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FINITE DIFFERENCE/H^1-GALERKIN MFE PROCEDURE FOR A FRACTIONAL WATER WAVE MODEL*

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Abstract In this article, an H^1-Galerkin mixed finite element (MFE) method for solving the time fractional water wave model is presented. First-order backward Euler difference method and $L1$ formula are applied to approximate integer derivative and Caputo fractional derivative with order $1/2$, respectively, and H^1-Galerkin mixed finite element method is used to approximate the spatial direction. The analysis of stability for fully discrete mixed finite element scheme is made and the optimal space-time orders of convergence for two unknown variables in both H^1-norm and L^2-norm are derived. Further, some computing results for a priori analysis and numerical figures based on four changed parameters in the studied problem are given to illustrate the effectiveness of the current method.

Keywords Time fractional water wave model, H^1-Galerkin MFE method, stability, optimal convergence rate, a priori error estimates.

MSC(2010) 65N30, 65M60, 35Q10.

1. Introduction

A lot of numerical methods and theories on fractional calculus [3,14,16,56] has been concerned by more and more scholars, especially paid attention to the problems of numerical solutions of the fractional partial differential equations (FPDEs). The numerical solutions of FPDEs are mainly obtained by using finite difference methods [1,2,7–9,12,17,27,34,39,40,42,44,45,49,51,53,58], finite element methods [5,13,18–20,23,24,28,32,38,50,54,55], spectral methods [22,26,52], LDG methods [11,46], finite volume (element) methods [6,48] and so forth. In [52], Zhang and Xu studied a spectral methods for a water wave model with a nonlocal viscous term, which is developed by RLW equations [4,10,15,25,31,33,35] with integer order derivatives. In this article, we consider a finite element method to solve the nonlinear time

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fractional water wave model [21, 52]

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}-\beta\frac{\partial^3 u}{\partial x^2\partial t}+\frac{\nu^{\frac{1}{2}}}{\Gamma(1/2)}\int_0^t\frac{\partial u(x,\tau)}{\partial\tau}\frac{d\tau}{(t-\tau)^{\frac{1}{2}}}+\gamma u\frac{\partial u}{\partial x}-\alpha\frac{\partial^2 u}{\partial x^2}=0,(x,t)\in\Omega\times J, \tag{1.1}$$

with boundary condition

$$u(x_L,t)=u(x_R,t)=0,t\in\bar{J}, \tag{1.2}$$

and initial condition

$$u(x,0)=u_0(x),x\in\Omega, \tag{1.3}$$

where $\Omega=[x_L,x_R](\subset R)$ is the spatial domain, $J=(0,T]$ is the time interval with bounded upper bound T. $u_0(x)$ is the initial function, $\alpha>0,\beta>0,\gamma>0,\nu\geq 0$ are some constants.

MFE method coving many types is a class of numerical methods for solving PDEs. In 1998, Pani [36] presented an H^1-Galerkin MFE method for the linear parabolic equations. Compared to classical mixed methods, this method mainly include three characteristics: It does not require the LBB consistency condition; The polynomial degree of the finite element space W_h is not limited by the one of V_h; The optimal error results in L^2 and H^1 norms for both the scalar unknown u and its gradient σ are arrived at. From then on, many authors began to give some numerical analysis on H^1-Galerkin methods for integer order PDEs [15, 30, 31, 37, 41, 43, 57]. But until recently, the method was used to look for the numerical solutions for the linear time-fractional PDEs [29]. However, the applications of the H^1-Galerkin MFE method for solving nonlinear fractional PDEs have not been carried out.

Here, the aim of our article is to discuss the numerical theoretical process and the numerical results of the H^1-Galerkin MFE method for a time fractional water wave model. We formulate a fully discrete scheme, then give the proof of optimal a priori error estimates for the scalar unknown u and the gradient term σ in the L^2-norm, we also obtain the optimal H^1-error results. We show some numerical results to verify our theoretical analysis.

The layout of the paper is as follows. In Section 2, we give an H^1-Galerkin MFE fully discrete scheme and the analysis of stability for a time fractional water wave model (1.1). In Section 3, we derive optimal a priori error results in L^2-and H^1-norms for both u and σ. In Section 4, we confirm our results of theorems by the calculations of some numerical data. In Section 5, we give some analysis of conclusions on H^1-Galerkin MFE method for time fractional water wave model.

2. Discrete scheme and stability

2.1. An H^1-Galerkin MFE formulation

In order to get the H^1-Galerkin mixed formulation, we first split equation (1.1) into the following lower-order system of two equations by introducing an auxiliary variable $\sigma=\frac{\partial u(x,t)}{\partial x}$

$$\begin{cases}(a)\ \dfrac{\partial u}{\partial t}+\sigma-\beta\dfrac{\partial^2\sigma}{\partial x\partial t}+\dfrac{\nu^{\frac{1}{2}}}{\Gamma(1/2)}\displaystyle\int_0^t\frac{\partial u(x,\tau)}{\partial\tau}\frac{d\tau}{(t-\tau)^{\frac{1}{2}}}+\gamma u\sigma-\alpha\dfrac{\partial\sigma}{\partial x},\\[2mm] (b)\ \sigma-\dfrac{\partial u}{\partial x}=0.\end{cases} \tag{2.1}$$

Now we multiply (2.1)(a) and (2.1)(b) by $-\frac{\partial w}{\partial x}, w \in H^1$ and $\frac{\partial v}{\partial x}, v \in H_0^1$, respectively, and integrate with respect to space from x_L to x_R to arrive at the mixed weak formulation for $(u, \sigma) \in H_0^1 \times H^1$

$$\begin{cases} (a)\ \Big(\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}\Big) = \Big(\sigma, \dfrac{\partial v}{\partial x}\Big), \\ (b)\ \Big(\dfrac{\partial \sigma}{\partial t}, w\Big) + \beta\Big(\dfrac{\partial^2 \sigma}{\partial x \partial t}, \dfrac{\partial w}{\partial x}\Big) + \alpha\Big(\dfrac{\partial \sigma}{\partial x}, \dfrac{\partial w}{\partial x}\Big) \\ \quad + \dfrac{\nu^{\frac{1}{2}}}{\Gamma(1/2)}\Big(\displaystyle\int_0^t \dfrac{\partial \sigma(x,\tau)}{\partial \tau}\dfrac{d\tau}{(t-\tau)^{\frac{1}{2}}}, w\Big) = \Big(\sigma, \dfrac{\partial w}{\partial x}\Big) + \gamma\Big(u\sigma, \dfrac{\partial w}{\partial x}\Big). \end{cases} \tag{2.2}$$

Introducing the mixed finite element spaces $V_h \subset H_0^1$ and $W_h \subset H^1$, with the following approximation properties: for $1 \leq p \leq \infty$ and $k,\ r$ positive integers [36]

$$\inf_{v_h \in V_h}\{\|v - v_h\|_{L^p} + h\|v - v_h\|_{W^{1,p}}\} \leq Ch^{k+1}\|v\|_{W^{k+1,p}}, v \in\ H_0^1 \cap W^{k+1,p},$$

$$\inf_{w_h \in W_h}\{\|w - w_h\|_{L^p} + h\|w - w_h\|_{W^{1,p}}\} \leq Ch^{r+1}\|w\|_{W^{r+1,p}}, w \in\ W^{r+1,p}.$$

Based on the finite element spaces $V_h \subset H_0^1$ and $W_h \subset H^1$, the semidiscrete finite element scheme with H^1-Galerkin mixed method is written as

$$\begin{cases} (a)\ \Big(\dfrac{\partial u_h}{\partial x}, \dfrac{\partial v_h}{\partial x}\Big) = \Big(\sigma_h, \dfrac{\partial v_h}{\partial x}\Big), \forall v_h \in V_h, \\ (b)\ \Big(\dfrac{\partial \sigma_h}{\partial t}, w_h\Big) + \beta\Big(\dfrac{\partial^2 \sigma_h}{\partial x \partial t}, \dfrac{\partial w_h}{\partial x}\Big) + \alpha\Big(\dfrac{\partial \sigma_h}{\partial x}, \dfrac{\partial w_h}{\partial x}\Big) \\ \quad + \dfrac{\nu^{\frac{1}{2}}}{\Gamma(1/2)}\Big(\displaystyle\int_0^t \dfrac{\partial \sigma_h(x,\tau)}{\partial \tau}\dfrac{d\tau}{(t-\tau)^{\frac{1}{2}}}, w_h\Big) \\ \quad = \Big(\sigma_h, \dfrac{\partial w_h}{\partial x}\Big) + \gamma\Big(u_h\sigma_h, \dfrac{\partial w_h}{\partial x}\Big), \forall w_h \in W_h. \end{cases} \tag{2.3}$$

Remark 2.1. Compared to standard finite element method, two important variables can be solved simultaneously.

2.2. Stability

For giving the fully discrete finite element scheme with 1/2-order fractional derivative, we need to make the partition of the time interval $[0, T]$. We insert the nodes $t_n = n\Delta t (n = 0, 1, 2, \cdots, M)$ satisfying $0 = t_0 < t_1 < t_2 < \cdots < t_M = T$ with mesh length $\Delta t = T/M$ for some positive integer M. For a smooth function ϕ on $[0, T]$, define $\phi^n = \phi(t_n)$. Now we need to approximate the 1/2-order fractional derivative at time $t = t_{n+1}$ by

$$\begin{aligned} &\frac{1}{\Gamma(1/2)}\int_0^{t_{n+1}} \frac{\partial \sigma(x,\tau)}{\partial \tau}\frac{d\tau}{(t_{n+1}-\tau)^{\frac{1}{2}}} \\ =&\frac{1}{\Gamma(3/2)}\sum_{k=0}^{n}\Big[(k+1)^{1/2} - (k)^{1/2}\Big]\frac{\sigma(t_{n+1-k}) - \sigma(t_{n-k})}{\Delta t^{1/2}} + \epsilon_0^{n+1}, \end{aligned} \tag{2.4}$$

where ϵ_0^{n+1} is the truncation error with the following norm inequality

$$\|\epsilon_0^{n+1}\| \leq C\Delta t^{3/2}. \tag{2.5}$$

Based on the discrete formula (2.4) of time-fractional derivative, we obtain the time semi-discrete scheme of (2.2) with the notation $B_k^{1/2} = (k+1)^{1/2} - (k)^{1/2}$

$$\begin{cases} (a) \ \Big(\dfrac{\partial u^{n+1}}{\partial x}, \dfrac{\partial v}{\partial x}\Big) = \Big(\sigma^{n+1}, \dfrac{\partial v}{\partial x}\Big), \forall v \in H_0^1, \\ (b) \ \Big(\dfrac{\sigma^{n+1} - \sigma^n}{\Delta t}, w\Big) + \beta\Big(\dfrac{\frac{\partial \sigma^{n+1}}{\partial x} - \frac{\partial \sigma^n}{\partial x}}{\Delta t}, \dfrac{\partial w}{\partial x}\Big) + \alpha\Big(\dfrac{\partial \sigma^{n+1}}{\partial x}, \dfrac{\partial w}{\partial x}\Big) \\ \quad + g(\nu)\displaystyle\sum_{k=0}^{n} B_k^{1/2}\Big(\dfrac{\sigma(t_{n+1-k}) - \sigma(t_{n-k})}{\Delta t^{1/2}}, w\Big) = \Big(\sigma^{n+1}, \dfrac{\partial w}{\partial x}\Big) + \gamma\Big(u^n\sigma^{n+1}, \dfrac{\partial w}{\partial x}\Big) \\ \quad + \nu^{\frac{1}{2}}(\epsilon_0^{n+1}, w) + (\epsilon_1^{n+1}, w) + \Big(\epsilon_2^{n+1}, \dfrac{\partial w}{\partial x}\Big), \forall w \in H^1, \end{cases} \tag{2.6}$$

where $g(\nu) = \frac{\nu^{\frac{1}{2}}}{\Gamma(3/2)}$ and the corresponding errors are

$$\epsilon_1^{n+1} = \frac{\sigma^{n+1} - \sigma^n}{\Delta t} - \sigma(t_{n+1}) = O(\Delta t), \tag{2.7}$$

and

$$\epsilon_2^{n+1} = \frac{\frac{\partial \sigma^{n+1}}{\partial x} - \frac{\partial \sigma^n}{\partial x}}{\Delta t} - \frac{\partial^2 \sigma}{\partial x \partial t}(t_{n+1}) + (u^{n+1} - u^n)\sigma^{n+1} = O(\Delta t). \tag{2.8}$$

Now, we formulate a fully discrete procedure: Find $(u_h^{n+1}, \sigma_h^{n+1}) \in V_h \times W_h, (n = 0, 1, \cdots, M-1)$ such that, for any $v_h \in V_h$ and $w_h \in W_h$

$$\begin{cases} (a) \ \Big(\dfrac{\partial u_h^{n+1}}{\partial x}, \dfrac{\partial v_h}{\partial x}\Big) = \Big(\sigma_h^{n+1}, \dfrac{\partial v_h}{\partial x}\Big), \\ (b) \ \Big(\dfrac{\sigma_h^{n+1} - \sigma_h^n}{\Delta t}, w_h\Big) + \beta\Big(\dfrac{\frac{\partial \sigma_h^{n+1}}{\partial x} - \frac{\partial \sigma_h^n}{\partial x}}{\Delta t}, \dfrac{\partial w_h}{\partial x}\Big) + \alpha\Big(\dfrac{\partial \sigma_h^{n+1}}{\partial x}, \dfrac{\partial w_h}{\partial x}\Big) \\ \quad + g(\nu)\displaystyle\sum_{k=0}^{n} B_k^{1/2}\Big(\dfrac{\sigma_h^{n+1-k} - \sigma_h^{n-k}}{\Delta t^{1/2}}, w_h\Big) = \Big(\sigma_h^{n+1}, \dfrac{\partial w_h}{\partial x}\Big) + \gamma\Big(u_h^n\sigma_h^{n+1}, \dfrac{\partial w_h}{\partial x}\Big). \end{cases} \tag{2.9}$$

Remark 2.2. In the scheme (2.9)(b), for the given initial values σ_h^n and u_h^n, the variable σ_h^{n+1} can be solved by iterative process. Further, the variable u_h^{n+1} can be obtained by the value of σ_h^{n+1} in the scheme (2.9)(a).

Now we analyze the stability for scheme (2.9).

Theorem 2.1. *Let the pair $(u_h^n, \sigma_h^n) \in V_h \times W_h$ be the solution of approximation scheme (2.9), then the following inequality of stability holds with a constant $\mathcal{K} > 0$ free of space-time step pair $(h, \Delta t)$*

$$\|u_h^n\|_1^2 + \boldsymbol{\Xi}(\sigma_h^n) \le \exp^{\mathcal{K}T}\Big[\boldsymbol{\Xi}(\sigma_h^0) + \frac{(T\nu)^{1/2}}{\Gamma(3/2)}\|\sigma_h^0\|^2\Big], \tag{2.10}$$

where $\boldsymbol{\Xi}(\sigma_h^n) \triangleq \|\sigma_h^n\|^2 + \beta\Big\|\frac{\partial \sigma_h^n}{\partial x}\Big\|^2 + g(\nu)\Delta t^{1/2}\sum_{k=0}^{n-1} B_k^{1/2}\|\sigma_h^{n-k}\|^2.$

Proof. For $u_h \in V_h \subset H_0^1$, we set $v_h = u_h^{n+1}$ in (2.9) and apply Cauchy-Schwarz inequality as well as Poincaré inequality to follow

$$\|u_h^{n+1}\| \le C\Big\|\frac{\partial u_h^{n+1}}{\partial x}\Big\| \le C\|\sigma_h^{n+1}\|. \tag{2.11}$$

For $\sigma_h, w_h \in W_h \subset H^1$, we take $w_h = \sigma_h^{n+1}$ in (2.9) and note that the inequality $(b-a)b = \frac{1}{2}[b^2 - a^2 + (b-a)^2] \ge \frac{1}{2}[b^2 - a^2]$ to arrive at

$$\begin{aligned}&\frac{\|\sigma_h^{n+1}\|^2 - \|\sigma_h^n\|^2}{2\Delta t} + \frac{\beta\Big\|\frac{\partial \sigma_h^{n+1}}{\partial x}\Big\|^2 - \beta\Big\|\frac{\partial \sigma_h^{n}}{\partial x}\Big\|^2}{2\Delta t} + \alpha\Big\|\frac{\partial \sigma_h^{n+1}}{\partial x}\Big\|^2\\ \le& -g(\nu)\sum_{k=0}^{n} B_k^{1/2}\Big(\frac{\sigma_h^{n+1-k} - \sigma_h^{n-k}}{\Delta t^{1/2}}, \sigma_h^{n+1}\Big) + \Big(\sigma_h^{n+1}, \frac{\partial \sigma_h^{n+1}}{\partial x}\Big) + \gamma\Big(u_h^n\sigma_h^{n+1}, \frac{\partial \sigma_h^{n+1}}{\partial x}\Big).\end{aligned} \tag{2.12}$$

For the next discussion, we need to rewrite the first term on the right hand side of inequality (2.12) as

$$\begin{aligned}&-g(\nu)\sum_{k=0}^{n} B_k^{1/2}\Big(\frac{\sigma_h^{n+1-k} - \sigma_h^{n-k}}{\Delta t^{1/2}}, \sigma_h^{n+1}\Big)\\ =& -g(\nu)\Delta t^{-1/2}\Big[\|\sigma_h^{n+1}\|^2 - \sum_{k=0}^{n-1}(B_k^{1/2} - B_{k+1}^{1/2})(\sigma_h^{n-k}, \sigma_h^{n+1}) - B_n^{1/2}(\sigma_h^0, \sigma_h^{n+1})\Big].\end{aligned} \tag{2.13}$$

Based on the changed formulation (2.13), we multiply (2.12) by $2\Delta t$, use Cauchy-Schwarz inequality and Young inequality and note that the equality $\sum_{k=0}^{n-1}(B_k^{1/2} - B_{k+1}^{1/2}) = B_0^{1/2} - B_n^{1/2}$ to follow

$$\begin{aligned}&\|\sigma_h^{n+1}\|^2 - \|\sigma_h^n\|^2 + \beta\Big(\Big\|\frac{\partial \sigma_h^{n+1}}{\partial x}\Big\|^2 - \Big\|\frac{\partial \sigma_h^{n}}{\partial x}\Big\|^2\Big) + \alpha\Big\|\frac{\partial \sigma_h^{n+1}}{\partial x}\Big\|^2\\ \le& -2g(\nu)\Delta t^{1/2}\|\sigma_h^{n+1}\|^2 + g(\nu)\Delta t^{1/2}\sum_{k=0}^{n-1}(B_k^{1/2} - B_{k+1}^{1/2})(\|\sigma_h^{n-k}\|^2 + \|\sigma_h^{n+1}\|^2)\\ &+ g(\nu)\Delta t^{1/2}B_n^{1/2}(\|\sigma_h^0\|^2 + \|\sigma_h^{n+1}\|^2) + \gamma\Delta t(1 + \|u_h^n\|_\infty^2)\Big(\|\sigma_h^{n+1}\|^2 + \Big\|\frac{\partial \sigma_h^{n+1}}{\partial x}\Big\|^2\Big)\\ =& -g(\nu)\Delta t^{1/2}\Big(2\|\sigma_h^{n+1}\|^2 + \sum_{k=0}^{n-1}B_{k+1}^{1/2}\|\sigma_h^{n-k}\|^2\Big) + g(\nu)\Delta t^{1/2}\sum_{k=0}^{n-1}B_k^{1/2}\|\sigma_h^{n-k}\|^2\\ &+ g(\nu)\Delta t^{1/2}(B_0^{1/2} - B_n^{1/2} + B_n^{1/2})\|\sigma_h^{n+1}\|^2 + g(\nu)\Delta t^{1/2}B_n^{1/2}\|\sigma_h^0\|^2\\ &+ \gamma\Delta t(1 + \|u_h^n\|_\infty^2)\Big(\|\sigma_h^{n+1}\|^2 + \Big\|\frac{\partial \sigma_h^{n+1}}{\partial x}\Big\|^2\Big).\end{aligned} \tag{2.14}$$

Noting that $B_0^{1/2} = 1$, we have

$$\begin{aligned}&\|\sigma_h^{n+1}\|^2 + \beta\Big\|\frac{\partial \sigma_h^{n+1}}{\partial x}\Big\|^2 + [\alpha - \gamma\Delta t(1 + \|u_h^n\|_\infty^2)]\Big\|\frac{\partial \sigma_h^{n+1}}{\partial x}\Big\|^2\\ &+ g(\nu)\Delta t^{1/2}\Big(\|\sigma_h^{n+1}\|^2 + \sum_{k=0}^{n-1}B_{k+1}^{1/2}\|\sigma_h^{n-k}\|^2\Big)\end{aligned}$$

$$
\begin{aligned}
\leq & \|\sigma_h^n\|^2 + \beta\Big\|\frac{\partial \sigma_h^n}{\partial x}\Big\|^2 + g(\nu)\Delta t^{1/2}\sum_{k=0}^{n-1} B_k^{1/2}\|\sigma_h^{n-k}\|^2 \\
& + g(\nu)\Delta t^{1/2} B_n^{1/2}\|\sigma_h^0\|^2 + \Delta t(1+\|u_h^n\|_\infty^2)\|\sigma_h^{n+1}\|^2.
\end{aligned} \tag{2.15}
$$

We easily see that

$$
g(\nu)\Delta t^{1/2}\Big(\|\sigma_h^{n+1}\|^2 + \sum_{k=0}^{n-1} B_{k+1}^{1/2}\|\sigma_h^{n-k}\|^2\Big) = g(\nu)\Delta t^{1/2}\sum_{k=0}^{n} B_k^{1/2}\|\sigma_h^{n+1-k}\|^2. \tag{2.16}
$$

Considering (2.18), we follow

$$
\begin{aligned}
& \mathbf{\Xi}(\sigma_h^{n+1}) + [\alpha - \Delta t(1+\|u_h^n\|_\infty^2)]\Big\|\frac{\partial \sigma_h^{n+1}}{\partial x}\Big\|^2 \\
\leq & \mathbf{\Xi}(\sigma_h^n) + g(\nu)\Delta t^{1/2} B_n^{1/2}\|\sigma_h^0\|^2 + \gamma\Delta t(1+\|u_h^n\|_\infty^2)\|\sigma_h^{n+1}\|^2.
\end{aligned} \tag{2.17}
$$

For sufficiently small Δt, we have $\alpha - \Delta t(1+\|u_h^n\|_\infty^2) > 0$, then remove the second positive term on the left hand side of (2.17) and consider the boundedness for $\|u_h^n\|_\infty^2$ to get

$$
\mathbf{\Xi}(\sigma_h^{n+1}) \leq \mathbf{\Xi}(\sigma_h^n) + g(\nu)\Delta t^{1/2} B_n^{1/2}\|\sigma_h^0\|^2 + \mathcal{K}\Delta t\|\sigma_h^{n+1}\|^2. \tag{2.18}
$$

Noting that $\|\sigma_h^{n+1}\|^2 \leq \mathbf{\Xi}(\sigma_h^{n+1})$, we have

$$
(1-\mathcal{K}\Delta t)\mathbf{\Xi}(\sigma_h^{n+1}) \leq \mathbf{\Xi}(\sigma_h^n) + g(\nu)\Delta t^{1/2} B_n^{1/2}\|\sigma_h^0\|^2. \tag{2.19}
$$

Further, we follow by the iterative process

$$
\begin{aligned}
& \mathbf{\Xi}(\sigma_h^{n+1}) \\
\leq & \frac{1}{(1-\mathcal{K}\Delta t)}\mathbf{\Xi}(\sigma_h^n) + \frac{g(\nu)\Delta t^{1/2} B_n^{1/2}}{(1-\mathcal{K}\Delta t)}\|\sigma_h^0\|^2 \\
\leq & \frac{1}{(1-\mathcal{K}\Delta t)^2}\mathbf{\Xi}(\sigma_h^{n-1}) + \Big[\frac{B_{n-1}^{1/2}}{(1-\mathcal{K}\Delta t)} + B_n^{1/2}\Big]\frac{g(\nu)\Delta t^{1/2}}{(1-\mathcal{K}\Delta t)}\|\sigma_h^0\|^2 \\
\leq & \frac{1}{(1-\mathcal{K}\Delta t)^3}\mathbf{\Xi}(\sigma_h^{n-2}) + \Big[\frac{B_{n-2}^{1/2}}{(1-\mathcal{K}\Delta t)^2} + \frac{B_{n-1}^{1/2}}{(1-\mathcal{K}\Delta t)} + B_n^{1/2}\Big]\frac{g(\nu)\Delta t^{1/2}}{(1-\mathcal{K}\Delta t)}\|\sigma_h^0\|^2 \\
\leq & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\leq & \frac{1}{(1-\mathcal{K}\Delta t)^{n+1}}\mathbf{\Xi}(\sigma_h^0) + \Big[\frac{B_0^{1/2}}{(1-\mathcal{K}\Delta t)^n} + \cdots + \frac{B_{n-1}^{1/2}}{(1-\mathcal{K}\Delta t)} + B_n^{1/2}\Big]\frac{g(\nu)\Delta t^{1/2}}{(1-\mathcal{K}\Delta t)}\|\sigma_h^0\|^2.
\end{aligned} \tag{2.20}
$$

Noting that $(1-\mathcal{K}\Delta t)^j > (1-\mathcal{K}\Delta t)^i > 0 (i > j)$ for sufficiently small Δt and $B_k^{1/2} = (k+1)^{1/2} - (k)^{1/2}$, we have

$$
\begin{aligned}
\mathbf{\Xi}(\sigma_h^{n+1}) \leq & \frac{1}{(1-\mathcal{K}\Delta t)^{n+1}}\mathbf{\Xi}(\sigma_h^0) + \frac{1}{(1-\mathcal{K}\Delta t)^{n+1}}\Big[B_0^{1/2} \\
& + \cdots + B_{n-1}^{1/2} + B_n^{1/2}\Big] g(\nu)\Delta t^{1/2}\|\sigma_h^0\|^2 \\
\leq & \frac{1}{(1-\mathcal{K}\Delta t)^{n+1}}\mathbf{\Xi}(\sigma_h^0) + \frac{1}{(1-\mathcal{K}\Delta t)^{n+1}}(n+1)^{1/2} g(\nu)\Delta t^{1/2}\|\sigma_h^0\|^2.
\end{aligned} \tag{2.21}
$$

Considering the chosen time mesh $\Delta t = T/M$ and noting that $n+1 \leq M$, we follow

$$\Xi(\sigma_h^{n+1}) \leq (1 - \frac{\mathcal{K}T}{M})^{-M}\Big[\Xi(\sigma_h^0) + T^{1/2}g(\nu)\|\sigma_h^0\|^2\Big]. \tag{2.22}$$

In view of the monotonicity of series $(1 - \frac{\mathcal{K}T}{M})^{-M}$ and $\lim_{M\to\infty}(1 - \frac{\mathcal{K}T}{M})^{-M} = \exp^{\mathcal{K}T}$, we easily follow

$$\Xi(\sigma_h^{n+1}) \leq \exp^{\mathcal{K}T}\Big[\Xi(\sigma_h^0) + \frac{(T\nu)^{1/2}}{\Gamma(3/2)}\|\sigma_h^0\|^2\Big]. \tag{2.23}$$

Make a combination for (2.11) and (2.23) to get

$$\|u_h^n\|_1^2 \leq \exp^{\mathcal{K}T}\Big[\Xi(\sigma_h^0) + \frac{(T\nu)^{1/2}}{\Gamma(3/2)}\|\sigma_h^0\|^2\Big]. \tag{2.24}$$

Based on the derived results (2.23) and (2.24), the theorem of stability is proved. □

Remark 2.3. From the conclusion, we can clearly see that $\|u_h^n\|_1$ and $\Xi(\sigma_h^n)$ are only dependent on the norm for initial value σ_0^h, so the mixed element scheme is unconditionally stable.

3. Error estimates and convergence rates

In the next analysis, we start to discuss the optimal a priori error estimates. For a priori error estimates for fully discrete scheme, we introduce two projection operators [36, 47] in Lemma 3.1 and Lemma 3.2.

Lemma 3.1. *For the variable $u \in H_0^1$, there exists an elliptic projection $\mathcal{P}_h u \in V_h$ satisfying, for any $v_h \in V_h$*

$$\Big(\frac{\partial u}{\partial x} - \mathcal{P}_h\frac{\partial u}{\partial x}, \frac{\partial v_h}{\partial x}\Big) = 0, \tag{3.1}$$

which covers the following estimates, for $j = 0, 1$

$$\|u - \mathcal{P}_h u\|_j \leq C_\star h^{k+1-j}\|u\|_{k+1}. \tag{3.2}$$

Lemma 3.2. *Further, for $\sigma \in H^1$, we also define another elliptic projection $\mathcal{Q}_h\sigma \in W_h$ such that, for $w_h \in W_h$*

$$\mathfrak{F}(\sigma - \mathcal{Q}_h\sigma, w_h) = 0, \tag{3.3}$$

where $\mathfrak{F}(\sigma, w) = \Big(\frac{\partial\sigma}{\partial x}, \frac{\partial w}{\partial x}\Big) + \lambda(\sigma, w)$. Here $\lambda > 0$ is taken to satisfy

$$\mathfrak{F}(w, w) \geq \mu_0\|w\|_1^2, w \in H^1, \mu_0 > 0.$$

Then the following estimates are obtained: for $j = 0, 1$

$$\|\sigma - \mathcal{Q}_h\sigma\|_j \leq C_* h^{r+1-j}\|\sigma\|_{r+1}, \ \Big\|\frac{\partial\sigma}{\partial t} - \mathcal{Q}_h\frac{\partial\sigma}{\partial t}\Big\| \leq C_* h^{r+1}\Big\|\frac{\partial\sigma}{\partial t}\Big\|_{r+1}. \tag{3.4}$$

Now we give the following estimate theorem.

Theorem 3.1. *Assuming that $\sigma_h^0 = \mathcal{Q}_h\sigma(0)$, then there exists a constant $C > 0$ free of space-time steps h and Δt satisfying, for $j = 0, 1$*

$$
\begin{aligned}
\|\sigma^J - \sigma_h^J\|_j \leq & C\Big[h^{r+1-j}\|\sigma\|_{L^\infty(H^{r+1})} + h^{r+1}\Big\|\frac{\partial\sigma}{\partial t}\Big\|_{L^2(H^{r+1})} \\
& + g(\nu)\Delta t^{-1/2}h^{r+1}\|\sigma\|_{L^\infty(H^{r+1})} + \Delta t + \nu^{\frac{1}{2}}\Delta t^{3/2} + h^{k+1}\|u\|_{L^\infty(H^{k+1})}\Big], \\
\|u^J - u_h^J\|_j \leq & C\Big[h^{r+1}\Big(\|\sigma\|_{L^\infty(H^{r+1})} + \Big\|\frac{\partial\sigma}{\partial t}\Big\|_{L^2(H^{r+1})}\Big) \\
& + g(\nu)\Delta t^{-1/2}h^{r+1}\|\sigma\|_{L^\infty(H^{r+1})} + \Delta t + \nu^{\frac{1}{2}}\Delta t^{3/2} + h^{k+1-j}\|u\|_{L^\infty(H^{k+1})}\Big].
\end{aligned}
\tag{3.5}
$$

Proof. With the help of the two projection operators $\mathcal{P}_h$ and $\mathcal{Q}_h$, we now split the errors as

$$
\begin{aligned}
u(t_n) - u_h^n &= (u(t_n) - \mathcal{P}_h u^n) + (\mathcal{P}_h u^n - u_h^n) = \hat{E}_u^n + \mathcal{E}_u^n, \\
\sigma(t_n) - \sigma_h^n &= (\sigma(t_n) - \mathcal{Q}_h\sigma^n) + (\mathcal{Q}_h\sigma^n - \sigma_h^n) = \hat{E}_\sigma^n + \mathcal{E}_\sigma^n.
\end{aligned}
$$

Subtracting (2.9) from (2.6) and using two projections (3.1) and (3.3), we get the error equations

$$
\begin{cases}
(a)\ \ \Big(\dfrac{\partial\mathcal{E}_u^{n+1}}{\partial x}, \dfrac{\partial v_h}{\partial x}\Big) = \Big(\hat{E}_\sigma^{n+1}, \dfrac{\partial v_h}{\partial x}\Big) + \Big(\mathcal{E}_\sigma^{n+1}, \dfrac{\partial v_h}{\partial x}\Big), \\
(b)\ \ \Big(\dfrac{\mathcal{E}_\sigma^{n+1} - \mathcal{E}_\sigma^n}{\Delta t}, w_h\Big) + \beta\Big(\dfrac{\frac{\partial\mathcal{E}_\sigma^{n+1}}{\partial x} - \frac{\partial\mathcal{E}_\sigma^n}{\partial x}}{\Delta t}, \dfrac{\partial w_h}{\partial x}\Big) + \alpha\Big(\dfrac{\partial\mathcal{E}_\sigma^{n+1}}{\partial x}, \dfrac{\partial w_h}{\partial x}\Big) \\
\quad + g(\nu)\displaystyle\sum_{k=0}^{n} B_k^{1/2}\Big(\dfrac{\mathcal{E}_\sigma^{n+1-k} - \mathcal{E}_\sigma^{n-k}}{\Delta t^{1/2}}, w_h\Big) = -\Big((1-\beta\lambda)\dfrac{\hat{E}_\sigma^{n+1} - \hat{E}_\sigma^n}{\Delta t} + \alpha\hat{E}_\sigma^{n+1}, w_h\Big) \\
\quad - g(\nu)\displaystyle\sum_{k=0}^{n} B_k^{1/2}\Big(\dfrac{\hat{E}_\sigma^{n+1-k} - \hat{E}_\sigma^{n-k}}{\Delta t^{1/2}}, w_h\Big) + \Big(\mathcal{E}_\sigma^{n+1}, \dfrac{\partial w_h}{\partial x}\Big) \\
\quad + \gamma\Big(u^n\sigma^{n+1} - u_h^n\sigma_h^{n+1}, \dfrac{\partial w_h}{\partial x}\Big) + \nu^{\frac{1}{2}}(\epsilon_0^{n+1}, w_h) + (\epsilon_1^{n+1}, w_h) + \Big(\epsilon_2^{n+1}, \dfrac{\partial w_h}{\partial x}\Big).
\end{cases}
\tag{3.6}
$$

Taking $v_h = \mathcal{E}_u^{n+1}$ in (3.6)(a) and using a similar discussion to (2.11), we arrive at

$$
\|\mathcal{E}_u^{n+1}\| \leq C\Big\|\frac{\partial\mathcal{E}_u^{n+1}}{\partial x}\Big\| \leq C(\|\hat{E}_\sigma^{n+1}\| + \|\mathcal{E}_\sigma^{n+1}\|). \tag{3.7}
$$

Choose $w_h = \mathcal{E}_\sigma^{n+1}$ in (3.6)(b) and note that the inequality $(b-a)b \geq \frac{1}{2}[b^2 - a^2]$ again to follow

$$
\begin{aligned}
& \frac{\|\mathcal{E}_\sigma^{n+1}\|^2 - \|\mathcal{E}_\sigma^n\|^2}{2\Delta t} + \frac{\beta\Big\|\frac{\partial\mathcal{E}_\sigma^{n+1}}{\partial x}\Big\|^2 - \beta\Big\|\frac{\mathcal{E}_\sigma^n}{\partial x}\Big\|^2}{2\Delta t} + \alpha\Big\|\frac{\partial\mathcal{E}_\sigma^{n+1}}{\partial x}\Big\|^2 \\
\leq & -g(\nu)\sum_{k=0}^{n} B_k^{1/2}\Big(\frac{\mathcal{E}_\sigma^{n+1-k} - \mathcal{E}_\sigma^{n-k}}{\Delta t^{1/2}}, \mathcal{E}_\sigma^{n+1}\Big) - \Big((1-\beta\lambda)\frac{\hat{E}_\sigma^{n+1} - \hat{E}_\sigma^n}{\Delta t} + \alpha\hat{E}_\sigma^{n+1}, \mathcal{E}_\sigma^{n+1}\Big) \\
& - g(\nu)\sum_{k=0}^{n} B_k^{1/2}\Big(\frac{\hat{E}_\sigma^{n+1-k} - \hat{E}_\sigma^{n-k}}{\Delta t^{1/2}}, \mathcal{E}_\sigma^{n+1}\Big) + \Big(\mathcal{E}_\sigma^{n+1}, \frac{\partial\mathcal{E}_\sigma^{n+1}}{\partial x}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\gamma\Big(u^n\sigma^{n+1}-u_h^n\sigma_h^{n+1},\frac{\partial\mathcal{E}_\sigma^{n+1}}{\partial x}\Big)+(\nu^{\frac{1}{2}}\epsilon_0^{n+1}+\epsilon_1^{n+1},\mathcal{E}_\sigma^{n+1})+\Big(\epsilon_2^{n+1},\frac{\partial\mathcal{E}_\sigma^{n+1}}{\partial x}\Big)\\
=&I_1+I_2+I_3+I_4+I_5+I_6+I_7. \qquad (3.8)
\end{aligned}
$$

In order to make the next analysis, now we have to give the estimates for $I_i, i = 1, 2, \cdots, 7$. For I_1, we make a simple calculation using Cauchy-Schwarz inequality and Young inequality to yield

$$
\begin{aligned}
I_1\le&-g(\nu)\Delta t^{-1/2}\|\mathcal{E}_\sigma^{n+1}\|^2+\frac{1}{2}g(\nu)\Delta t^{-1/2}\sum_{k=0}^{n-1}(B_k^{1/2}-B_{k+1}^{1/2})(\|\mathcal{E}_\sigma^{n-k}\|^2+\|\mathcal{E}_\sigma^{n+1}\|^2)\\
&+\frac{1}{2}g(\nu)\Delta t^{-1/2}B_n^{1/2}(\|\mathcal{E}_\sigma^{0}\|^2+\|\mathcal{E}_\sigma^{n+1}\|^2)\\
=&\frac{1}{2}g(\nu)\Delta t^{-1/2}\Big[-\|\mathcal{E}_\sigma^{n+1}\|^2-\sum_{k=0}^{n-1}B_{k+1}^{1/2}\|\mathcal{E}_\sigma^{n-k}\|^2+\sum_{k=0}^{n-1}B_k^{1/2}\|\mathcal{E}_\sigma^{n-k}\|^2+B_n^{1/2}\|\mathcal{E}_\sigma^{0}\|^2\Big].
\end{aligned}
\tag{3.9}
$$

For I_2, we use Cauchy-Schwarz inequality and Young inequality to get

$$
\begin{aligned}
I_2\le&\frac{1}{2}\Big(|1-\beta\lambda|^2\Big\|\frac{\hat{E}_\sigma^{n+1}-\hat{E}_\sigma^{n}}{\Delta t}\Big\|^2+\alpha^2\|\hat{E}_\sigma^{n+1}\|^2+2\|\mathcal{E}_\sigma^{n+1}\|^2\Big)\\
\le&\frac{1}{2}\Big(\frac{|1-\beta\lambda|^2}{\Delta t^2}\Big\|\int_{t_n}^{t_{n+1}}\frac{\partial\hat{E}_\sigma}{\partial t}ds\Big\|^2+\alpha^2\|\hat{E}_\sigma^{n+1}\|^2+2\|\mathcal{E}_\sigma^{n+1}\|^2\Big)\\
\le&\frac{1}{2}\Big(\frac{|1-\beta\lambda|^2}{\Delta t}\int_{t_n}^{t_{n+1}}\Big\|\frac{\partial\hat{E}_\sigma}{\partial t}\Big\|^2ds+\alpha^2\|\hat{E}_\sigma^{n+1}\|^2+2\|\mathcal{E}_\sigma^{n+1}\|^2\Big).
\end{aligned}
\tag{3.10}
$$

For I_3, we use the estimate (3.4) combining Cauchy-Schwarz inequality and Young inequality to follow

$$
\begin{aligned}
I_3=&-g(\nu)\Delta t^{-1/2}\Big[(\hat{E}_\sigma^{n+1}-\sum_{k=0}^{n-1}(B_k^{1/2}-B_{k+1}^{1/2})\hat{E}_\sigma^{n-k}-B_n^{1/2}\hat{E}_\sigma^{0},\mathcal{E}_\sigma^{n+1})\Big]\\
\le&g(\nu)\Delta t^{-1/2}\Big\|\hat{E}_\sigma^{n+1}-\sum_{k=0}^{n-1}(B_k^{1/2}-B_{k+1}^{1/2})\hat{E}_\sigma^{n-k}-B_n^{1/2}\hat{E}_\sigma^{0}\Big\|\|\mathcal{E}_\sigma^{n+1}\|\\
\le&Cg(\nu)^2\Delta t^{-1}\Big[(1+\sum_{k=0}^{n-1}(B_k^{1/2}-B_{k+1}^{1/2})+B_n^{1/2}\Big]^2h^{2r+2}\|\sigma\|^2_{L^\infty(H^{r+1})}+\frac{1}{2}\|\mathcal{E}_\sigma^{n+1}\|^2\\
\le&Cg(\nu)^2\Delta t^{-1}h^{2r+2}\|\sigma\|^2_{L^\infty(H^{r+1})}+\frac{1}{2}\|\mathcal{E}_\sigma^{n+1}\|^2.
\end{aligned}
\tag{3.11}
$$

For I_4, I_6, I_7, consider (2.5), (2.7) and (2.8) and apply Cauchy-Schwarz inequality and Young inequality to follow

$$
\begin{aligned}
I_4+I_6+I_7\le&\Big(\mathcal{E}_\sigma^{n+1},\frac{\partial\mathcal{E}_\sigma^{n+1}}{\partial x}\Big)+(\nu^{\frac{1}{2}}\epsilon_0^{n+1}+\epsilon_1^{n+1},\mathcal{E}_\sigma^{n+1})+\Big(\epsilon_2^{n+1},\frac{\partial\mathcal{E}_\sigma^{n+1}}{\partial x}\Big)\\
\le&\frac{1}{2}(\|\epsilon_0^{n+1}+\epsilon_1^{n+1}\|^2+\|\epsilon_2^{n+1}\|^2+\|\mathcal{E}_\sigma^{n+1}\|^2)+\Big\|\frac{\partial\mathcal{E}_\sigma^{n+1}}{\partial x}\Big\|^2\\
\le&C(\Delta t^2+\nu\Delta t^3)+\frac{1}{2}\|\mathcal{E}_\sigma^{n+1}\|^2+\Big\|\frac{\partial\mathcal{E}_\sigma^{n+1}}{\partial x}\Big\|^2.
\end{aligned}
\tag{3.12}
$$

Finally, we give the estimate for I_5. Now we use some important inequalities to get

$$
\begin{aligned}
I_5 =&\gamma\Big(u^n(\hat{E}_\sigma^{n+1}+\mathcal{E}_\sigma^{n+1})+(\hat{E}_u^n+\mathcal{E}_u^n)\sigma_h^{n+1},\frac{\partial\mathcal{E}_\sigma^{n+1}}{\partial x}\Big)\\
\le&\Big[\|u^n\|_\infty(\|\hat{E}_\sigma^{n+1}\|+\|\mathcal{E}_\sigma^{n+1}\|)+(\|\hat{E}_u^n\|+\|\mathcal{E}_u^n)\|\sigma_h^{n+1}\|_\infty\Big]\Big\|\frac{\partial\mathcal{E}_\sigma^{n+1}}{\partial x}\Big\|\\
\le&C\Big[\|\hat{E}_\sigma^{n+1}\|^2+\|\mathcal{E}_\sigma^{n+1}\|^2+\|\hat{E}_u^n\|^2+\|\mathcal{E}_u^n\|^2+\Big\|\frac{\partial\mathcal{E}_\sigma^{n+1}}{\partial x}\Big\|^2\Big]. \qquad (3.13)
\end{aligned}
$$

Now we substitute (3.9)-(3.13) into (3.8) to arrive at

$$
\begin{aligned}
&\frac{\|\mathcal{E}_\sigma^{n+1}\|^2-\|\mathcal{E}_\sigma^n\|^2}{2\Delta t}+\frac{\beta\Big\|\frac{\partial\mathcal{E}_\sigma^{n+1}}{\partial x}\Big\|^2-\beta\Big\|\frac{\mathcal{E}_\sigma^n}{\partial x}\Big\|^2}{2\Delta t}+\alpha\Big\|\frac{\partial\mathcal{E}_\sigma^{n+1}}{\partial x}\Big\|^2\\
\le&\frac{1}{2}g(\nu)\Delta t^{-1/2}\Big[-\|\mathcal{E}_\sigma^{n+1}\|^2-\sum_{k=0}^{n-1}B_{k+1}^{1/2}\|\mathcal{E}_\sigma^{n-k}\|^2+\sum_{k=0}^{n-1}B_k^{1/2}\|\mathcal{E}_\sigma^{n-k}\|^2+B_n^{1/2}\|\mathcal{E}_\sigma^0\|^2\Big]\\
&+\frac{|1-\beta\lambda|^2}{2\Delta t}\int_{t_n}^{t_{n+1}}\Big\|\frac{\partial\hat{E}_\sigma}{\partial t}\Big\|^2ds+Cg(\nu)^2\Delta t^{-1}h^{2r+2}\|\sigma\|^2_{L^\infty(H^{r+1})}+C(\Delta t^2+\Delta t^3)\\
&+C\Big[\|\hat{E}_\sigma^{n+1}\|^2+\|\mathcal{E}_\sigma^{n+1}\|^2+\|\hat{E}_u^n\|^2+\|\mathcal{E}_u^n\|^2+\Big\|\frac{\partial\mathcal{E}_\sigma^{n+1}}{\partial x}\Big\|^2\Big]. \qquad (3.14)
\end{aligned}
$$

Note that

$$
-\|\mathcal{E}_\sigma^{n+1}\|^2-\sum_{k=0}^{n-1}B_{k+1}^{1/2}\|\mathcal{E}_\sigma^{n-k}\|^2=-\sum_{k=0}^{n}B_k^{1/2}\|\mathcal{E}_\sigma^{n+1-k}\|^2, \qquad (3.15)
$$

which combines the resulting inequality (3.14) to yield

$$
\begin{aligned}
&\boldsymbol{\Xi}(\mathcal{E}_\sigma^{n+1})-\boldsymbol{\Xi}(\mathcal{E}_\sigma^n)+2\alpha\Delta t\Big\|\frac{\partial\mathcal{E}_\sigma^{n+1}}{\partial x}\Big\|^2\\
\le&g(\nu)\Delta t^{1/2}B_n^{1/2}\|\mathcal{E}_\sigma^0\|^2+|1-\beta\lambda|^2\int_{t_n}^{t_{n+1}}\Big\|\frac{\partial\hat{E}_\sigma}{\partial t}\Big\|^2ds+Cg(\nu)^2h^{2r+2}\|\sigma\|^2_{L^\infty(H^{r+1})}\\
&+C(\Delta t^3+\Delta t^4)+C\Delta t\Big[\|\hat{E}_\sigma^{n+1}\|^2+\|\mathcal{E}_\sigma^{n+1}\|^2+\|\hat{E}_u^n\|^2+\|\mathcal{E}_u^n\|^2+\Big\|\frac{\partial\mathcal{E}_\sigma^{n+1}}{\partial x}\Big\|^2\Big]. \qquad (3.16)
\end{aligned}
$$

Sum from 0 to $J-1$ for n and note that $\mathcal{E}_\sigma^0=0$ and (3.7) to arrive at

$$
\begin{aligned}
&\boldsymbol{\Xi}(\mathcal{E}_\sigma^J)+2\alpha\Delta t\sum_{n=0}^{J-1}\Big\|\frac{\partial\mathcal{E}_\sigma^{n+1}}{\partial x}\Big\|^2\\
\le&|1-\beta\lambda|^2\int_{t_0}^{t_J}\Big\|\frac{\partial\hat{E}_\sigma}{\partial t}\Big\|^2ds+CJg(\nu)^2h^{2r+2}\|\sigma\|^2_{L^\infty(H^{r+1})}+CJ\Delta t^3\\
&+C\Delta t\sum_{n=0}^{J-1}\Big[\|\hat{E}_\sigma^{n+1}\|^2+\|\mathcal{E}_\sigma^{n+1}\|^2+\|\hat{E}_u^n\|^2+\Big\|\frac{\partial\mathcal{E}_\sigma^{n+1}}{\partial x}\Big\|^2\Big]\\
\le&|1-\beta\lambda|^2\int_{t_0}^{t_J}\Big\|\frac{\partial\hat{E}_\sigma}{\partial t}\Big\|^2ds+CJg(\nu)^2h^{2r+2}\|\sigma\|^2_{L^\infty(H^{r+1})}\\
&+CJ(\Delta t^3+\nu\Delta t^4)+C\Delta t\sum_{n=0}^{J-1}\Big[\|\hat{E}_\sigma^{n+1}\|^2+\|\hat{E}_u^n\|^2+\boldsymbol{\Xi}(\mathcal{E}_\sigma^J)\Big]. \qquad (3.17)
\end{aligned}
$$

Noting that $J \le M = T\Delta t^{-1}$ and using Gronwall lemma, we have

$$\begin{aligned}\Xi(\mathcal{E}_\sigma^J) + 2\alpha\Delta t\sum_{n=0}^{J-1}\left\|\frac{\partial \mathcal{E}_\sigma^{n+1}}{\partial x}\right\|^2 \le& C\int_{t_0}^{t_J}\left\|\frac{\partial \hat{E}_\sigma}{\partial t}\right\|^2 ds + Cg(\nu)^2\Delta t^{-1}h^{2r+2}\|\sigma\|^2_{L^\infty(H^{r+1})}\\ &+ C(\Delta t^2+\nu\Delta t^3) + C\Delta t\sum_{n=0}^{J-1}\Big[\|\hat{E}_\sigma^{n+1}\|^2 + \|\hat{E}_u^n\|^2\Big],\end{aligned} \tag{3.18}$$

which combines the triangle inequality with (3.2) and (3.4) to follow

$$\begin{aligned}\|\sigma^J - \sigma_h^J\|_j^2 \le& C\Big[h^{2r+2-2j}\|\sigma\|^2_{L^\infty(H^{r+1})} + h^{2r+2}\left\|\frac{\partial\sigma}{\partial t}\right\|^2_{L^2(H^{r+1})}\\ &+ g(\nu)^2\Delta t^{-1}h^{2r+2}\|\sigma\|^2_{L^\infty(H^{r+1})} + \Delta t^2 + \nu\Delta t^3 + h^{2k+2}\|u\|^2_{L^\infty(H^{k+1})}\Big].\end{aligned} \tag{3.19}$$

Combines (3.7) and (3.18) with triangle inequality to follow

$$\begin{aligned}&\|u^J - u_h^J\|_j^2\\ \le& C\Big[h^{2r+2}\Big(\|\sigma\|^2_{L^\infty(H^{r+1})} + \left\|\frac{\partial\sigma}{\partial t}\right\|^2_{L^2(H^{r+1})}\Big)\\ &+ g(\nu)^2\Delta t^{-1}h^{2r+2}\|\sigma\|^2_{L^\infty(H^{r+1})} + \Delta t^2 + \nu\Delta t^3 + h^{2k+2-2j}\|u\|^2_{L^\infty(H^{k+1})}\Big].\end{aligned} \tag{3.20}$$

Based on (3.19) and (3.20), we use inequality $a_1^2 + a_2^2 + \cdots + a_i^2 \le (a_1 + a_2 + \cdots + a_i)^2, a_k \ge 0, k = 1, 2, \cdots i$ to obtain the conclusions of theorem 3.1. □

Remark 3.1. (i). It is easy to check from theorem 3.1 that the optimal convergence rate $O(\Delta t + h^{r+1-j} + \nu^{\frac{1}{2}}\Delta t^{-\frac{1}{2}}h^{r+1} + h^{k+1})$ for norms $\|\sigma^J - \sigma_h^J\|_j$, the corresponding optimal convergence result $O(\Delta t + (1+\nu^{\frac{1}{2}}\Delta t^{-\frac{1}{2}})h^{r+1} + h^{k+1-j})$ for norms $\|u^J - u_h^J\|_j$ is also be derived. When $\nu = 0$, the fractional wave problem is changed as integer order RLW-Burgers equation, then theorem 3.1 has the optimal convergence rates $O(\Delta t + h^{r+1} + h^{k+1-j})$ and $O(\Delta t + h^{r+1-j} + h^{k+1})$.

(ii). Here, Compared to standard finite element method, our method can obtain the optimal a error results in H^1-norm for both u and σ.

4. Some numerical results

Now we study some numerical results to verify the results of analysis. In equation (1.1)-(1.3), we choose different parameters α, β, γ, ν and the initial value $u(x, 0) = 3\mathrm{sech}^2[(0.5)^{\frac{3}{2}}(x - x_0)]$, where x_0 is the midpoint of spatial domain $[x_L, x_R]$.

Tables 1-6 show some numerical results of a priori errors for variables u and σ in L^2 and H^1-norms. When taking $T = 10$ and parameters $\alpha = \beta = \gamma = \nu = 1$, we obtained the calculated a priori error results based on $N_h = 2M = 200, 400, 800$ in Table 1 with $[x_L, x_R] = [0, 400]$, which indicate the numerical calculations for long space interval are also effective. Based on the data computed in Tables 2-6, we declare that our method can numerically solved well the fractional water wave model, although some changed parameters α, β, γ, ν are chosen in different forms.

Table 1. Convergence results with $[x_L, x_R] = [0, 400]$, $T = 10$, $\alpha = \beta = \gamma = \nu = 1$

(N_h, M)	$u - L^2$	$\sigma - L^2$	$u - H^1$	$\sigma - H^1$
(200,100)	8.17707E-02	2.72931E-02	8.96280E-02	3.37991E-02
(400,200)	2.71078E-02	8.88547E-03	2.94119E-02	1.11323E-02
(800,400)	1.00551E-02	3.34442E-03	1.06843E-02	4.07641E-03
R_{ATE}	1.59287	1.61902	1.60755	1.60223
R_{ATE}	1.43079	1.40969	1.46091	1.44938

Table 2. Convergence results with $[x_L, x_R] = [0, 40]$, $T = 10$, $\alpha = \beta = \gamma = \nu = 1$

(N_h, M)	$u - L^2$	$\sigma - L^2$	$u - H^1$	$\sigma - H^1$
(100,100)	4.24903E-02	1.40950E-02	4.47335E-02	1.66648E-02
(200,200)	1.85606E-02	6.10288E-03	1.95306E-02	7.21177E-03
(400,400)	6.23459E-03	2.03967E-03	6.55826E-03	2.40859E-03
R_{ATE}	1.19489	1.20762	1.19562	1.20838
R_{ATE}	1.57387	1.58116	1.57435	1.58216

Table 3. Convergence results with $[x_L, x_R] = [0, 40]$, $T = 10$, $\beta = \gamma = \nu = 1$

α	(N_h, M)	$u - L^2$	$\sigma - L^2$	$u - H^1$	$\sigma - H^1$
	(100,100)	1.21504E-01	1.06686E-01	1.62419E-01	1.69733E-01
0.01	(200,200)	5.44929E-02	4.71780E-02	7.24772E-02	7.48241E-02
	(400,400)	1.85495E-02	1.59550E-02	2.45682E-02	2.52097E-02
	R_{ATE}	1.15687	1.17718	1.16412	1.18169
	R_{ATE}	1.55468	1.56411	1.56074	1.56952
	(100,100)	7.91653E-03	1.70890E-03	8.09640E-03	1.79706E-03
5	(200,200)	3.33054E-03	7.37648E-04	3.40419E-03	7.72801E-04
	(400,400)	1.10402E-03	2.47591E-04	1.12920E-03	2.59398E-04
	R_{ATE}	1.24911	1.21206	1.24997	1.21747
	R_{ATE}	1.59299	1.57497	1.59201	1.57493

Table 4. Convergence results with $[x_L, x_R] = [0, 40]$, $T = 10$, $\alpha = \gamma = \nu = 1$

β	(N_h, M)	$u - L^2$	$\sigma - L^2$	$u - H^1$	$\sigma - H^1$
	(100,100)	4.03555E-02	1.14850E-02	4.19229E-02	1.29192E-02
0.01	(200,200)	1.76860E-02	4.99119E-03	1.83676E-02	5.60950E-03
	(400,400)	5.95292E-03	1.67185E-03	6.18130E-03	1.87775E-03
	R_{ATE}	1.19015	1.20229	1.19058	1.20358
	R_{ATE}	1.57094	1.57794	1.57118	1.57887
	(100,100)	5.10672E-02	1.95127E-02	5.46243E-02	2.28009E-02
5	(200,200)	2.19945E-02	8.17889E-03	2.34592E-02	9.50623E-03
	(400,400)	7.33251E-03	2.68959E-03	7.80908E-03	3.11687E-03
	R_{ATE}	1.21525	1.25444	1.21939	1.26214
	R_{ATE}	1.58476	1.60452	1.58693	1.60878

In Figs.1-2, when taking parameters $\alpha = \beta = \gamma = \nu = 1$, we plot the solutions u_h and σ_h, respectively, at time $t = 100, 200, 300, 400, 500$ based on space interval $[x_L, x_R] = [0, 400]$ and time interval $[0, T] = [0, 500]$. It is easy to find that the Fig. 1 of numerical solution u_h in this article is in agreement with the Fig. 3 in

Table 5. Convergence results with $[x_L, x_R] = [0, 40]$, $T = 10$, $\alpha = \beta = \nu = 1$

γ	(N_h, M)	$u - L^2$	$\sigma - L^2$	$u - H^1$	$\sigma - H^1$
0.001	(100,100)	8.04983E-03	3.22055E-03	8.70293E-03	3.63283E-03
	(200,200)	3.16763E-03	1.34479E-03	3.36959E-03	1.50603E-03
	(400,400)	1.06721E-03	4.49895E-04	1.13602E-03	5.04898E-04
	R_{ATE}	1.34555	1.25992	1.36893	1.27034
	R_{ATE}	1.56956	1.57972	1.56859	1.57669
1000	(100,100)	1.23639E-03	6.83785E-04	1.29195E-03	9.72158E-04
	(200,200)	7.52058E-04	3.13071E-04	7.74822E-04	5.24458E-04
	(400,400)	3.42558E-04	1.28767E-04	3.52489E-04	2.47144E-04
	R_{ATE}	0.71722	1.12705	0.73762	0.89036
	R_{ATE}	1.13450	1.28172	1.13629	1.08548

Table 6. Convergence results with $[x_L, x_R] = [0, 40]$, $T = 10$, $\alpha = \beta = \gamma = 1$

ν	(N_h, M)	$u - L^2$	$\sigma - L^2$	$u - H^1$	$\sigma - H^1$
0.01	(100,100)	2.66867E-01	6.74138E-02	2.74944E-01	7.75527E-02
	(200,200)	1.28306E-01	3.17147E-02	1.32020E-01	3.70245E-02
	(400,400)	4.54948E-02	1.10993E-02	4.67725E-02	1.30590E-02
	R_{ATE}	1.05654	1.08789	1.05839	1.06670
	R_{ATE}	1.49581	1.51468	1.49702	1.50344
10	(100,100)	1.37217E-02	5.99427E-03	1.48296E-02	7.89724E-03
	(200,200)	5.62191E-03	2.54766E-03	6.09224E-03	3.32476E-03
	(400,400)	1.85852E-03	8.45216E-04	2.02113E-03	1.09766E-03
	R_{ATE}	1.28732	1.23441	1.28344	1.24810
	R_{ATE}	1.59690	1.59178	1.59181	1.59881

Ref. [52]. The result suggest that the numerical scheme proposed here is effective and feasible.

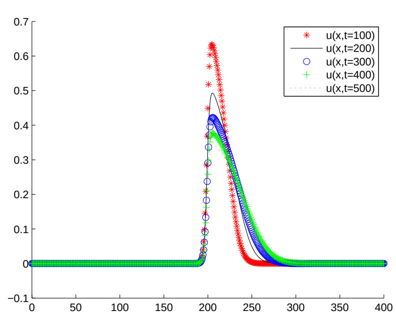

Figure 1. The solutions u_h with temporal evolution for $\alpha = \beta = \gamma = \nu = 1$.

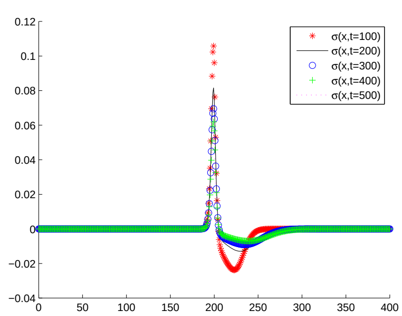

Figure 2. The solutions σ_h with temporal evolution for $\alpha = \beta = \gamma = \nu = 1$.

In Figs. 3-20, we show some numerical figures on variables u_h and σ_h at time $t = 100, 200, 300, 400$ based on space interval $[x_L, x_R] = [0, 100]$, time interval $[0, T] = [0, 400]$, $N_h = 200, M = 400$ and changed parameters $\alpha, \beta, \gamma, \nu$. By the performances of numerical solutions in Figs. 3-20, we can easily see that the studied problem is affected by the terms with parameters $\alpha, \beta, \gamma, \nu$. Figs. 3-8 describe

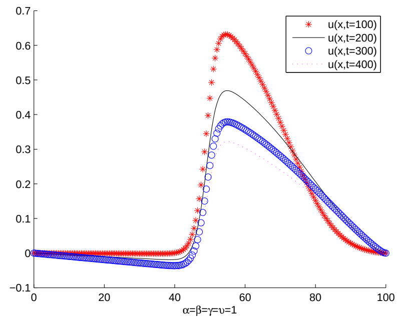

Figure 3. The solutions u_h with temporal evolution for $\alpha = \beta = \gamma = \nu = 1$.

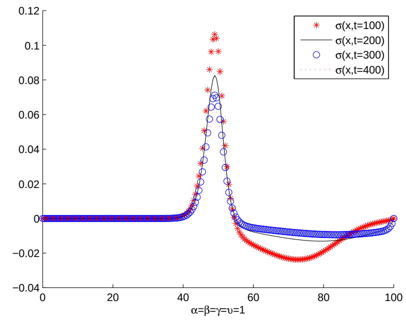

Figure 4. The solutions σ_h with temporal evolution for $\alpha = \beta = \gamma = \nu = 1$.

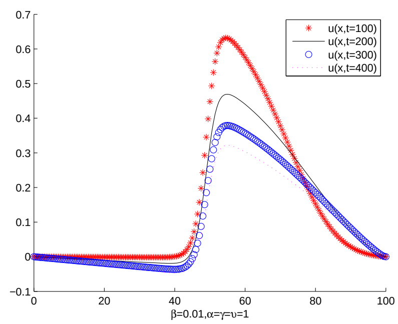

Figure 5. The solutions u_h with temporal evolution for $\beta = 0.01$, $\alpha = \gamma = \nu = 1$.

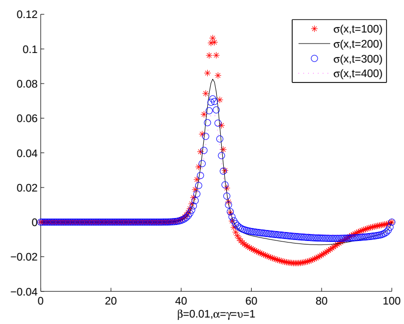

Figure 6. The solutions σ_h with temporal evolution for $\beta = 0.01$, $\alpha = \gamma = \nu = 1$.

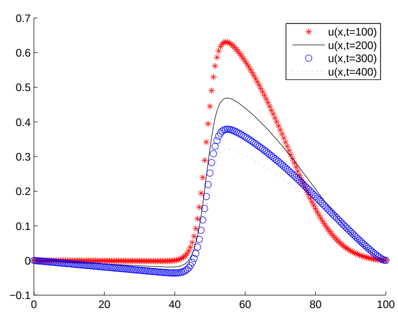

Figure 7. The solutions u_h with temporal evolution for $\beta = 5$, $\alpha = \gamma = \nu = 1$.

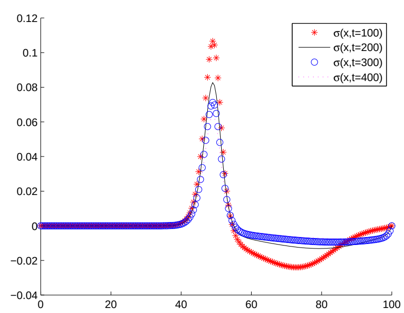

Figure 8. The solutions σ_h with temporal evolution for $\beta = 5$, $\alpha = \gamma = \nu = 1$.

some performances with changed parameter $\beta = 1, 0.01, 5$ and fixed parameters $\alpha = \gamma = \nu = 1$. However, the numerical solutions u_h and σ_h are almost not affected by the changed parameter β. When choosing fixed parameters $\beta = \gamma = \nu = 1$, from Figs. 3-4, 9-12, we can see that the performances of numerical solutions are obviously affected by the changed parameter $\alpha = 1, 0.01, 5$, especially for the case

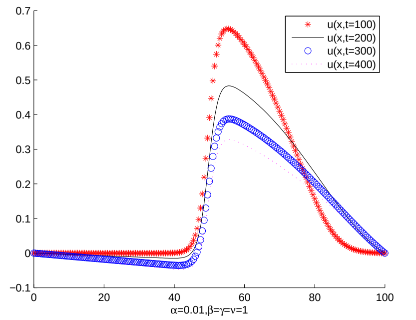

Figure 9. The solutions u_h with temporal evolution for $\alpha = 0.01$, $\beta = \gamma = \nu = 1$.

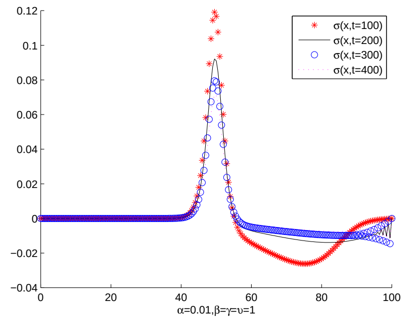

Figure 10. The solutions σ_h with temporal evolution for $\alpha = 0.01$, $\beta = \gamma = \nu = 1$.

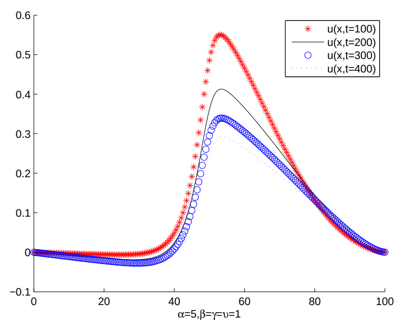

Figure 11. The solutions u_h with temporal evolution for $\alpha = 5$, $\beta = \gamma = \nu = 1$.

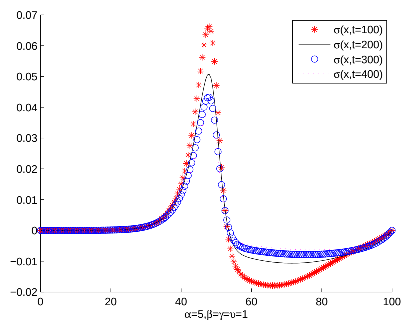

Figure 12. The solutions σ_h with temporal evolution for $\alpha = 5$, $\beta = \gamma = \nu = 1$.

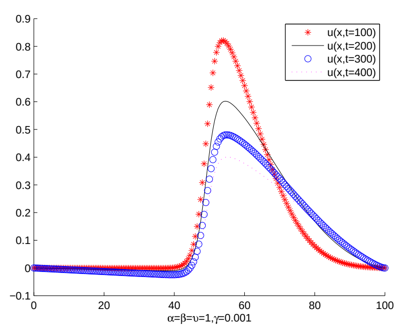

Figure 13. The solutions u_h with temporal evolution for $\gamma = 0.001$, $\alpha = \beta = \nu = 1$.

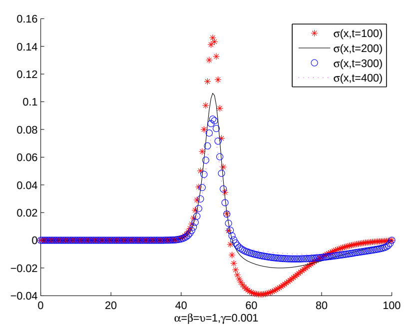

Figure 14. The solutions σ_h with temporal evolution for $\gamma = 0.001$, $\alpha = \beta = \nu = 1$.

$\alpha = 5$. When making some comparisons between Figs. 13-14, Figs. 15-16 and Figs. 3-4, we find that there are obvious changes for the performances for numerical solutions with changed parameters $\gamma = 1, 0.001, 1000$ and fixed parameters $\alpha = \beta = \nu = 1$. Finally, we take changed parameters $\nu = 1, 0.1, 10$ respectively,

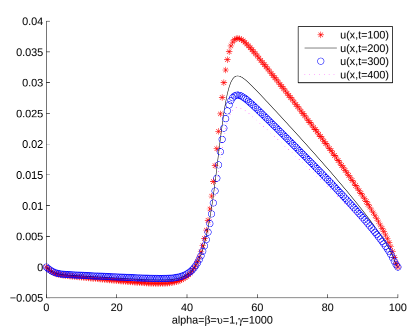

Figure 15. The solutions u_h with temporal evolution for $\gamma = 1000$, $\alpha = \beta = \nu = 1$.

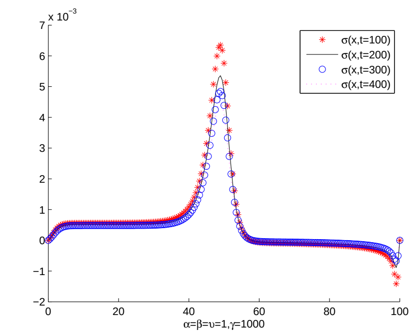

Figure 16. The solutions σ_h with temporal evolution for $\gamma = 1000$, $\alpha = \beta = \nu = 1$.

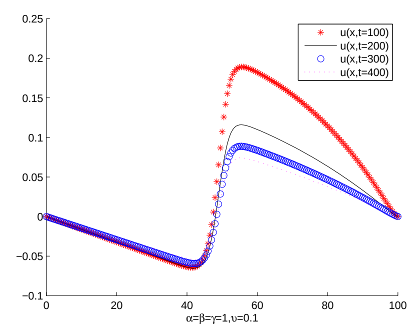

Figure 17. The solutions u_h with temporal evolution for $\nu = 0.1$, $\alpha = \beta = \gamma = 1$.

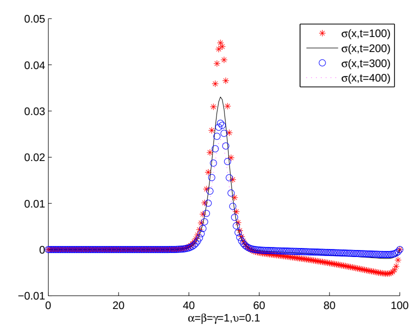

Figure 18. The solutions σ_h with temporal evolution for $\nu = 0.1$, $\alpha = \beta = \gamma = 1$.

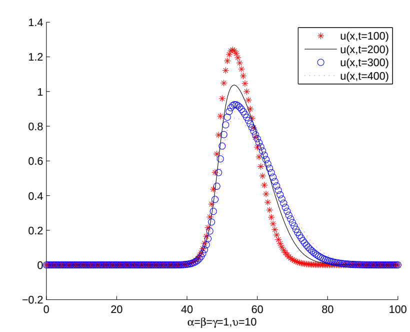

Figure 19. The solutions u_h with temporal evolution for $\nu = 10$, $\alpha = \beta = \gamma = 1$.

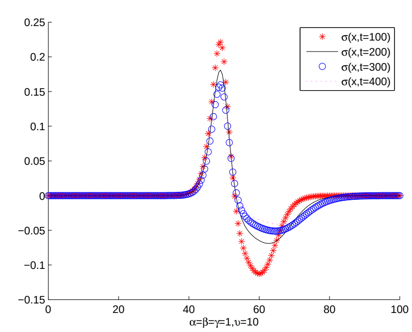

Figure 20. The solutions σ_h with temporal evolution for $\nu = 10$, $\alpha = \beta = \gamma = 1$.

and fixed parameters $\alpha = \beta = \gamma = 1$, then from Figs. 3-4 and Figs. 17-20 we see that the performances for numerical solutions also varies greatly. This shows that the nonlocal viscous term with parameter ν plays an important role in the current problem.

5. Some concluding remarks and extensions

In this article, our aim is to study an H^1-Galerkin MFE method for solving nonlinear time fractional problems. Some optimal convergence results based on a priori estimates in both L^2-norm and H^1-norm are derived. From the numerical results and analysis, ones easily see that an H^1-Galerkin MFE method is successfully applied to solve a water wave model with time fractional derivative.

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