

# A NEW VERSION OF THE SMITH METHOD FOR SOLVING SYLVESTER EQUATION AND DISCRETE-TIME SYLVESTER EQUATION\*

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**Abstract** Recently, Xue etc. [37] discussed the Smith method for solving Sylvester equation  $AX + XB = C$ , where one of the matrices  $A$  and  $B$  is at least a nonsingular  $M$ -matrix and the other is an (singular or nonsingular)  $M$ -matrix. Furthermore, in order to find the minimal non-negative solution of a certain class of non-symmetric algebraic Riccati equations, Gao and Bai [17] considered a doubling iteration scheme to inexactly solve the Sylvester equations. This paper discusses the iterative error of the standard Smith method used in [17] and presents the prior estimations of the accurate solution  $X$  for the Sylvester equation. Furthermore, we give a new version of the Smith method for solving discrete-time Sylvester equation or Stein equation  $AXB + X = C$ , while the new version of the Smith method can also be used to solve Sylvester equation  $AX + XB = C$ , where both  $A$  and  $B$  are positive definite. We also study the convergence rate of the new Smith method. At last, numerical examples are given to illustrate the effectiveness of our methods.

**Keywords** Sylvester equation, Discrete-time Sylvester equation, Smith method, Positive definite matrix, Convergence.

**MSC(2010)** 65F10, 65F50.

## 1. Introduction

The Sylvester equation and the discrete-time Sylvester equation or Stein equation take the forms

$$AX + XB = C, \quad (1.1)$$

and

$$AXB + X = C, \quad (1.2)$$

respectively, where  $A$ ,  $B$ , and  $C$  are known complex matrices of size  $m \times m$ ,  $n \times n$ , and  $m \times n$ , respectively, and  $X$  is the unknown matrix of size  $m \times n$ . It is well known that if  $B = A^*$ , (1.1) and (1.2) are Lyapunov equations, where  $A^*$  denotes the complex conjugate transpose of  $A$ . The equation (1.1) has a unique solution if and only if  $A$  and  $-B$  have no common eigenvalues [26]. The equation (1.2) has a

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unique solution if and only if  $\lambda_i \lambda_j \neq -1, i = 1, 2, \dots, m; j = 1, 2, \dots, n$ , where  $\lambda_i$  and  $\lambda_j$  are the eigenvalues of  $A$  and  $B$ , respectively [31, 34].

The Sylvester equations and the discrete-time Sylvester equations appear frequently in many areas of applied mathematics such as matrix eigen-decompositions [19], control theory [11, 12], model reduction [1, 4, 31], numerical solutions of the matrix differential Riccati equations [10, 13, 16, 17], image processing [9], and so on. Especially, Bhatia & Rosenthal gave an elegant survey [5] including the history of the equation and many interesting and important theoretical results.

This paper is concerned with the numerical solutions of the Sylvester equation and the discrete-time Sylvester equation [7, 23, 24, 38–40]. For the Sylvester equation, the standard ones are the Bartels-Stewart algorithm [3] and the Hessenberg-Schur method [18]. By the Schur decomposition, these methods transformed the original equation into a system that is easily solved by a forward substitution. The Bartels-Stewart algorithm [3] transforms  $A$  and  $B$  into real Schur forms, while in the Hessenberg-Schur algorithm [18], the matrix  $A$  is reduced to upper Hessenberg form and the matrix  $B$  is transformed into real Schur form. All these methods are efficient when  $A$  and  $B$  are dense. When  $A$  and  $B$  are large and sparse, the iterative solution of (1.1) by the alternating-direction-implicit (ADI) method might be more attractive. See [6, 12, 15, 27, 28, 32, 34–36]. The Krylov-subspace Galerkin and minimal residual algorithms [22] for solving the Sylvester equation (1.1) were presented by Hu & Reichel. In [29], Simoncini extended the work of Hu & Reichel to block form by using the idea developed in [22]. Based on block-Arnoldi and nonsymmetric block-Lanczos algorithms, El Guennouniet etc. [14] introduced some new Krylov methods to solve (1.1). In [25], Robbe & Sadkane discussed the convergence properties of the block GMRES and FOM methods for solving large Sylvester equations of the form (1.1). For the discrete-time Sylvester equation, the analytical solution of the matrix equation (1.2) has been considered by many authors, see [18]. Direct methods for solving the matrix equation (1.2) such as those proposed in [18, 19] are attractive if both matrices  $A$  and  $B$  are of small size. When both  $A$  and  $B$  are large and sparse, the iterative solution of (1.2) by the alternating-direction-implicit (ADI) method might also be more attractive, see [8, 33]. The Krylov-subspace Galerkin and minimal residual algorithms for solving the discrete-time Sylvester equation (1.2) were presented in [2]. Recently, Xue etc. [37] discussed the Smith method for solving Sylvester equation  $AX + XB = C$ , where one of the matrices  $A$  and  $B$  is at least a nonsingular  $M$ -matrix and the other is an (singular or nonsingular)  $M$ -matrix. Furthermore, Gao & Bai [17] considered a doubling iteration scheme to inexactly solve the Sylvester equations in order to find the minimal non-negative solution of a certain class of non-symmetric algebraic Riccati equations. In this paper, we will discuss a new version of the Smith method for solving the discrete-time Sylvester equation  $AXB + X = C$ , while the new version of the Smith method can also be used to solve Sylvester equation  $AX + XB = C$ , when both  $A$  and  $B$  are positive definite.

Customarily, the notations of the paper are arranged as follows :  $\rho(A)$ ,  $\|A\|_2$ , and  $\|A\|_F$  denote the spectral radius, 2-norm, and Frobenius norm of the matrix  $A \in C^{m \times m}$ , respectively [19].

The rest of this paper is organized as follows. In Section 2, we review the Smith method for solving Sylvester equation  $AX + XB = C$ , where both  $A$  and  $B$  are positive definite. By considering the convergence rate of the Smith method, we give some results about the upper bounds of the error  $\|X - X_k\|_2$  and the residual error

$\|R(X_k)\|_2 = \|C - AX_k - X_kB\|_2$ , where  $X_k$  is the  $k$ -order approximate solution of (1.1). In addition, two prior estimations of the accurate solution  $X$  are given. In Section 3, we present a new version of the Smith method for solving discrete-time Sylvester equation  $AXB + X = C$ , where both  $A$  and  $B$  are positive definite. The convergence analysis and a prior estimation of the accurate solution  $X$  are presented. In Section 4, some numerical examples are tested to illustrate the reliability and effectiveness of the method we discussed. Finally, we summarize our findings in Section 5.

## 2. The Smith method for solving Sylvester equations and error analysis

In this section, we review the Smith method described in [17,37] for solving Sylvester equations  $AX + XB = C$ , where both  $A$  and  $B$  are positive definite. Furthermore, we give some results about the upper bounds of the iterative error and the residual error, respectively.

Let  $I_m$  and  $I_n$  be  $m \times m$  and  $n \times n$  identity matrices, respectively. For any scalar  $\alpha > 0$ , the equation (1.1) can be rewritten as

$$(A + \alpha I_m)X(B + \alpha I_n) - (A - \alpha I_m)X(B - \alpha I_n) = 2\alpha C. \quad (2.1)$$

Under the assumption that both  $A$  and  $B$  are positive definite matrices, naturally, both  $A + \alpha I_m$  and  $B + \alpha I_n$  are nonsingular, hence, the system (2.1) can be simply described as

$$X - UXV = W, \quad (2.2)$$

where  $U = (A + \alpha I_m)^{-1}(A - \alpha I_m)$ ,  $V = (B - \alpha I_n)(B + \alpha I_n)^{-1}$  and  $W = 2\alpha(A + \alpha I_m)^{-1}C(B + \alpha I_n)^{-1}$ .

According to the iterative idea of the Smith method [17], we can get the iterative scheme

$$X_k = \sum_{i=0}^k U^i W V^i \quad \text{for } k \geq 0. \quad (2.3)$$

Although solving (1.1) by (2.3) is a practical method, its convergence rate is slow. In general, (2.3) can be quickly approximated by

$$X_0 = W, \quad X_{k+1} = X_k + U^{2^k} X_k V^{2^k} \quad \text{for } k \geq 0. \quad (2.4)$$

By induction, we get

$$X_k = \sum_{i=1}^{2^k} U^{i-1} W V^{i-1} \quad \text{for } k \geq 0.$$

By using (2.4) and a simple transformation, we can also obtain an accelerated Smith method for solving the discrete-time Sylvester equation  $AXB + X = C$ . That is

$$X_0 = \tilde{W}, \quad X_{k+1} = X_k + \tilde{U}^{2^k} X_k \tilde{V}^{2^k} \quad \text{for } k \geq 0, \quad (2.5)$$

where  $\tilde{U} = (A + \alpha I_m)^{-1}(A - \alpha I_m)$ ,  $\tilde{V} = (I_n - \alpha B)(I_n + \alpha B)^{-1}$  and  $\tilde{W} = 2\alpha(A + \alpha I_m)^{-1}C(I_n + \alpha B)^{-1}$ .

In the next subsection, we will study the convergence error of the Smith method for solving the Sylvester equation, while the convergence error of the Smith method for solving the discrete-time Sylvester equation can be discussed in a similar way.

## 2.1. Analysis of convergence errors

Firstly, for the upper bound of the error  $\|X - X_k\|_2$ , the following conclusion can be established, where  $X_k$  is the  $k$ -order approximate solution of (1.1) by the Smith method (2.4).

**Theorem 2.1.** *Let  $X_k$  be the  $k$ -order approximate solution of (1.1) by the Smith method (2.4) and  $X$  be the accurate solution of (1.1). Then*

$$\|X - X_k\|_2 \leq M\|W\|_2(1-r)^{-1}r^{2^k}. \quad (2.6)$$

**Proof.** From (2.4), we get

$$\begin{aligned} \|X - X_k\|_2 &= \left\| \sum_{i=2^k+1}^{+\infty} U^{i-1}WV^{i-1} \right\|_2 \\ &\leq \sum_{i=2^k+1}^{+\infty} \|U^{i-1}\|_2 \|V^{i-1}\|_2 \|W\|_2 \\ &\leq M\|W\|_2 \sum_{i=2^k+1}^{+\infty} r^{i-1} \\ &= M\|W\|_2 r^{2^k} \sum_{i=0}^{+\infty} r^i \\ &= M\|W\|_2(1-r)^{-1}r^{2^k}. \end{aligned}$$

□

Through Theorem 2.1, one sees that the sequence  $\{X_k\}$  converges to  $X$  very rapidly. Furthermore, Theorem 2.2 gives an upper bound of the residual error  $\|R(X_k)\|_2 = \|C - AX_k - X_k B\|_2$ , that can be used to stop the iteration process without computing extra matrix products  $AX_k$  and  $X_k B$ .

**Theorem 2.2.** *Let  $X_k$  be the  $k$ -order approximate solution of (1.1) by the Smith method (2.4) and  $X$  be the accurate solution of (1.1). Let  $R(X_k) = C - AX_k - X_k B$  be the corresponding residual error. Then*

$$\|R(X_k)\|_2 \leq M(\|A\|_2 + \|B\|_2)\|W\|_2(1-r)^{-1}r^{2^k}.$$

**Proof.** By (2.6), we have

$$\begin{aligned} \|R(X_k)\|_2 &= \|C - AX_k - X_k B\|_2 \\ &= \|A(X - X_k) + (X - X_k)B\|_2 \\ &\leq \|A\|_2\|X - X_k\|_2 + \|B\|_2\|X - X_k\|_2 \\ &= (\|A\|_2 + \|B\|_2)\|X - X_k\|_2 \\ &\leq M(\|A\|_2 + \|B\|_2)\|W\|_2(1-r)^{-1}r^{2^k}. \end{aligned}$$

□

## 2.2. Two prior estimations of the accurate solution $X$

For the Sylvester equation and the Lyapunov equation, in this section, we present two prior estimations for the accurate solutions  $X$ , respectively.

**Theorem 2.3.** *If both  $A$  and  $B$  are positive definite matrices, then the solution  $X$  of (1.1) satisfies*

$$\|X\|_2 \leq \frac{1}{2} \|C\|_2 \|P_1\|_2^{1/2} \|P_2\|_2^{1/2}, \quad (2.7)$$

where  $P_1$  and  $P_2$  satisfy

$$AP_1 + P_1A^* = 2I_m, \quad B^*P_2 + P_2B = 2I_n. \quad (2.8)$$

**Proof.** For any nonzero vectors  $x \in C^m$  and  $y \in C^n$ , let

$$\eta_i^* = x^*U^i(A + \alpha I_m)^{-1}, \quad \zeta_i = (B + \alpha I_n)^{-1}V^i y,$$

where  $\eta_i^*$  denotes the conjugate transpose of the vector  $\eta_i$ .

From the Theorem 2.1 and the expression of  $W$  in (2.2), we have

$$\begin{aligned} |x^*Xy| &= \left| \sum_{i=0}^{+\infty} x^*U^i(A + \alpha I_m)^{-1}2\alpha C(B + \alpha I_n)^{-1}V^i y \right| \\ &= \left| \sum_{i=0}^{+\infty} \eta_i^*(2\alpha C)\zeta_i \right| \\ &\leq \|2\alpha C\|_2 \sum_{i=0}^{+\infty} \|\eta_i\|_2 \|\zeta_i\|_2. \end{aligned}$$

By the Schwarz's inequality, we have

$$|x^*Xy| \leq \|2\alpha C\|_2 \left( \sum_{i=0}^{+\infty} \|\eta_i\|_2^2 \right)^{1/2} \left( \sum_{i=0}^{+\infty} \|\zeta_i\|_2^2 \right)^{1/2}. \quad (2.9)$$

Since the two equations in (2.8) have the same forms as (1.1), their solutions  $P_1$  and  $P_2$  can be expressed in the following form

$$P_1 = \sum_{i=0}^{+\infty} U^i W_1 V_1^i,$$

where  $U = (A + \alpha I_m)^{-1}(A - \alpha I_m)$ ,  $V_1 = (A^* - \alpha I_m)(A^* + \alpha I_m)^{-1}$  and  $W_1 = 4\alpha(A + \alpha I_m)^{-1}(A^* + \alpha I_m)^{-1}$ . Then

$$\begin{aligned} x^*P_1x &= \sum_{i=0}^{+\infty} 4\alpha x^*U^i(A + \alpha I_m)^{-1}(A^* + \alpha I_m)^{-1}V_1^i x \\ &= 4\alpha \sum_{i=0}^{+\infty} \eta_i^* \eta_i \\ &= 4\alpha \sum_{i=0}^{+\infty} \|\eta_i\|_2^2. \end{aligned} \quad (2.10)$$

Similarly,

$$y^*P_2y = 4\alpha \sum_{i=0}^{+\infty} \zeta_i^* \zeta_i = 4\alpha \sum_{i=0}^{+\infty} \|\zeta_i\|_2^2. \quad (2.11)$$

Substituting (2.10) and (2.11) into (2.9), we get

$$|x^* X y| \leq \frac{1}{2} \|C\|_2 (x^* P_1 x)^{1/2} (y^* P_2 y)^{1/2}. \quad (2.12)$$

Divide both sides of (2.12) by  $\|x\|_2 \|y\|_2$ , we get

$$\frac{|x^* X y|}{\|x\|_2 \|y\|_2} \leq \frac{1}{2} \|C\|_2 \frac{(x^* P_1 x)^{1/2}}{\|x\|_2} \frac{(y^* P_2 y)^{1/2}}{\|y\|_2}.$$

Let  $\max_{\substack{0 \neq x \in C^m \\ 0 \neq y \in C^n}} \frac{|x^* X y|}{\|x\|_2 \|y\|_2} = \frac{|x_0^* X y_0|}{\|x_0\|_2 \|y_0\|_2}$ , then

$$\|X\|_2 = \frac{|x_0^* X y_0|}{\|x_0\|_2 \|y_0\|_2} \leq \frac{1}{2} \|C\|_2 \frac{(x_0^* P_1 x_0)^{1/2}}{\|x_0\|_2} \frac{(y_0^* P_2 y_0)^{1/2}}{\|y_0\|_2}. \quad (2.13)$$

Note that,

$$\begin{aligned} \left| \frac{x_0^* P_1 x_0}{x_0^* x_0} \right| &\leq \frac{\|x_0\|_2 \|P_1 x_0\|_2}{x_0^* x_0} = \frac{\|P_1 x_0\|_2}{\|x_0\|_2} = \frac{(x_0^* P_1^* P_1 x_0)^{1/2}}{\|x_0\|_2} \\ &\leq \rho(P_1^* P_1)^{1/2} \frac{(x_0^* x_0)^{1/2}}{\|x_0\|_2} = \|P_1\|_2, \end{aligned}$$

similarly, we have

$$\left| \frac{y_0^* P_2 y_0}{y_0^* y_0} \right| \leq \|P_2\|_2. \quad (2.14)$$

From (2.13) – (2.14), we get

$$\|X\|_2 = \frac{|x_0^* X y_0|}{\|x_0\|_2 \|y_0\|_2} \leq \frac{1}{2} \|C\|_2 \|P_1\|_2^{1/2} \|P_2\|_2^{1/2}.$$

Hence, (2.7) is obtained.  $\square$

**Remark 2.1.** In the above theorem, we give a prior estimation of the accurate solution  $X$  of the Sylvester equation whose coefficient matrices are positive definite. It should be noted that if the real part of all eigenvalues of  $A$  and  $B$  are positive, then the conclusion still maintains. For the Sylvester equation  $AX + XB = C$ , where both  $A$  and  $B$  are Hurwitzian matrices (i.e., all their eigenvalues have negative real parts), [30] had given a similar result.

For the Lyapunov equation, we are going to present a more practical prior estimation of the accurate solution  $X$ .

**Lemma 2.1.** Let  $A \in C^{m \times m}$  be a nonsingular matrix and  $B \in C^{m \times m}$ , then,

$$\|AB\|_F \geq \|A^{-1}\|_2^{-1} \|B\|_F.$$

**Proof.** Let  $B = (b_1, b_2, \dots, b_m)$ , where  $b_i \in C^{m \times 1}$ ,  $i = 1, 2, \dots, m$ , then

$$\begin{aligned} \|AB\|_F^2 &= \|(Ab_1, Ab_2, \dots, Ab_m)\|_F^2 \\ &= \|Ab_1\|_2^2 + \|Ab_2\|_2^2 + \dots + \|Ab_m\|_2^2 \\ &\leq \|A\|_2^2 (\|b_1\|_2^2 + \|b_2\|_2^2 + \dots + \|b_m\|_2^2) \\ &= \|A\|_2^2 \|B\|_F^2, \end{aligned}$$

i.e.

$$\|AB\|_F \leq \|A\|_2 \|B\|_F.$$

Since  $A$  is nonsingular, we have

$$\|B\|_F = \|A^{-1}AB\|_F \leq \|A^{-1}\|_2 \|AB\|_F,$$

i.e.  $\|AB\|_F \geq \|A^{-1}\|_2^{-1} \|B\|_F$ .  $\square$

By Lemma 2.1, one can prove the following conclusion.

**Theorem 2.4.** *For a symmetric positive definite matrix  $A \in C^{m \times m}$  and a symmetric matrix  $C \in C^{m \times m}$ , if the Lyapunov equation  $AX + XA = C$  is valid over symmetric matrix  $X$ , then*

$$\|X\|_F \leq \sqrt{\frac{m}{2}} \frac{\rho(C)}{\lambda_{\min}(A)}, \quad (2.15)$$

where  $\lambda_{\min}(A)$  is the minimum eigenvalue of  $A$ .

**Proof.** The square of the both sides of the equation  $AX + XA = C$  is

$$AXAX + AXXA + XAAX + XAXA = C^2.$$

Then, it holds

$$tr(C^2) = tr(AXAX) + tr(AXXA) + tr(XAAX) + tr(XAXA),$$

where  $tr(\cdot)$  denotes the trace of a matrix. Since  $A$ ,  $X$ , and  $C$  are symmetric matrices and  $\|C\|_F = \sqrt{tr(C^T C)}$ , we have

$$\begin{aligned} tr(C^2) &= tr(C^T C) = \|C\|_F^2, \\ tr(AXAX) &= tr((XAXA)^T) = tr(XAXA), \\ tr(AXXA) &= tr(AX(AX)^T) = \|AX\|_F^2, \\ tr(XAAX) &= tr(XA(XA)^T) = \|XA\|_F^2 = \|(XA)^T\|_F^2 = \|AX\|_F^2. \end{aligned}$$

It follows

$$2\|AX\|_F^2 + 2tr(XAXA) = \|C\|_F^2. \quad (2.16)$$

Under the assumption that  $A$  is a symmetric positive definite matrix, we get

$$A^{\frac{1}{2}}XAXAA^{-\frac{1}{2}} = A^{\frac{1}{2}}XA^{\frac{1}{2}}A^{\frac{1}{2}}XA^{\frac{1}{2}} = (A^{\frac{1}{2}}XA^{\frac{1}{2}})^T A^{\frac{1}{2}}XA^{\frac{1}{2}}.$$

Hence,  $XAXA$  is similar to  $(A^{\frac{1}{2}}XA^{\frac{1}{2}})^T A^{\frac{1}{2}}XA^{\frac{1}{2}}$ . It follows

$$tr(XAXA) = tr((A^{\frac{1}{2}}XA^{\frac{1}{2}})^T A^{\frac{1}{2}}XA^{\frac{1}{2}}) = \|A^{\frac{1}{2}}XA^{\frac{1}{2}}\|_F^2 \geq 0.$$

Thus (2.16) implies

$$\|AX\|_F \leq \frac{\|C\|_F}{\sqrt{2}}. \quad (2.17)$$

Furthermore, the matrix  $A$  is nonsingular because  $A$  is symmetric positive definite matrix, from Lemma 2.1, we also have

$$\|AX\|_F \geq \|A^{-1}\|_2^{-1} \|X\|_F.$$

Combined with (2.17), we have

$$\|X\|_F \leq \frac{\|C\|_F \|A^{-1}\|_2}{\sqrt{2}}. \tag{2.18}$$

Noting that  $A^{-1}$  is also symmetric positive definite, it follows

$$\|A^{-1}\|_2 = \rho(A^{-1}) = \frac{1}{\lambda_{\min}(A)}. \tag{2.19}$$

In addition, we have

$$\|C\|_F \leq \sqrt{m} \|C\|_2 = \sqrt{m} \rho(C). \tag{2.20}$$

Substituting (2.19) and (2.20) into (2.18), we get

$$\|X\|_F \leq \sqrt{\frac{m}{2}} \frac{\rho(C)}{\lambda_{\min}(A)}.$$

Hence, (2.15) is obtained. □

### 3. A new version of Smith method for solving discrete-time Sylvester equations and Sylvester equations

In this section, through a simple computation, we present a new version of the Smith method for solving discrete-time Sylvester equation  $AXB + X = C$ , while the new version of the Smith method can also be used to solve Sylvester equation,  $AX + XB = C$ , where both  $A$  and  $B$  are positive definite.

For any scalar  $\alpha > 0$ , the equation (1.2) can be rewritten as

$$(A + \alpha I_m)X(B + \alpha I_n) - (\alpha I_m - A)X(B - \alpha I_n) + 2X = 2\alpha^2 X + 2C. \tag{3.1}$$

Considering the case that both  $A + \alpha I_m$  and  $B + \alpha I_n$  are nonsingular, we can rewrite (3.1) as

$$X - \bar{U}X\bar{V} + 2(A + \alpha I_m)^{-1}X(B + \alpha I_n)^{-1} = 2\bar{W},$$

where  $\bar{U} = (A + \alpha I_m)^{-1}(\alpha I_m - A)$ ,  $\bar{V} = (B - \alpha I_n)(B + \alpha I_n)^{-1}$  and  $\bar{W} = (A + \alpha I_m)^{-1}(\alpha^2 X + C)(B + \alpha I_n)^{-1}$ .

Recursively, for any natural number  $l$ , we get

$$\begin{aligned} \bar{U}X\bar{V} - \bar{U}^2X\bar{V}^2 + 2\bar{U}(A + \alpha I_m)^{-1}X(B + \alpha I_n)^{-1}\bar{V} &= 2\bar{U}\bar{W}\bar{V}; \\ \bar{U}^2X\bar{V}^2 - \bar{U}^3X\bar{V}^3 + 2\bar{U}^2(A + \alpha I_m)^{-1}X(B + \alpha I_n)^{-1}\bar{V}^2 &= 2\bar{U}^2\bar{W}\bar{V}^2; \\ \bar{U}^3X\bar{V}^3 - \bar{U}^4X\bar{V}^4 + 2\bar{U}^3(A + \alpha I_m)^{-1}X(B + \alpha I_n)^{-1}\bar{V}^3 &= 2\bar{U}^3\bar{W}\bar{V}^3; \\ \dots & \\ \bar{U}^{l-1}X\bar{V}^{l-1} - \bar{U}^lX\bar{V}^l + 2\bar{U}^{l-1}(A + \alpha I_m)^{-1}X(B + \alpha I_n)^{-1}\bar{V}^{l-1} &= 2\bar{U}^{l-1}\bar{W}\bar{V}^{l-1}. \end{aligned}$$

By adding them up, we can get the following system

$$X - \bar{U}^lX\bar{V}^l + 2 \sum_{i=0}^{l-1} \bar{U}^i (A + \alpha I_m)^{-1}X(B + \alpha I_n)^{-1}\bar{V}^i = 2 \sum_{i=0}^{l-1} \bar{U}^i \bar{W} \bar{V}^i,$$



i.e.

$$X = \bar{U}^l X \bar{V}^l + 2 \sum_{i=0}^{l-1} \bar{U}^i (A + \alpha I_m)^{-1} (\alpha^2 X - X + C) (B + \alpha I_n)^{-1} \bar{V}^i.$$

Let  $X_0 = 0$ . We continue to consider the iteration system

$$\begin{aligned} X_k &= \bar{U}^l X_{k-1} \bar{V}^l + 2 \sum_{i=0}^{l-1} \bar{U}^i (A + \alpha I_m)^{-1} (\alpha^2 X_{k-1} - X_{k-1} + C) (B + \alpha I_n)^{-1} \bar{V}^i, \\ k &= 1, 2, \dots \end{aligned} \quad (3.2)$$

The iterative scheme (3.2) is a new version of Smith method for solving discrete-time Sylvester equation  $AXB + X = C$ . Through a simple transformation, we can also obtain the following iterative scheme for the Sylvester equation  $AX + XB = C$ ,

$$\begin{aligned} X_k &= \hat{U}^l X_{k-1} \hat{V}^l + 2 \sum_{i=0}^{l-1} \hat{U}^i (A + \alpha I_m)^{-1} (\alpha^2 X_{k-1} B - X_{k-1} B + C) (I_n + \alpha B)^{-1} \hat{V}^i, \\ k &= 1, 2, \dots, \end{aligned} \quad (3.3)$$

where  $X_0 = 0$ ,  $\hat{U} = (A + \alpha I_m)^{-1} (\alpha I_m - A)$ , and  $\hat{V} = (I_n - \alpha B) (I_n + \alpha B)^{-1}$ .

For simplicity, we just consider the iterative scheme (3.2), and the iterative scheme (3.3) can be discussed in a similar way.

### 3.1. Convergence analysis

Before we present the convergence analysis, we first review a lemma.

**Lemma 3.1** ([21], p.183). *Given a discrete-time Sylvester equation  $AXB + X = C$ , where both  $A$  and  $B$  are positive definite, the spectral radii  $\rho(\bar{U})$  and  $\rho(\bar{V})$  are less than 1, and there exist constants  $\bar{M} > 0$  and  $\bar{r} > 0$ , such that*

$$\rho(\bar{U})\rho(\bar{V}) < \bar{r} < 1,$$

and for all  $i$

$$\|\bar{U}^i\|_2 \cdot \|\bar{V}^i\|_2 \leq \bar{M}\bar{r}^i.$$

**Theorem 3.1.** *Let  $X_k$  be the  $k$ -order approximate solution of (1.2) by the Smith method (3.2), then we can get*

$$\|X_{k+1} - X_k\|_2 < L \|X_k - X_{k-1}\|_2, \quad (3.4)$$

where  $L = \bar{M}\bar{r}^l + 2\bar{M}|\alpha^2 - 1| \frac{1-\bar{r}^{l-1}}{1-\bar{r}} \sqrt{\frac{1}{\lambda_{\min}(AA^*)}} \sqrt{\frac{1}{\lambda_{\min}(BB^*)}}$ .

**Proof.** According to the iterative scheme (3.2), we can get

$$X_{k+1} = \bar{U}^l X_k \bar{V}^l + 2 \sum_{i=0}^{l-1} \bar{U}^i (A + \alpha I_m)^{-1} (\alpha^2 X_k - X_k + C) (B + \alpha I_n)^{-1} \bar{V}^i.$$

Combined with (3.2), we have

$$X_{k+1} - X_k = \bar{U}^l (X_k - X_{k-1}) \bar{V}^l + 2(\alpha^2 - 1) \sum_{i=0}^{l-1} \bar{U}^i (A + \alpha I_m)^{-1} (X_k - X_{k-1}) (B + \alpha I_n)^{-1} \bar{V}^i$$

and

$$\begin{aligned} & \|X_{k+1} - X_k\|_2 \\ & \leq \|\bar{U}^l\|_2 \|\bar{V}^l\|_2 \|X_k - X_{k-1}\|_2 \\ & \quad + 2|\alpha^2 - 1| \sum_{i=0}^{l-1} \|\bar{U}^i\|_2 \|(A + \alpha I_m)^{-1}\|_2 \|(B + \alpha I_n)^{-1}\|_2 \|\bar{V}^i\|_2 \|X_k - X_{k-1}\|_2. \end{aligned}$$

Noting that the matrix  $A$  is positive definite and the matrix  $\alpha(A + A^*) + \alpha^2 I_m$  is also positive definite, hence

$$\begin{aligned} \|(A + \alpha I_m)^{-1}\|_2 &= \sqrt{\lambda_{\max}[(A + \alpha I_m)^{-1}(A^* + \alpha I_m)^{-1}]} \\ &= \sqrt{\lambda_{\max}[(AA^* + \alpha(A + A^*) + \alpha^2 I_m)^{-1}]} \\ &= \sqrt{\frac{1}{\lambda_{\min}(AA^* + \alpha(A + A^*) + \alpha^2 I_m)}} \\ &< \sqrt{\frac{1}{\lambda_{\min}(AA^*)}}. \end{aligned} \tag{3.5}$$

Similarly,

$$\|(B + \alpha I_n)^{-1}\|_2 < \sqrt{\frac{1}{\lambda_{\min}(BB^*)}}. \tag{3.6}$$

By Lemma 3.1, (3.5) and (3.6), we have

$$\begin{aligned} & \|X_{k+1} - X_k\|_2 \\ & < \left[ \bar{M}\bar{r}^l + 2\bar{M}|\alpha^2 - 1| \sqrt{\frac{1}{\lambda_{\min}(AA^*)}} \sqrt{\frac{1}{\lambda_{\min}(BB^*)}} \sum_{i=0}^{l-1} \bar{r}^i \right] \|X_k - X_{k-1}\|_2 \\ & = \left[ \bar{M}\bar{r}^l + 2\bar{M}|\alpha^2 - 1| \frac{1 - \bar{r}^{l-1}}{1 - \bar{r}} \sqrt{\frac{1}{\lambda_{\min}(AA^*)}} \sqrt{\frac{1}{\lambda_{\min}(BB^*)}} \right] \|X_k - X_{k-1}\|_2 \\ & = L \|X_k - X_{k-1}\|_2. \end{aligned}$$

Hence, (3.4) is obtained. □

From the Theorem 3.1, we can find that  $0 < L < 1$  when  $l$  is large enough and  $\alpha$  is close to 1. Combined with the Cauchy criterion for convergence, it is easily to prove that the new Smith method is convergent.

### 3.2. The prior estimation of the accurate solution $X$

In this section, we present a prior estimation of the accurate solution  $X$  of the Lyapunov equation.

**Theorem 3.2.** *For a symmetric positive definite matrix  $A \in C^{m \times m}$  and a symmetric matrix  $C \in C^{m \times m}$ , if the Lyapunov equation  $AXA + X = C$  is valid over symmetric matrix  $X$ , then*

$$\|X\|_F \leq \sqrt{\frac{m}{1 + \lambda_{\min}^4(A)}} \rho(C). \tag{3.7}$$

**Proof.** The square of the both sides of the equation  $AXA + X = C$  is

$$AXAAXA + AXAX + XAXA + X^2 = C^2,$$

then,

$$tr(C^2) = tr(AXAAXA) + tr(AXAX) + tr(XAXA) + tr(X^2).$$

Since  $A$ ,  $X$ , and  $C$  are symmetric matrices and  $\|C\|_F = \sqrt{tr(C^T C)}$ , we have

$$\begin{aligned} tr(C^2) &= tr(C^T C) = \|C\|_F^2, \\ tr(AXAX) &= tr((XAXA)^T) = tr(XAXA), \\ tr(X^2) &= tr(X^T X) = \|X\|_F^2, \\ tr(AXAAXA) &= tr((AXA)^T(AXA)) = \|AXA\|_F^2. \end{aligned}$$

So,

$$\|AXA\|_F^2 + 2tr(XAXA) + \|X\|_F^2 = \|C\|_F^2. \quad (3.8)$$

As

$$A^{\frac{1}{2}}XAXAA^{-\frac{1}{2}} = A^{\frac{1}{2}}XA^{\frac{1}{2}}A^{\frac{1}{2}}XA^{\frac{1}{2}} = (A^{\frac{1}{2}}XA^{\frac{1}{2}})^T A^{\frac{1}{2}}XA^{\frac{1}{2}},$$

$XAXA$  is similar to  $(A^{\frac{1}{2}}XA^{\frac{1}{2}})^T A^{\frac{1}{2}}XA^{\frac{1}{2}}$ , and we get

$$tr(XAXA) = tr((A^{\frac{1}{2}}XA^{\frac{1}{2}})^T A^{\frac{1}{2}}XA^{\frac{1}{2}}) = \|A^{\frac{1}{2}}XA^{\frac{1}{2}}\|_F^2 \geq 0.$$

Thus (3.8) implies

$$\|AXA\|_F^2 + \|X\|_F^2 \leq \|C\|_F^2. \quad (3.9)$$

Furthermore, the matrix  $A$  is nonsingular because  $A$  is symmetric positive definite matrix. From the Lemma 2.1, we have

$$\|AXA\|_F \geq \|A^{-1}\|_2^{-1} \|XA\|_F = \|A^{-1}\|_2^{-1} \|AX\|_F \geq (\|A^{-1}\|_2^{-1})^2 \|X\|_F.$$

Combined with (3.9), we also have

$$\|X\|_F^2 \leq \frac{\|C\|_F^2}{1 + \|A^{-1}\|_2^{-4}}. \quad (3.10)$$

Noting that  $A^{-1}$  is symmetric positive definite, we get

$$\|A^{-1}\|_2 = \rho(A^{-1}) = \frac{1}{\lambda_{\min}(A)}. \quad (3.11)$$

In addition, we have

$$\|C\|_F \leq \sqrt{m}\|C\|_2 = \sqrt{m}\rho(C). \quad (3.12)$$

Substituting (3.11) and (3.12) into (3.10), we get

$$\|X\|_F \leq \sqrt{\frac{m}{1 + \lambda_{\min}^4(A)}} \rho(C).$$

Hence, (3.7) is obtained.  $\square$

## 4. Numerical examples

In this section, we present some numerical examples to illustrate the effectiveness of the method proposed in the paper for solving the discrete-time Sylvester equation and the Sylvester equation whose coefficient matrices are large positive definite matrices. The computation is carried out by MATLAB R2008b on a personal computer, equipped with a dual core 3.30 GHz processor and 2GByte of memory.

**Example 4.1.** Consider the discrete-time Sylvester equation  $AXB + X = C$ , where the matrices  $A, B \in C^{100 \times 100}$  are given by

$$A = \begin{pmatrix} \tilde{d}_1 & -1 & & & \\ & \tilde{d}_2 & -1 & & \\ & & \ddots & \ddots & \\ & & & \tilde{d}_{99} & -1 \\ -1 & & & & \tilde{d}_{100} \end{pmatrix}, \quad B = \begin{pmatrix} \hat{d}_1 & & & & -1 \\ -1 & \hat{d}_2 & & & \\ & -1 & \ddots & & \\ & & \ddots & \hat{d}_{99} & \\ & & & -1 & \hat{d}_{100} \end{pmatrix},$$

respectively, where the diagonal elements  $\tilde{d}_i$  and  $\hat{d}_i$  of  $A$  and  $B$ , respectively, are generated by the uniform distribution on the interval  $[3, 10]$ . We choose the suitable matrix  $C$  such that the solution  $X$  of the equation  $AXB + X = C$  is

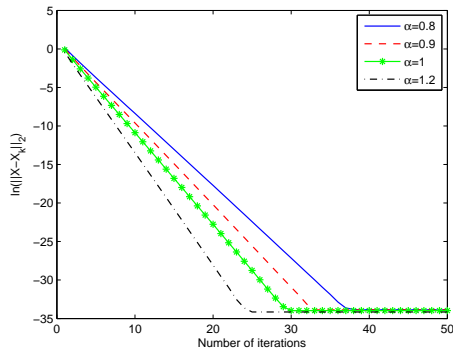
$$X = \begin{pmatrix} 0 & 2 & & & \\ 2 & 0 & 2 & & \\ & 2 & \ddots & \ddots & \\ & & \ddots & 0 & 2 \\ & & & 2 & 0 \end{pmatrix}. \quad (4.1)$$

With the assumption above, it is easy to verify that the discrete-time Sylvester equation  $AXB + X = C$  has a unique solution  $X$ . We apply the new Smith method (3.2) to seek for the unique solution  $X$ . Fig. 1, Fig. 2, and Fig. 3 display the convergence of the Smith method when the parameter  $l$  takes 3, 6, and 9, respectively.

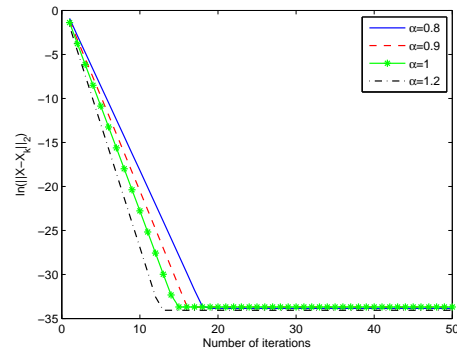
In addition, we use the the Smith method (2.5) to seek for the unique solution  $X$ . Fig. 4 displays the convergence of the Smith method. Furthermore, the convergence results of the new Smith method with  $l = 6$  and Smith accelerative method (2.5) for Example 4.1 are displayed in Fig. 5. From Fig. 5, we see that although the convergence speed for the iteration (3.2) is linearly and the convergence speed for Smith accelerative method (2.5) is quadratically, the convergence speed for the iteration (3.2) is faster than the Smith accelerative method (2.5) for the first several iterative steps. Hence, if the requirement on the computation precision for the solution is not high, the new Smith method (3.2) is a good choice. For example, for the case that the discrete-time Sylvester equation is not need to be solved exactly [17], then the new Smith method (3.2) can be used.

We continue to consider the Sylvester equation

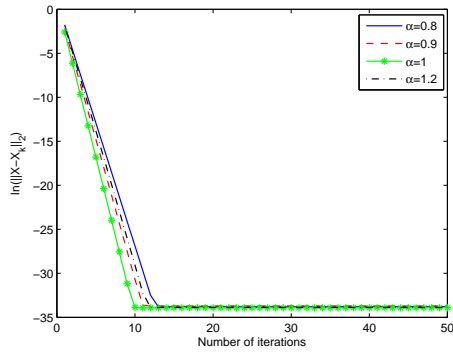
$$AX + XB = C.$$



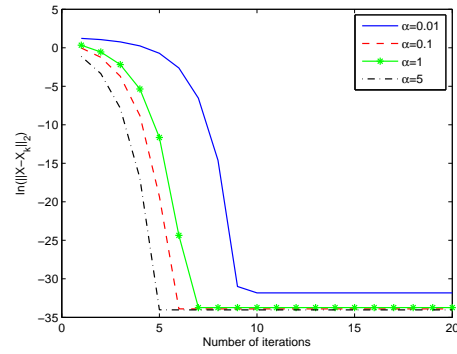
**Figure 1.** Convergence of the new Smith method,  $l = 3$ .



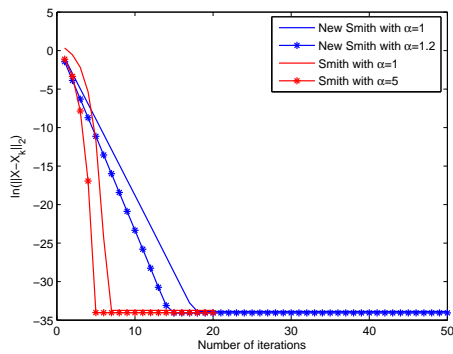
**Figure 2.** Convergence of the new Smith method,  $l = 6$ .



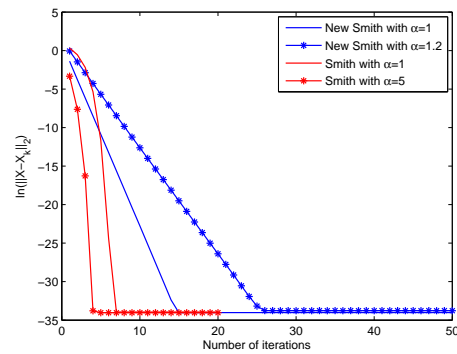
**Figure 3.** Convergence of the new Smith method,  $l = 9$ .



**Figure 4.** Convergence of the standard Smith method.



**Figure 5.** Convergence results of the new Smith method and standard Smith method.



**Figure 6.** Convergence results of the new Smith method and standard Smith method.

Similarly, we choose the suitable matrix  $C$  such that the matrix  $X$  defined by (4.1) is the the solution of the equation  $AX + XB = C$ . With the assumption above, it is easy to verify that both matrices  $A$  and  $B$  are positive definite, so the Sylvester

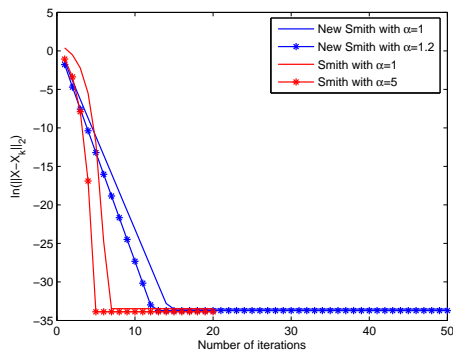
equation  $AX + XB = C$  has a unique solution  $X$ . We apply the Smith methods (2.4) and (3.3) with  $l = 6$  to seek for the unique solution  $X$ . Fig. 6 displays the convergence of the two Smith methods. Similar to the discussion above, if the requirement on the computation precision for the solution is not high, the new Smith method (3.3) can be used.

**Example 4.2.** Consider the discrete-time Sylvester equation  $AXB + X = C$ , where the matrices  $A, B \in C^{100 \times 100}$  are given by

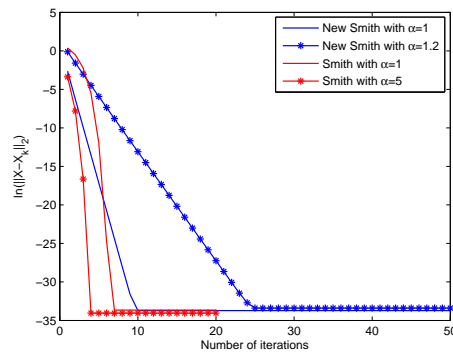
$$A = \begin{pmatrix} \tilde{d}_1 & 1 & -1 & & & \\ & \tilde{d}_2 & 1 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \tilde{d}_{98} & 1 & -1 \\ & & & & \tilde{d}_{99} & 1 \\ & & & & & \tilde{d}_{100} \end{pmatrix}, B = \begin{pmatrix} \hat{d}_1 & & & & & \\ 1 & \hat{d}_2 & & & & \\ -1 & 1 & \ddots & & & \\ & -1 & \ddots & \hat{d}_{98} & & \\ & & \ddots & 1 & \hat{d}_{99} & \\ & & & -1 & 1 & \hat{d}_{100} \end{pmatrix},$$

the diagonal entries  $\tilde{d}_i$  and  $\hat{d}_i$  of the coefficient matrices  $A$  and  $B$ , respectively, are generated by the uniform distribution on the interval  $[3, 10]$ . We choose the suitable matrix  $C$  such that the solution  $X$  is

$$X = \begin{pmatrix} 0 & 2i & & & \\ 2 & 0 & 2i & & \\ 2 & \ddots & \ddots & & \\ & \ddots & & 0 & 2i \\ & & & 2 & 0 \end{pmatrix}.$$



**Figure 7.** Convergence results of the new Smith method and standard Smith method.



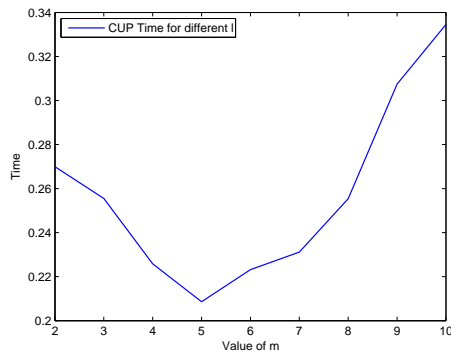
**Figure 8.** Convergence results of the new Smith method and standard Smith method.

The convergence results of the new Smith method with  $l = 6$  and Smith accelerative method (2.5) for Example 4.2 are displayed in Fig. 7. Similar to the discussion above, we choose the suitable matrix  $C$  such that the matrix  $X$  defined by (4.1)

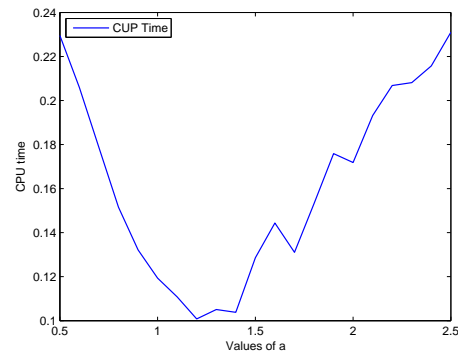
is the the solution of the equation  $AX + XB = C$ . We apply the Smith methods (2.4) and (3.3) with  $l = 6$  to seek for the unique solution  $X$ . Fig. 8 displays the convergence of the two Smith methods.

From the above two examples, one sees that the parameter selection affects the convergence rate of the Smith method. Furthermore, the next experiment will reveal the relationship between  $l$  and  $\alpha$  when we use the Smith method (3.2) for solving the discrete-time Sylvester equation.

For the Example 4.1, we continue to use the the Smith method (3.2) to solve the discrete-times Sylvester equation  $AXB + X = C$ . Let  $\alpha = 1.2$ , we compute the CPU-time when the parameter  $l$  takes different values and the common logarithm of the error satisfies  $\ln\|X - X_k\|_2 < -12$ . The result is shown in Fig. 9.



**Figure 9.** CPU time for the different values of the parameter  $l$ .



**Figure 10.** CPU time for the different values of the parameter  $\alpha$ .

Let  $l = 5$ , we also compute the CPU-time when the parameter  $\alpha$  takes different values and the common logarithm of the error satisfies  $\ln\|X - X_k\|_2 < -12$ . The result is shown in Fig. 10.

From the above two experiments, we also find that the Smith method (3.2) gets the desired solution rapidly and the parameter selection not only affects the convergence rate but also affects the CPU-time.

## 5. Conclusions

We have discussed the new version of the Smith method for solving the the discrete-time Sylvester equation  $AXB + X = C$  and Sylvester equation  $AX + XB = C$ , whose coefficient matrices are positive definite, and obtained some new results on the convergence rate and the prior estimations of the accurate solution  $X$ . Numerical examples are presented to show the effectiveness of the method we discussed.

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