 3 by 3 and two 2 by 2 systems without any additional function values, so that it can reduce the total computational costs and storage costs. This is the other remarkable improvement of the proposed scheme.

### 3.2.3. 3rd level

We need to determine the remaining unknown coefficients by considering

$$K_{3,i} = f(t_m + \alpha_{3,i}h, y_m + h\rho_{3,i}), \tag{3.53}$$

where

$$\rho_{3,i} = \beta_{3,i}f^1 + \omega_{3,i}K_{1,i} + (\alpha_{3,i} - \beta_{3,i} - \omega_{3,i})K_{2,i}. \tag{3.54}$$

Here, $\alpha_{3,i}, \beta_{3,i}$ and $\omega_{3,i}$ are arbitrary parameters to be determined. Similarly above, one may check that

$$\begin{aligned} K_{3,i} = f^1 &+ \alpha_{3,i}h\gamma_1 + h^2\Big(\frac{\alpha_{3,i}^2}{2}\gamma_2 + (\alpha_{1,i}\omega_{3,i} + \tau_i\alpha_{2,i})\gamma_9\Big) \\ &+ h^3\Big(\frac{\alpha_{3,i}^3}{6}\gamma_3 + \frac{1}{2}(\alpha_{1,i}^2\omega_{3,i} + \tau_i\alpha_{2,i}^2)\gamma_5 + \tau_i\mu_i\alpha_{1,i}f_j^1f_k^jf_l^kf^l \\ &+ \alpha_{3,i}(\alpha_{1,i}\omega_{3,i} + \tau_i\alpha_{2,i})\gamma_6\Big) + h^4\Big(\frac{\alpha_{3,i}^4}{24}\gamma_4 + \frac{1}{6}(\alpha_{1,i}^3\omega_{3,i} + \tau_i\alpha_{2,i}^3)\gamma_{10} \\ &+ \tau_i\mu_i\frac{\alpha_{1,i}^2}{2}f_j^1f_k^jf_{lm}^kf^lf^m + \tau_i\mu_i\alpha_{1,i}\alpha_{2,i}f_j^1f_{kl}^jf^kf_m^lf^m \\ &+ \frac{1}{2}(\alpha_{1,i}^2\omega_{3,i} + \tau_i\alpha_{2,i}^2)\alpha_{3,i}\gamma_7 + \frac{1}{2}(\alpha_{1,i}\omega_{3,i} + \tau_i\alpha_{2,i})^2\gamma_8 \\ &+ \tau_i\mu_i\alpha_{1,i}\alpha_{3,i}f_{jk}^1f^jf_l^kf_m^lf^m + \frac{\alpha_{3,i}^2}{2}(\alpha_{1,i}\omega_{3,i} + \tau_i\alpha_{2,i})\gamma_{11}\Big), \end{aligned} \tag{3.55}$$

where $\tau_i = \alpha_{3,i} - \beta_{3,i} - \omega_{3,i}$ and $\mu_i = \alpha_{2,i} - \beta_{2,i}$. Since $f_j^1f_{kl}^jf^kf_m^lf^m = f_{jk}^1f^jf_l^kf_m^lf^m$, there are 3 unknowns to be solved. Similarly above, we introduce a new notation $\widehat{K_{3,i}} = K_{3,i}(h) - K_{3,i}(-h)$, then

$$\widehat{K_{3,i}} = \hat{D} + 2h^3\tau_i\mu_i\alpha_{1,i}f_j^1f_k^jf_l^kf^l, \tag{3.56}$$

where $\hat{D} = 2(\alpha_{3,i}h)\gamma_1 + \frac{1}{3}(\alpha_{3,i}h)^3\gamma_3 + h^3(\alpha_{1,i}^2\omega_{3,i} + \tau_i\alpha_{2,i}^2)\gamma_5 + 2h^3(\alpha_{1,i}\omega_{3,i} + \tau_i\alpha_{2,i})\gamma_6$. Hence, $f_j^1f_k^jf_l^kf^l$ can be obtained by

$$f_j^1f_k^jf_l^kf^l = \gamma_{12} + O(h^2), \tag{3.57}$$

where

$$\gamma_{12} = (\widehat{K_{3,0}} - \hat{D})/2\tau_0\mu_0\alpha_{1,0}h^3, \quad \text{provided } \ \tau_0\mu_0\alpha_{1,0} \neq 0. \tag{3.58}$$

Similarly, for $\widetilde{K_{3,i}} = K_{3,i}(h) + K_{3,i}(-h)$, $\widetilde{K_{3,i}}$ can be rewritten as follows:

$$\begin{aligned} \widetilde{K_{3,i}} = \widetilde{D_{3,i}} &+ \tau_i\mu_i\frac{\alpha_{1,i}^2}{2}f_j^1f_k^jf_{lm}^kf^lf^m \\ &+ \tau_i\mu_i\alpha_{1,i}(\alpha_{2,i} + \alpha_{3,i})f_{jk}^1f^jf_l^kf_m^lf^m, \end{aligned} \tag{3.59}$$

where $\widetilde{D_{3,i}} = 2f^1 + h^2\Big(\alpha_{3,i}^2\gamma_2 + 2(\alpha_{1,i}\omega_{3,i} + \tau_i\alpha_{2,i})\gamma_9\Big) + h^4\Big(\frac{\alpha_{3,i}^4}{12}\gamma_4 + \frac{1}{3}(\alpha_{1,i}^3\omega_{3,i} + \tau_i\alpha_{2,i}^3)\gamma_{10} + (\alpha_{1,i}^2\omega_{3,i} + \tau_i\alpha_{2,i}^2)\alpha_{3,i}\gamma_7 + (\alpha_{1,i}\omega_{3,i} + \tau_i\alpha_{2,i})^2\gamma_8 + \alpha_{3,i}^2(\alpha_{1,i}\omega_{3,i} + \tau_i\alpha_{2,i})\gamma_{11}\Big)$.
Since

$$\begin{pmatrix} f_j^1 f_k^j f_{lm}^k f^l f^m \\ f_{jk}^1 f^j f_l^k f_m^l f^m \end{pmatrix} = \begin{pmatrix} \gamma_{13} + O(h) \\ \gamma_{14} + O(h) \end{pmatrix}, \tag{3.60}$$

the coefficients can be determined by solving the following system

$$\begin{pmatrix} \gamma_{13} \\ \gamma_{14} \end{pmatrix} = C^{-1} \begin{pmatrix} \widetilde{K_{3,0}} - \widetilde{D_{3,0}} \\ \widetilde{K_{3,1}} - \widetilde{D_{3,1}} \end{pmatrix}, \tag{3.61}$$

where $C = \begin{pmatrix} \tau_0\mu_0\alpha_{1,0}^2 h^4 & 2\tau_0\mu_0\alpha_{1,0}(\alpha_{2,0} + \alpha_{3,0})h^4 \\ \tau_1\mu_1\alpha_{1,1}^2 h^4 & 2\tau_1\mu_1\alpha_{1,1}(\alpha_{2,1} + \alpha_{3,1})h^4 \end{pmatrix}$.

### 3.2.4. 4th level

Finally, for the determination of the remaining unknown coefficient $f_j^1 f_k^j f_l^k f_m^l f^m$ in the expression, we consider

$$K_4 := f(t_m + \alpha_4 h, y_m + h\sigma), \quad i = 0, 1, 2, \tag{3.62}$$

where

$$\sigma = \beta_{4,0}f^1 + \beta_{4,1}K_{1,0} + \beta_{4,2}K_{2,0} + (\alpha_4 - \beta_{4,0} - \beta_{4,1} - \beta_{4,2})K_{3,0},$$

where $\alpha_4$ and $\beta_{4,i}$ are arbitrary parameters to be determined. Similarly above, one may check that

$$\begin{aligned} K_4 = f^1 + \alpha_4 h\gamma_1 + h^2\left(\frac{\alpha_4^2}{2}\gamma_2 + \nu_1\gamma_9\right) + h^3\Big(\frac{\alpha_4^3}{6}\gamma_3 + \nu_2\gamma_5 \\ + \nu_3\gamma_{12} + \alpha_4\nu_1\gamma_6\Big) + h^4\Big(\frac{\alpha_4^4}{24}\gamma_4 + \nu_4\gamma_{10} + \nu_5\gamma_{13} + \frac{\alpha_4^2}{2}\nu_1\gamma_{11} \\ + \alpha_4\nu_2\gamma_7 + \frac{1}{2}\nu_1^2\gamma_8 + \nu_7\gamma_{14} + \alpha_4\nu_3\gamma_{14} + \nu_6 f_j^1 f_k^j f_l^k f_m^l f^m\Big), \end{aligned} \tag{3.63}$$

where

$$\begin{aligned} \nu_1 &= \beta_{4,1}\alpha_{1,0} + \beta_{4,2}\alpha_{2,0} + \kappa\alpha_{3,0}, \\ \nu_2 &= (\beta_{4,1}\alpha_{1,0}^2 + \beta_{4,2}\alpha_{2,0}^2 + \kappa\alpha_{3,0}^2)/2, \\ \nu_3 &= \beta_{4,2}\mu_0\alpha_{1,0} + \kappa(\tau_0\alpha_{2,0} + \alpha_{1,0}\omega_{3,0}), \\ \nu_4 &= (\beta_{4,1}\alpha_{1,0}^3 + \beta_{4,2}\alpha_{2,0}^3 + \kappa\alpha_{3,0}^3)/6, \\ \nu_5 &= (\beta_{4,2}\mu_0\alpha_{1,0}^2 + \kappa(\tau_0\alpha_{2,0}^2 + \alpha_{1,0}^2\omega_{3,0}))/2, \\ \nu_6 &= \kappa\tau_0\mu_0\alpha_{1,0}, \\ \nu_7 &= \beta_{4,2}\mu_0\alpha_{1,0}\alpha_{2,0} + \kappa\alpha_{3,0}(\tau_0\alpha_{2,0} + \alpha_{1,0}\omega_{3,0}), \end{aligned} \tag{3.64}$$

with $\kappa = \alpha_4 - \beta_{4,0} - \beta_{4,1} - \beta_{4,2}$. Since $f^1_j f^j_k f^k_l f^l_m f^m = \gamma_{15} + O(h)$,

$$\begin{aligned}\gamma_{15} == \frac{1}{h^4\nu_6}\Big(K_4 - f^1 - \alpha_4 h\gamma_1 - h^2\Big(\frac{\alpha_4^2}{2}\gamma_2 + \nu_1\gamma_9\Big) - h^3\Big(\frac{\alpha_4^3}{6}\gamma_3 \\ + \nu_2\gamma_5 + \nu_3\gamma_{12} + \alpha_4\nu_1\gamma_6\Big) - h^4\Big(\frac{\alpha_4^4}{24}\gamma_4 + \nu_4\gamma_{10} + \nu_5\gamma_{13} \\ + \frac{\alpha_4^2}{2}\nu_1\gamma_{11} + \alpha_4\nu_2\gamma_7 + \frac{1}{2}\nu_1^2\gamma_8 + (\nu_7 + \alpha_4\nu_3)\gamma_{14}\Big)\Big).\end{aligned} \tag{3.65}$$

By substituting into (3.3), and one may define the local approximation of degree 5 by

$$\begin{aligned}\phi(t) = \phi(t_m) + (t - t_m)f^1 + \frac{(t-t_m)^2}{2}\gamma_1 + \frac{(t-t_m)^3}{6}(\gamma_2 + \gamma_9) \\ + \frac{(t-t_m)^4}{24}(\gamma_3 + 3\gamma_6 + \gamma_5 + \gamma_{12}) + \frac{(t-t_m)^5}{120}(\gamma_4 + \gamma_{10} \\ +6\gamma_{11} + 4\gamma_7 + \gamma_{13} + 7\gamma_{14} + 3\gamma_8 + \gamma_{15}) + O(h^6),\end{aligned} \tag{3.66}$$

where $\gamma_i$, $i = 1, \cdots, 15$, are defined in (3.34), (3.39), (3.45), (3.49), (3.52), (3.57), (3.61) and (3.65).

## 3.3. Analysis

We will show that a special choice of the parameters appeared in (3.27) gives the classical fourth order RK method. We recall the fourth order RK method described by

$$\begin{aligned}&y_{m+1} = y_m + \frac{h}{6}\Big(k_1 + 2k_2 + 2k_3 + k_4\Big), \\ &k_1 = f(t_m, y_m), \quad k_2 = f(t_m + \frac{h}{2}, y_m + \frac{h}{2}k_1), \\ &k_3 = f(t_m + \frac{h}{2}, y_m + \frac{h}{2}k_2), \quad k_4 = f(t_m + h, y_m + hk_3).\end{aligned} \tag{3.67}$$

**Lemma 3.1.** *For the local approximation $y(t)$ in (3.27), the value $y(t_{m+1})$ is the same with the approximation $y_{m+1}$ in (3.67) provided we take*

$$\alpha_3 = 1, \quad \alpha_{1,0} = \alpha_{2,0} = \frac{1}{2}, \quad \beta_{2,0} = \beta_{3,i} = 0, \quad i = 0, 1, \tag{3.68}$$

*and the other parameters $\alpha_{i,j}$ and $\beta_{i,j}$ are arbitrary parameters so that the matrices $A_i$ and $B$ in (3.10) and (3.19) are nonsingular.*

**Proof.** Using the coefficients defined in (3.68), we can easily check that

$$K_{1,0} = k_2, \quad K_{2,0} = k_3, \quad K_3 = k4.$$

□

**Remark 3.3.** Compared the 4th order Runge-Kutta(RK) methods to our formulation, the proposed scheme needs more function evaluations or more stages. However unlike the traditional RK formulation which is calculated sequentially using all values in previous stages, our formulation can be calculated all function values in each stage simultaneously using vector calculations [12]. That is, using vector calculations, function values at each level count as only one function evaluation. Therefore, the calculation time for both traditional RK scheme and our formulation are exactly the same. From this point of view, one can say that each local approximation of degree $p$ requires $p$ level values for the slope function $f$.

**Remark 3.4.** For a simple ODE equation

$$y' = \lambda y, \qquad y(0) = 1, \tag{3.69}$$

the fourth order local polynomial approximation is calculated as follows: Since

$$\begin{aligned} K_{1,i}(h) &= \lambda(1 + \alpha_{1,i}\lambda h)y_m, \\ K_{1,i}(-h) &= \lambda(1 - \alpha_{1,i}\lambda h)y_m, \end{aligned} \tag{3.70}$$

for any coefficients $\alpha_{1,0}$ and $\alpha_{1,1}$, the unknowns $\gamma_1, \gamma_2$ and $\gamma_3$ in (3.10) and (3.14) are calculated by

$$\begin{aligned} \begin{pmatrix} \gamma_1 \\ \gamma_3 \end{pmatrix} &= A^{-1} \begin{pmatrix} K_{1,0}(h) - K_{1,0}(-h) \\ K_{1,1}(h) - K_{1,1}(-h) \end{pmatrix} \\ &= A^{-1} \begin{pmatrix} 2\alpha_{1,0}\lambda^2 h y_m \\ 2\alpha_{1,1}\lambda^2 h y_m \end{pmatrix} = \begin{pmatrix} \lambda^2 y_m \\ 0 \end{pmatrix}, \end{aligned} \tag{3.71}$$

and

$$\gamma_2 = \Big(2K_{1,0} - 2f - 2\alpha_{1,0}h\gamma_1 - \frac{\alpha_{1,0}^3 h^3}{3}\Big)/\alpha_{1,0}^2 h^2 = 0. \tag{3.72}$$

For any coefficients $\alpha_{2,i}$ and $\beta_{2,i}$,

$$\begin{aligned} K_{2,i}(h) &= \lambda y_m\Big(1 + \alpha_{2,i}\lambda h + (\alpha_{2,i} - \beta_{2,i})\alpha_{1,i}\lambda^2 h^2\Big), \\ K_{2,i}(-h) &= \lambda y_m\Big(1 - \alpha_{2,i}\lambda h + (\alpha_{2,i} - \beta_{2,i})\alpha_{1,i}\lambda^2 h^2\Big). \end{aligned} \tag{3.73}$$

Based on the proposed algorithm, $\gamma_5$ and $\gamma_6$ are calculated by

$$\begin{pmatrix} \gamma_5 \\ \gamma_6 \end{pmatrix} = B_1^{-1} \begin{pmatrix} K_{2,0}(h) - K_{2,0}(-h) - \hat{D}_0 \\ K_{2,1}(h) - K_{2,1}(-h) - \hat{D}_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.74}$$

where $B_1$ and $\hat{D}_i$ are defined in (3.19) and $\gamma_4$ is

$$\begin{aligned} \gamma_4 = &\Big(K_{2,0} - f^1 - (\alpha_{2,0}h)\gamma_1 + \frac{1}{2}(\alpha_{2,0}h)^2\gamma_2 - \frac{1}{6}(\alpha_{2,0}h)^3\gamma_3 \\ &+ h^3\mu_0\frac{\alpha_{1,0}^2}{2}\gamma_5 + h^3\mu_0\alpha_{1,0}\alpha_{2,0}\gamma_6\Big)/\mu_0\alpha_{1,0}h^2 = \lambda^3 y_m, \end{aligned} \tag{3.75}$$

$$\begin{aligned} K_3 = &\lambda y_m\Big(1 + \alpha_3\lambda h + \beta_{3,1}\alpha_{1,0}\lambda^2 h^2 \\ &+ (\alpha_3 - \beta_{3,0} - \beta_{3,1})\lambda h(\alpha_{2,0}\lambda h + \mu_0\alpha_{1,0}\lambda^2 h^2)\Big). \end{aligned} \tag{3.76}$$

$\gamma_7$ is obtained by

$$\gamma_7 = \frac{1}{h^3\nu_3}\Big(K_3 - f - \gamma_1 h\alpha_3 - h^2\Big(\nu_1\gamma_4 + \frac{\alpha_3^2}{2}\gamma_2\Big) - h^3\Big(\nu_2\gamma_5 + \alpha_3\nu_1\gamma_6 + \frac{\alpha_3^3}{6}\gamma_3\Big)\Big) = \lambda^4 y_m. \tag{3.77}$$

Summarizing all calculations above, the approximation can be written as follows:

$$y(t) = y_m + (t-t_m)(\lambda y_m) + \frac{(t-t_m)^2}{2}(\lambda^2 y_m) + \frac{(t-t_m)^3}{6}(\lambda^3 y_m) + \frac{(t-t_m)^4}{24}(\lambda^4 y_m) + O(h^5). \tag{3.78}$$

This is the exactly the same as the Taylor expansion of $\exp(\lambda t)$ at $t = t_m$ of (3.69) up to 4th degree.

# 4. Numerical Results

In this section, we show two numerical results for both non-stiff and stiff problem-s. For the comparison of the numerical results, the maximum error $Err(h)$ and convergence rates are presented, which are defined by

$$Err(h) = \max_{1\le i\le n} ||\phi(t_i) - y_i||_\infty, \quad \text{rate} = \frac{\log(Err(h_1)/Err(h_2))}{\log(h_1/h_2)}, \tag{4.1}$$

respectively, where $||\cdot||_\infty$ denotes the maximum norm and $h_i$, $i = 1, 2$ are given two time step sizes. For the convenience, we denote the ECM algorithm based on the $p$ degree local approximation by ECM$p$.

## 4.1. A simple nonlinear problem

As the first example, we consider a nonlinear initial value problem

$$\frac{d\phi}{dt} = \frac{\kappa\phi(t)(1-\phi(t))}{2\phi(t)-1} \qquad t \in (0, 2]; \qquad \phi(0) = \frac{5}{6}, \tag{4.2}$$

whose solution is $\phi(t) = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{5}{36}\exp(-\kappa t)}$ with a parameter $\kappa$.

To investigate the convergence of ECM based on the proposed local platform-s, the problem (4.2) is solved on the interval $[0, 2]$ with different step sizes $h = 2^{-n}, n = 1, 2, 3, 4$ with the parameter $\kappa = 1$ which is represented for non-stiffness. As seen in Table 1, the numerical convergence order of ECM based on the fourth and fifth degree local platforms are the theoretical orders 10 and 12 stated in Sec. 2, respectively.

To examine the effectiveness of the proposed algorithm, we compare it with the traditional algorithm based on the 4th degree polynomial for the same setting described above. Fig. 1 shows that the convergence behaviors of both algorithms are quite similar and the convergence orders are about 10. It can be also seen that the accuracy of the proposed scheme is lower than that of the traditional scheme at the lower accuracy, but the difference is getting smaller and even the accuracy of the proposed scheme is better at the higher accuracy requirement.

**Table 1.** Numerical comparison of ECM4 and ECM5 for non-stiffness ($\kappa = 1$)

| | ECM4 | | ECM5 | |
|---|---|---|---|---|
| $n$ | $Err(h)$ | rate | $Err(h)$ | rate |
| 1 | $5.0417 \cdot 10^{-6}$ | – | $2.8407 \cdot 10^{-4}$ | $--$ |
| 2 | $9.8011 \cdot 10^{-10}$ | 12.3287 | $1.6905 \cdot 10^{-8}$ | 14.0365 |
| 3 | $4.9449 \cdot 10^{-13}$ | 10.9528 | $7.2580 \cdot 10^{-12}$ | 11.1856 |
| 4 | $2.2204 \cdot 10^{-16}$ | 11.1209 | $1.5543 \cdot 10^{-15}$ | 12.1891 |

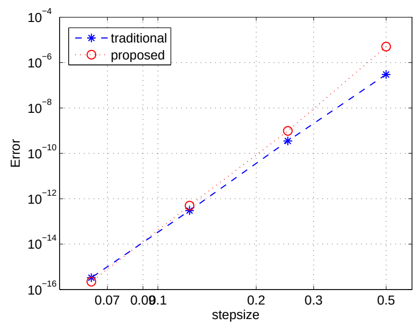


**Figure 1.** Comparison of convergence for traditional method and proposed scheme

Additionally, to examine the behavior of ECM for stiffness, the problem (4.2) is solved on the interval [0, 0.5] with the parameter $\kappa = 20$, which gives a mild stiffness. The numerical results with different step size, $h = 2^{-n}, n = 4, 5, 6$, are reported in Table 2. Table 1 shows that the numerical evidence can support the theoretical convergence analyzed in Sec. 2.

**Table 2.** Numerical comparison of ECM4 and ECM5 for stiffness ($\kappa = 20$)

| | ECM4 | | ECM5 | |
|---|---|---|---|---|
| $n$ | $Err(h)$ | rate | $Err(h)$ | rate |
| 4 | $1.6894 \cdot 10^{-7}$ | – | $3.9505 \cdot 10^{-1}$ | $--$ |
| 5 | $3.6406 \cdot 10^{-11}$ | 12.1801 | $5.1899 \cdot 10^{-10}$ | 29.504 |
| 6 | $2.1538 \cdot 10^{-14}$ | 10.7231 | $1.4633 \cdot 10^{-13}$ | 11.792 |

To investigate the efficiency of the proposed scheme, the algorithm ECM5 is compared with the existing stiff solvers - Radau and ode15s with respect to CPU time. Similar above, we solve the problem on the time interval $[0, 0.5]$ with the fixed uniform step size $h = 1/64$ and the parameter $\kappa = 20$ and report it in Table 3. It can be seen that ECM5 is superior to other existing methods in the sense of CPU time.

**Table 3.** CPU time comparison using ECM5, Radau and ode15s

| ECM5 | | Radau | | ode15s | |
|---|---|---|---|---|---|
| $Err(h)$ | cpu | $Err(h)$ | cpu | $Err(h)$ | cpu |
| $2.5346 \cdot 10^{-13}$ | 0.0176 | $4.2133 \cdot 10^{-13}$ | 0.0689 | $6.0485 \cdot 10^{-13}$ | 0.3678 |

## 4.2. Oregonator Model

In the next example, we consider the oregonator model [1] which originates from chemical reactions. It is formulated for the most important parts of the kinetic mechanism that gives rise to oscillation in the chemical reaction. The model is represented by a stiff ODE system consisting of 3 equations given by

$$\frac{d\phi}{dt} = f(\phi), \quad \phi(0) = \phi_0, \quad \phi \in \mathcal{R}^3. \tag{4.3}$$

The function $f$ is defined by

$$f(\phi) = \begin{pmatrix} s(\phi_2 - \phi_1\phi_2 + \phi_1 - q\phi_1^2) \\ \frac{1}{s}(-\phi_2 - \phi_1\phi_2 + \phi_3) \\ w(\phi_1 - \phi_3) \end{pmatrix}. \tag{4.4}$$

The problem is solved with the parameters $s = 77.27$, $w = 0.161$ and $q = 8.375 \times 10^{-6}$ and the initial values $(1, 2, 3)^T$. Since the analytic solution does not exist, we take the numerical reference values obtained by ECM5 with fixed step size $h = 2^{-7}$ as the analytic solution. The numerical results with different step size $h = [1/12, 1/24, 1/36]$ are reported in Table 4. The result show the numerical evidences for the theoretical convergence order.

**Table 4.** Numerical comparison of ECM4 and ECM5

| | ECM4 | | ECM5 | |
|---|---|---|---|---|
| $h$ | $Err(h)$ | rate | $Err(h)$ | rate |
| 1/12 | $3.8609 \cdot 10^{-4}$ | – | $4.5158 \cdot 10^{-4}$ | – – |
| 1/24 | $3.2061 \cdot 10^{-7}$ | 10.2339 | $8.4498 \cdot 10^{-8}$ | 12.3838 |
| 1/36 | $4.7661 \cdot 10^{-9}$ | 10.38 | $6.4875 \cdot 10^{-10}$ | 12.0095 |

To examine the efficiency of the proposed scheme, we compare the error as a function of CPU time for ECM scheme using the traditional platform and the proposed platform based on the 4th degree polynomial with Radau5 scheme. Note that we use a vector calculation for parallel computation to get function values in each level, so the CPU time is more suitable for comparison than the number of function calls. The CPU time is taken an average value after executing 100 times to minimize random computer execution factors in the operating system.

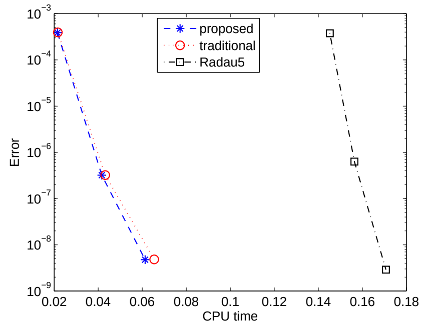


**Figure 2.** Comparison of error as a function of CPU time for traditional method and proposed scheme

For comparison, we march from $t = 0$ to $t_F = 6$ with different time step sizes $h = 1/12, 1/24$ and $1/36$. Fig. 2 shows that regardless of the local platform order, ECM methods are superior to the Radau5 scheme, as shown in Table. 3. Also, it can be seen that the method based on the proposed local platform needs less CPU time and time differences between these schemes are bigger as the number of time intervals is increasing, compared to ECM based on the traditional local platform.

## 5. Conclusion and further discussion

In this paper, we develop a new methodology for construction of local platforms based on the explicit one-step error correction method (ECM) for solving initial value problems. By adding more step function values on the traditional scheme to construct the fourth and fifth local platforms, we can prevent the excess of the number of unknowns and reduce the system size to solve at each level. Numerical simulations show that the proposed scheme is promising and competitive.

Currently, we are generalizing the proposed idea to construction of local platforms having arbitrary order, so that ECMs can have excellent super convergence order. Also, besides polynomial-based local platforms, we are investigating other types of local platforms, such as exponential functions [8]. Progress will be reported soon.

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