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CONVOLUTION PROPERTIES FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS*

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Abstract The authors introduce two new subclasses of analytic functions. The object of the present paper is to investigate some convolution properties of functions in these subclasses.

Keywords Analytic function, Hadamard product (or convolution), subordinate.

MSC(2010) 30C45, 30C55.

1. Introduction

In this paper we assume that

$$N = \{1, 2, 3, \cdots\}, k \in N \setminus \{1\}, -1 \leq B < A \leq 1, B \leq 0 \text{ and } 0 \leq \lambda \leq 1. \tag{1.1}$$

For two functions f and g analytic in the open unit disk $U = \{z : |z| < 1\}$, the function f is said to be subordinate to g, written $f(z) \prec g(z)$ $(z \in U)$, if there exists an analytic function w in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$.

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.2}$$

which are analytic in U.

Let

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \in \mathcal{A} \quad (j = 1, 2).$$

Then the Hadamard product (or convolution) of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n.$$

We shall require the following lemma in our investigation.

Lemma 1.1. *Let $f \in \mathcal{A}$ defined by* (1.2) *satisfy*

$$\sum_{n=2}^{\infty} [(1 - \lambda + \lambda n)(1 - B) - (1 - A)\delta_{n,k}]|a_n| \leq A - B, \tag{1.3}$$

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where

$$\delta_{n,k} = \begin{cases} 0 \left(\frac{n-1}{k} \notin N\right), \\ 1 \left(\frac{n-1}{k} \in N\right). \end{cases} \tag{1.4}$$

Then

$$\frac{(1-\lambda)f(z) + \lambda z f'(z)}{f_k(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in U), \tag{1.5}$$

where

$$f_k(z) = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{-j} f(\varepsilon_k^j z) \quad \textit{and} \quad \varepsilon_k = \textit{exp}\left(\frac{2\pi i}{k}\right). \tag{1.6}$$

Proof. For $f \in \mathcal{A}$ defined by (1.2), the function $f_k(z)$ in (1.6) can be expressed as

$$f_k(z) = z + \sum_{n=2}^{\infty} \delta_{n,k} a_n z^n, \tag{1.7}$$

with

$$\delta_{n,k} = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{j(n-1)} = \begin{cases} 0 \left(\frac{n-1}{k} \notin N\right), \\ 1 \left(\frac{n-1}{k} \in N\right). \end{cases}$$

In view of (1.1) and (1.4), we see that

$$A\delta_{n,k} - B(1-\lambda+\lambda n) \geq 0 \quad (n \geq 2). \tag{1.8}$$

Assume that the inequality (1.3) holds. Then from (1.7) and (1.8) we deduce that

$$\begin{aligned} \left| \frac{\frac{(1-\lambda)f(z)+\lambda z f'(z)}{f_k(z)} - 1}{A - B\frac{(1-\lambda)f(z)+\lambda z f'(z)}{f_k(z)}} \right| &= \left| \frac{\sum_{n=2}^{\infty}(1-\lambda+\lambda n-\delta_{n,k})a_n z^{n-1}}{A - B + \sum_{n=2}^{\infty}[A\delta_{n,k} - B(1-\lambda+\lambda n)]a_n z^{n-1}} \right| \\ &\leq \frac{\sum_{n=2}^{\infty}(1-\lambda+\lambda n-\delta_{n,k})|a_n|}{A - B - \sum_{n=2}^{\infty}[A\delta_{n,k} - B(1-\lambda+\lambda n)]|a_n|} \\ &\leq 1 \quad (|z| = 1). \end{aligned}$$

Thus, by the maximum modulus theorem, we have (1.5). □

Now we introduce the following two subclasses of $\mathcal{A}$.

Definition 1.1. A function $f \in \mathcal{A}$ defined by (1.2) is said to be in the class $H_k(\lambda, A, B)$ if and only if it satisfies the coefficient inequality (1.3).

It follows from Lemma 1.1 that, if $f \in H_k(\lambda, A, B)$, then the subordination relation (1.5) holds.

Definition 1.2. A function $f \in \mathcal{A}$ defined by (1.2) is said to be in the class $M_k(\lambda, A, B)$ if and only if it satisfies

$$\sum_{n=2}^{\infty} n[(1-\lambda+\lambda n)(1-B) - (1-A)\delta_{n,k}]|a_n| \leq A - B. \tag{1.9}$$

It is clear that for $f \in \mathcal{A}$,

$$f \in M_k(\lambda, A, B) \Longleftrightarrow zf' \in H_k(\lambda, A, B). \tag{1.10}$$

In particular, by taking $\lambda = 1$ and the Lemma 1.1, we see that each function in the classes $H_k(1, A, B)$ and $M_k(1, A, B)$ is starlike with respect to k-symmetric points. Analytic (and meromorphic) functions which are starlike with respect to symmetric points and related functions have been extensively studied by several authors (see, e.g., [1] to [5], [8] to [10] and [13] to [25]; see also the recent works [8, 18, 23, 24]).

In the present paper, we derive several convolution properties for each of the above-defined classes $H_k(\lambda, A, B)$, $M_k(\lambda, A, B)$. Our results are motivated by a number of recent works (see, for example, [1] to [25]).

2. Convolution properties

In this section we assume that

$$-1 \leq B_j < A_j \leq 1 \text{ and } B_j \leq 0 \quad (j = 1, 2). \tag{2.1}$$

Theorem 2.1. *Let $f_j \in H_k(\lambda, A_j, B_j)$ $(j = 1, 2)$.*

(i) If

$$(1-B_1)(A_2-B_2)+(1-B_2)(A_1-B_1) \geq (1-B_1)(1-B_2) \text{ and } 0 \leq \lambda \leq 1, \tag{2.2}$$

then $f_1 * f_2 \in H_k(\lambda, A(B), B)$, where

$$A(B) = B + \frac{1-B}{1+\lambda} \prod_{j=1}^{2} \frac{A_j - B_j}{1 - B_j}, \tag{2.3}$$

and for each B the number $A(B)$ cannot be replaced by a smaller one.

(ii) If

$$(1-B_1)(A_2-B_2)+(1-B_2)(A_1-B_1) < (1-B_1)(1-B_2) \text{ and } \lambda_1 \leq \lambda \leq 1, \tag{2.4}$$

where

$$\lambda_1 = \frac{(1-B_1)(1-B_2) - [(1-B_1)(A_2-B_2) + (1-B_2)(A_1-B_1)]}{(k-1)(1-B_1)(1-B_2)} \in (0, 1), \tag{2.5}$$

then $f_1 * f_2 \in H_k(\lambda, A(B), B)$ and for each B the number $A(B)$ cannot be replaced by a smaller one.

(iii) If

$$(1-B_1)(A_2-B_2)+(1-B_2)(A_1-B_1) < (1-B_1)(1-B_2) \text{ and } 0 \leq \lambda < \lambda_1, \tag{2.6}$$

then $f_1 * f_2 \in H_k(\lambda, \widetilde{A}(B), B)$, where

$$\widetilde{A}(B) = B + \frac{1-B}{\lambda k \prod_{j=1}^{2} \frac{1-B_j}{A_j - B_j} + \sum_{j=1}^{2} \frac{1-B_j}{A_j - B_j}}, \tag{2.7}$$

and for each B the number $\widetilde{A}(B)$ cannot be replaced by a smaller one.

Proof. It is obvious that $B < A(B) < 1$ and $B < \widetilde{A}(B) < 1$. Let

$$f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \in H_k(\lambda, A_j, B_j) \quad (j = 1, 2).$$

Then

$$\begin{aligned} &\sum_{n=2}^{\infty} \left\{ \prod_{j=1}^{2} \frac{(1-\lambda+\lambda n)(1-B_j) - (1-A_j)\delta_{n,k}}{A_j - B_j} \right\} |a_{n,1} a_{n,2}| \\ &\le \prod_{j=1}^{2} \left\{ \sum_{n=2}^{\infty} \frac{(1-\lambda+\lambda n)(1-B_j) - (1-A_j)\delta_{n,k}}{A_j - B_j} |a_{n,j}| \right\} \le 1. \end{aligned} \tag{2.8}$$

Also, $f_1 * f_2 \in H_k(\lambda, A, B)$ if and only if

$$\sum_{n=2}^{\infty} \frac{(1-\lambda+\lambda n)(1-B) - (1-A)\delta_{n,k}}{A - B} |a_{n,1} a_{n,2}| \le 1. \tag{2.9}$$

In order to prove Theorem 2.1, it follows from (2.8) and (2.9) that we need only to find the smallest A such that

$$\frac{(1-\lambda+\lambda n)(1-B) - (1-A)\delta_{n,k}}{A - B} \le \prod_{j=1}^{2} \frac{(1-\lambda+\lambda n)(1-B_j) - (1-A_j)\delta_{n,k}}{A_j - B_j} \tag{2.10}$$

for all $n \ge 2$, that is, that

$$A \ge B + \frac{(1-B)(1-\lambda+\lambda n - \delta_{n,k})}{\prod_{j=1}^{2} \left\{ \frac{(1-\lambda+\lambda n - \delta_{n,k})(1-B_j)}{A_j - B_j} + \delta_{n,k} \right\} - \delta_{n,k}} \quad (n \ge 2). \tag{2.11}$$

For $n \ge 2$ and $\frac{n-1}{k} \in N$, we have $\delta_{n,k} = 1$, $n = 1 + mk (m \in N)$ and

$$A \ge B + \frac{1-B}{\lambda(n-1) \prod_{j=1}^{2} \frac{1-B_j}{A_j - B_j} + \sum_{j=1}^{2} \frac{1-B_j}{A_j - B_j}} = \varphi(n). \tag{2.12}$$

The function $\varphi(n)$ is decreasing in n and hence

$$\varphi(n) \le \varphi(1+k) = B + \frac{1-B}{\lambda k \prod_{j=1}^{2} \frac{1-B_j}{A_j - B_j} + \sum_{j=1}^{2} \frac{1-B_j}{A_j - B_j}}. \tag{2.13}$$

For $n \ge 2$ and $\frac{n-1}{k} \notin N$, we have $\delta_{n,k} = 0$, $\delta_{1+m,k} = 0$ $(1 \le m \le k-1)$ and

$$A \ge B + \frac{1-B}{(1-\lambda+\lambda n) \prod_{j=1}^{2} \frac{1-B_j}{A_j - B_j}} = \psi(n). \tag{2.14}$$

It is clear that

$$\psi(n) \le \psi(2) = B + \frac{1-B}{(1+\lambda) \prod_{j=1}^{2} \frac{1-B_j}{A_j - B_j}}. \tag{2.15}$$

From (2.13) and (2.15), we have

$$\begin{aligned}
&\lambda k \prod_{j=1}^{2} \frac{1-B_j}{A_j-B_j} + \sum_{j=1}^{2} \frac{1-B_j}{A_j-B_j} - (1+\lambda) \prod_{j=1}^{2} \frac{1-B_j}{A_j-B_j} \\
&= \frac{h(\lambda)}{(A_1-B_1)(A_2-B_2)},
\end{aligned} \tag{2.16}$$

where

$$h(\lambda) = (\lambda k - \lambda - 1)(1-B_1)(1-B_2) + (1-B_1)(A_2-B_2) + (1-B_2)(A_1-B_1). \tag{2.17}$$

Note that $h(1) > 0$ and

$$h(0) = (1-B_1)(A_2-B_2) + (1-B_2)(A_1-B_1) - (1-B_1)(1-B_2). \tag{2.18}$$

If either (2.2) or (2.4) is satisfied, then it follows from (2.10) to (2.18) that $h(\lambda) \geq 0$, $\varphi(1+k) \leq \psi(2)$ and $f_1 * f_2 \in H_k(\lambda, A(B), B)$.

Furthermore, for $B < A_0 < A(B)$, we have

$$\frac{(1+\lambda)(1-B)}{A_0-B} \prod_{j=1}^{2} \frac{A_j-B_j}{(1+\lambda)(1-B_j)} > \frac{(1+\lambda)(1-B)}{A(B)-B} \prod_{j=1}^{2} \frac{A_j-B_j}{(1+\lambda)(1-B_j)} = 1.$$

Therefore the functions

$$f_j(z) = z - \frac{A_j-B_j}{(1+\lambda)(1-B_j)} z^2 \in H_k(\lambda, A_j, B_j) \quad (j = 1, 2),$$

show that $f_1 * f_2 \notin H_k(\lambda, A_0, B)$. This proves (i) and (ii).

(iii) If (2.6) is satisfied, then we have $h(\lambda) < 0$ $(0 \leq \lambda < \lambda_1)$, $\psi(2) < \varphi(1+k)$ and $f_1 * f_2 \in H_k(\lambda, \widetilde{A}(B), B)$. Furthermore, the number $\widetilde{A}(B)$ cannot be replaced by a smaller one as can be seen from the functions

$$f_j(z) = z - \frac{A_j-B_j}{(1+\lambda k)(1-B_j)-(1-A_j)} z^{1+k} \in H_k(\lambda, A_j, B_j) \quad (j = 1, 2).$$

The proof of Theorem 2.1 is completed. □

Theorem 2.2. *Let $f_1 \in H_k(\lambda, A_1, B_1)$, $f_2 \in M_k(\lambda, A_2, B_2)$ and let $A(B), \widetilde{A}(B), \lambda_1$ be given as in Theorem 2.1.*

(i) If $(1-B_1)(A_2-B_2) + (1-B_2)(A_1-B_1) \geq (1-B_1)(1-B_2)$ and $0 \leq \lambda \leq 1$, then $f_1 * f_2 \in M_k(\lambda, A(B), B)$ and for each B the number $A(B)$ cannot be replaced by a smaller one.

(ii) If $(1-B_1)(A_2-B_2) + (1-B_2)(A_1-B_1) < (1-B_1)(1-B_2)$ and $\lambda_1 \leq \lambda \leq 1$, then $f_1 * f_2 \in M_k(\lambda, A(B), B)$ and for each B the number $A(B)$ cannot be replaced by a smaller one.

(iii) If $(1-B_1)(A_2-B_2) + (1-B_2)(A_1-B_1) < (1-B_1)(1-B_2)$ and $0 \leq \lambda < \lambda_1$, then $f_1 * f_2 \in M_k(\lambda, \widetilde{A}(B), B)$ and for each B the number $\widetilde{A}(B)$ cannot be replaced by a smaller one.

Proof. Since

$$f_1 \in H_k(\lambda, A_1, B_1), \quad zf_2' \in H_k(\lambda, A_2, B_2) \quad (see (1.10)),$$

and

$$f_1(z) * zf_2'(z) = z(f_1 * f_2)'(z) \quad (z \in U),$$

the assertion of the theorem follows from Theorem 2.1. □

Theorem 2.3. *Let $f_1 \in H_k(\lambda, A_1, B_1)$ and $f_2 \in M_k(\lambda, A_2, B_2)$.*

(i) If the equation

$$h_1(\lambda) = a_1\lambda^2 + b_1\lambda + c_1 = 0 \tag{2.19}$$

has no root in (0,1), where

$$\begin{cases} a_1 = (k^2 + k - 2)(1 - B_1)(1 - B_2), \\ b_1 = (1 + k)[(1 - B_1)(A_2 - B_2) + (1 - B_2)(A_1 - B_1)] - 2(1 - B_1)(1 - B_2), \\ c_1 = (A_1 - B_1)(A_2 - B_2), \end{cases} \tag{2.20}$$

then $f_1 * f_2 \in H_k(\lambda, A_1(B), B)$, where

$$A_1(B) = B + \frac{1 - B}{2(1 + \lambda)} \prod_{j=1}^{2} \frac{A_j - B_j}{1 - B_j}, \tag{2.21}$$

and for each B the number $A_1(B)$ cannot be replaced by a smaller one.

(ii) If the equation (2.19) has two roots λ_1, λ_2 $(\lambda_1 \leq \lambda_2)$ in $(0, 1)$ and $0 \leq \lambda \leq \lambda_1$ or $\lambda_2 \leq \lambda \leq 1$, then $f_1 * f_2 \in H_k(\lambda, A_1(B), B)$ and for each B the number $A_1(B)$ cannot be replaced by a smaller one.

(iii) If the equation (2.19) has two roots λ_1, λ_2 $(\lambda_1 < \lambda_2)$ in $(0, 1)$ and $\lambda_1 < \lambda < \lambda_2$,then $f_1 * f_2 \in H_k(\lambda, \widetilde{A}_1(B), B)$, where

$$\widetilde{A_1}(B) = B + \frac{1 - B}{\lambda k(1 + k) \prod_{j=1}^{2} \frac{1-B_j}{A_j-B_j} + (1 + k) \sum_{j=1}^{2} \frac{1-B_j}{A_j-B_j} + \frac{1}{\lambda}}, \tag{2.22}$$

and for each B the number $\widetilde{A_1}(B)$ cannot be replaced by a smaller one.

Proof. Clearly $B < A_1(B) < 1$ and $B < \widetilde{A_1}(B) < 1$. In order to prove Theorem 2.3, we need only to find the smallest A such that

$$\frac{(1 - \lambda + \lambda n)(1 - B) - (1 - A)\delta_{n,k}}{A - B} \leq n \prod_{j=1}^{2} \frac{(1 - \lambda + \lambda n)(1 - B_j) - (1 - A_j)\delta_{n,k}}{A_j - B_j}, \tag{2.23}$$

for all $n \geq 2$, that is, that

$$A \geq B + \frac{(1 - \lambda + \lambda n - \delta_{n,k})(1 - B)}{n \prod_{j=1}^{2} \left\{ \frac{(1-\lambda+\lambda n-\delta_{n,k})(1-B_j)}{A_j-B_j} + \delta_{n,k} \right\} - \delta_{n,k}} \quad (n \geq 2). \tag{2.24}$$

For $n \geq 2$ and $\frac{n-1}{k} \in N$, (2.24) reduces to

$$A \geq B + \frac{1-B}{\lambda n(n-1)\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + n\sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + \frac{1}{\lambda}} = \varphi_1(n). \tag{2.25}$$

The function $\varphi_1(n)$ is decreasing in n and hence

$$\begin{aligned}\varphi_1(n) &\leq \varphi_1(1+k)\\ &= B + \frac{1-B}{\lambda k(1+k)\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + (1+k)\sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + \frac{1}{\lambda}}\\ &= \widetilde{A_1}(B). \end{aligned} \tag{2.26}$$

For $n \geq 2$ and $\frac{n-1}{k} \notin N$, (2.24) becomes

$$A \geq B + \frac{1-B}{n(1-\lambda+\lambda n)\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}} = \psi_1(n) \tag{2.27}$$

and we have

$$\psi_1(n) \leq \psi_1(2) = B + \frac{1-B}{2(1+\lambda)\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}} = A_1(B). \tag{2.28}$$

Now

$$\begin{aligned}&\lambda k(1+k)\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + (1+k)\sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + \frac{1}{\lambda} - 2(1+\lambda)\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}\\ &= \frac{h_1(\lambda)}{\lambda(A_1-B_1)(A_2-B_2)},\end{aligned} \tag{2.29}$$

where

$$\begin{aligned}h_1(\lambda) &= \lambda^2 k(1+k)(1-B_1)(1-B_2) + \lambda(1+k)[(1-B_1)(A_2-B_2)\\ &\quad + (1-B_2)(A_1-B_1)] + (A_1-B_1)(A_2-B_2) - 2\lambda(1+\lambda)(1-B_1)(1-B_2)\\ &= a_1\lambda^2 + b_1\lambda + c_1,\end{aligned} \tag{2.30}$$

and a_1, b_1, c_1 are given by (2.20). Note that $a_1 > 0$, $h_1(0) = c_1 > 0$ and $h_1(1) = (k^2+k-4)(1-B_1)(1-B_2)+(1+k)[(1-B_1)(A_2-B_2)+(1-B_2)(A_1-B_1)]+(A_1-B_1)(A_2-B_2) > 0$ (see (1.1)). Therefore, the equation $h_1(\lambda) = a_1\lambda^2 + b_1\lambda + c_1 = 0$ has no root in $(0,1)$ or has two roots in $(0,1)$.

The remaining part of the proof is similar to that as in Theorem 2.1 and hance we omit it.

Furthermore, the number $A_1(B)$ is best possible for the functions

$$\begin{aligned}f_1(z) &= z - \frac{A_1-B_1}{(1+\lambda)(1-B_1)}z^2 \in H_k(\lambda, A_1, B_1),\\ f_2(z) &= z - \frac{A_2-B_2}{2(1+\lambda)(1-B_2)}z^2 \in M_k(\lambda, A_2, B_2),\end{aligned}$$

and the number $\widetilde{A_1}(B)$ is best possible for the functions

$$f_1(z) = z - \frac{A_1 - B_1}{(1+\lambda k)(1-B_1)-(1-A_1)} z^{k+1} \in H_k(\lambda, A_1, B_1),$$
$$f_2(z) = z - \frac{A_2 - B_2}{(1+k)[(1+\lambda k)(1-B_2)-(1-A_2)]} z^{k+1} \in M_k(\lambda, A_2, B_2).$$

The proof of the theorem is completed. □

From Theorem 2.3 we have the following theorem immediately.

Theorem 2.4. *Let $f_j \in M_k(\lambda, A_j, B_j)$ $(j = 1, 2)$ and let $A_1(B), \widetilde{A_1}(B), h_1(\lambda)$ be given as in Theorem 2.3.*

(i) If the equation (2.19) has no roots in $(0,1)$, then $f_1 * f_2 \in M_k(\lambda, A_1(B), B)$ and for each B the number $A_1(B)$ cannot be replaced by a smaller one.

(ii) If the equation (2.19) has two roots λ_1, λ_2 $(\lambda_1 \le \lambda_2)$ in $(0,1)$ and $0 \le \lambda \le \lambda_1$ or $\lambda_2 \le \lambda \le 1$, then $f_1 * f_2 \in M_k(\lambda, A_1(B), B)$ and for each B the number $A_1(B)$ cannot be replaced by a smaller one.

(iii) If the equation (2.19) has two roots λ_1, λ_2 $(\lambda_1 < \lambda_2)$ in $(0,1)$ and $\lambda_1 < \lambda < \lambda_2$, then $f_1 * f_2 \in M_k(\lambda, \widetilde{A}_1(B), B)$ and for each B the number $\widetilde{A_1}(B)$ cannot be replaced by a smaller one.

Theorem 2.5. *Let $f_j \in M_k(\lambda, A_j, B_j)$ $(j = 1, 2)$.*

(i) If the equation

$$h_2(\lambda) = a_2\lambda^2 + b_2\lambda + c_2, \tag{2.31}$$

has no root in (0,1), where

$$\begin{cases} a_2 = [k(1+k)^2 - 4](1-B_1)(1-B_2), \\ b_2 = (1+k)^2[(1-B_1)(A_2-B_2) + (1-B_2)(A_1-B_1)] - 4(1-B_1)(1-B_2), \\ c_2 = (2+k)(A_1-B_1)(A_2-B_2), \end{cases} \tag{2.32}$$

then $f_1 * f_2 \in H_k(\lambda, A_2(B), B)$, where

$$A_2(B) = B + \frac{1-B}{4(1+\lambda)} \prod_{j=1}^{2} \frac{A_j - B_j}{1-B_j},$$

and for each B the number $A_2(B)$ cannot be replaced by a smaller one.

(ii) If the equation (2.31) has two roots λ_1, λ_2 $(\lambda_1 \le \lambda_2)$ in $(0,1)$ and $0 \le \lambda \le \lambda_1$ or $\lambda_2 \le \lambda \le 1$, then $f_1 * f_2 \in H_k(\lambda, A_2(B), B)$ and for each B the number $A_2(B)$ cannot be replaced by a smaller one.

(iii) If the equation (2.31) has two roots λ_1, λ_2 $(\lambda_1 < \lambda_2)$ in $(0,1)$ and $\lambda_1 < \lambda < \lambda_2$,then $f_1 * f_2 \in H_k(\lambda, \widetilde{A}_2(B), B)$, where

$$\widetilde{A_2}(B) = B + \frac{1-B}{\lambda k(1+k)^2 \prod_{j=1}^{2} \frac{1-B_j}{A_j-B_j} + (1+k)^2 \sum_{j=1}^{2} \frac{1-B_j}{A_j-B_j} + \frac{2+k}{\lambda}} \ ,$$

and for each B the number $\widetilde{A_2}(B)$ cannot be replaced by a smaller one.

Proof. We have $B < A_2(B) < 1$ and $B < \widetilde{A_2}(B) < 1$. In order to prove Theorem 2.5, we need only to find the smallest A such that

$$\begin{aligned}&\frac{(1-\lambda+\lambda n)(1-B)-(1-A)\delta_{n,k}}{A-B}\\ &\leq n^2\prod_{j=1}^{2}\frac{(1-\lambda+\lambda n)(1-B_j)-(1-A_j)\delta_{n,k}}{A_j-B_j}\quad (n\geq 2).\end{aligned}\tag{2.33}$$

For $n \geq 2$ and $\frac{n-1}{k} \in N$, (2.33) becomes

$$A \geq B + \frac{1-B}{\lambda n^2(n-1)\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + n^2\sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + \frac{n+1}{\lambda}} = \varphi_2(n).$$

The function $\varphi_2(n)$ is decreasing in n and

$$\begin{aligned}\varphi_2(n) \leq \varphi_2(1+k) &= B + \frac{1-B}{\lambda k(1+k)^2\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + (1+k)^2\sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + \frac{2+k}{\lambda}}\\ &= \widetilde{A_2}(B).\end{aligned}$$

For $n \geq 2$ and $\frac{n-1}{k} \notin N$, (2.33) reduces to

$$A \geq B + \frac{1-B}{n^2(1-\lambda+\lambda n)\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}} = \psi_2(n),$$

and

$$\psi_2(n) \leq \psi_2(2) = B + \frac{1-B}{4(1+\lambda)\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}} = A_2(B).$$

Now

$$\begin{aligned}&\lambda k(1+k)^2\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + (1+k)^2\sum_{j=1}^{2}\frac{1-B_j}{A_j-B_j} + \frac{2+k}{\lambda} - 4(1+\lambda)\prod_{j=1}^{2}\frac{1-B_j}{A_j-B_j}\\ &= \frac{h_2(\lambda)}{\lambda(A_1-B_1)(A_2-B_2)},\end{aligned}$$

where

$$\begin{aligned}h_2(\lambda) &= \lambda^2 k(1+k)^2(1-B_1)(1-B_2) + \lambda(1+k)^2[(1-B_1)(A_2-B_2)\\ &\quad + (1-B_2)(A_1-B_1)] + (2+k)(A_1-B_1)(A_2-B_2) - 4\lambda(1+\lambda)\\ &\quad \cdot (1-B_1)(1-B_2) = a_2\lambda^2 + b_2\lambda + c_2,\end{aligned}$$

and a_2, b_2, c_2 are given by (2.32). Note that $a_2 > 0$, $h_2(0) = c_2 > 0$ and $h_2(1) = [k(1+k)^2-8](1-B_1)(1-B_2) + (1+k)^2[(1-B_1)(A_2-B_2) + (1-B_2)(A_1-B_1)] + (2+k)(A_1-B_1)(A_2-B_2) > 0$ (see (1.1)).

The remaining part of the proof is similar to that as in Theorem 2.1 and hence we omit it. $\square$

Theorem 2.6. *Let $f \in H_k(\lambda, A, B)$. Then*

$$(f * h_\sigma)(z) \neq 0 \quad (z \in U \setminus \{0\};\ \sigma \in C,\ |\sigma| = 1), \tag{2.34}$$

where

$$h_\sigma(z) = z - \frac{1+B\sigma}{1+A\sigma}\left[\frac{(1-\lambda)z}{1-z} + \frac{\lambda z}{(1-z)^2}\right] + \frac{z^{1+k}}{1-z^k}.$$

Proof. For $f \in H_k(\lambda, A, B)$, we have (1.4), which is equivalent to

$$\frac{(1-\lambda)f(z) + \lambda z f'(z)}{f_k(z)} \neq \frac{1+A\sigma}{1+B\sigma} \quad (z \in U;\ \sigma \in C,\ |\sigma| = 1, 1 + B\sigma \neq 0), \tag{2.35}$$

or to

$$(1+B\sigma)[(1-\lambda)f(z) + \lambda z f'(z)] - (1+A\sigma)f_k(z) \neq 0 \quad (z \in U \setminus \{0\}; \sigma \in C, |\sigma| = 1). \tag{2.36}$$

Note that

$$\begin{aligned} zf'(z) &= f(z) * (z + \sum_{n=2}^{\infty} n z^n) \\ &= f(z) * \frac{z}{(1-z)^2}. \end{aligned} \tag{2.37}$$

If we put

$$f_k(z) = f(z) * (z + g(z)), \tag{2.38}$$

then

$$g(z) = \sum_{n=2}^{\infty} \delta_{n,k} z^n = \sum_{m=1}^{\infty} z^{1+mk} = \frac{z^{1+k}}{1-z^k}. \tag{2.39}$$

Making use of (2.35)-(2.39), we have

$$f(z) * \left\{(1+B\sigma)\left[\frac{(1-\lambda)z}{1-z} + \frac{\lambda z}{(1-z)^2}\right] - (1+A\sigma)\left(z + \frac{z^{1+k}}{1-z^k}\right)\right\} \neq 0,$$

for $z \in U \setminus \{0\},\ \sigma \in C$ and $|\sigma| = 1$. This leads to the desired result (2.34). □

Theorem 2.7. *Let $f \in M_k(\lambda, A, B)$ and $h_\sigma(z)$ be the same as in Theorem 2.6. Then*

$$f(z) * zh'_\sigma(z) \neq 0 \quad (z \in U \setminus \{0\};\ \sigma \in C,\ |\sigma| = 1).$$

Proof. Since

$$f \in M_k(\lambda, A, B) \Longleftrightarrow zf' \in H_k(\lambda, A, B),$$

it follows from Theorem 2.6 that

$$f(z) * zh'_\sigma(z) = zf'(z) * h_\sigma(z) \neq 0,$$

for $z \in U \setminus \{0\},\ \sigma \in C$ and $|\sigma| = 1$. □

3. Applications

To make objects invisible to human eyes has been long a subject of science fiction. But just in 2006, this imagination has been materialized in the range of microwave radiation. This is attributed two pioneering papers published in Science by Leonhardt [6] and Pendry et al. [11] in 2006, in which they proposed an ingenious idea to control electromagnetic waves by a specially designed materials. They suggest that a cloak, made of metamaterial (in which the refractive index spatially varies), can be designed so that an incident electromagnetic wave can be guided through the cloak giving an impression of free space when viewed from outside. This ensures that the cloak neither reflects nor scatters waves nor casts a shadow in the transmitted field. The cloak remains undetected by a viewing device. At the same time the cloak reduces scattering of radiation from the object where the imperfections are exponentially small. Hence the object becomes invisible to the detector.

Reports are available in the published literature (see, e.g., [6, 11]) that electromagnetic cloaking, which seemed impossible earlier, is technologically realizable when the cloak and the cloaked object have a circular symmetry in at least one plane, namely: spheres and cylinders. The cross section is a laminar or two dimensional cloaking.

Mathematically, the two dimensional cloak and the cloaked object are simply connected regions in the complex plane, the later being a subset of the former. By the Riemann mapping theorem both the regions are equivalent to conformal maps on the unit disk U. If we denote the cloaked object by the function $g(z)$ and the cloak by the function $q(z)$ then it is required that

$$g(z) \prec q(z) \quad (z \in U). \tag{3.1}$$

Very recently Mishra, Panigrahi and Mishra [7] have given some applications of subordination relationship (3.1) to electromagnetic cloaking. In the present paper, we found certain sufficient conditions under which relationship of the form (3.1) holds for functions which are more general than circular maps. For example, in Lemma 1.1, we have taken the cloak function

$$q(z) = \frac{1+Az}{1+Bz}, \tag{3.2}$$

to be an analytic univalent convex map. If a function $f(z)$ satisfies the condition (1.3), then the subordination relationship (3.1) holds true. All results in this paper are based on the condition (1.3).

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