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RIEMANN PROBLEM WITH DELTA INITIAL DATA FOR THE RELATIVISTIC CHAPLYGIN EULER EQUATIONS*

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Abstract In this paper, we study the Riemann problem with the initial data containing the Dirac delta function for the relativistic Chaplygin Euler equations. Under the generalized Rankine-Hugoniot conditions and entropy condition, we constructively obtain the global existence of generalized solutions including delta shock waves that explicitly exhibit four kinds of different structures. Moreover, we obtain the stability of generalized solutions by making use of the perturbation of the initial data.

Keywords Relativistic Euler equations, Riemann problem, delta shock wave, generalized-Rankine-Hugoniot conditions, entropy condition.

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1. Introduction

The Euler system of conservation laws of energy and momentum for a Chaplygin gas in special relativity reads (cf. [4, 6, 19])

$$\begin{cases} \left((p+\rho c^2)\frac{v^2}{c^2(c^2-v^2)}+\rho\right)_t+\left((p+\rho c^2)\frac{v}{c^2-v^2}\right)_x=0, \\ \left((p+\rho c^2)\frac{v}{c^2-v^2}\right)_t+\left((p+\rho c^2)\frac{v^2}{c^2-v^2}+p\right)_x=0, \end{cases} \tag{1.1}$$

where ρ and v represent the proper energy density, the pressure and the particle speed, respectively, and the constant c is the speed of light; the equations of state is

$$p(\rho)=-\frac{1}{\rho} \tag{1.2}$$

with $\rho>0$. System (1.1) models the dynamics of plane waves in special relativistic fluids in a two dimensional Minkowski time-space (x^0, x^1) :

$$divT=0, \tag{1.3}$$

where T is the stress-energy tensor for the fluid:

$$T^{ij}=(p+\rho c^2)v^iv^j+p\eta^{ij}, \tag{1.4}$$

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with all indices running from 0 to 1 with $x^0 = ct$. In (1.4),

$$\eta^{ij} = \eta_{ij} = diag(-1, 1),$$

denotes the flat Minkowski metric, v the 2-velocity of the fluid particle, and ρ the mass-energy density of the fluid as measured in units of mass in a reference frame moving with the fluid particle.

The Newtonian limit ($\frac{v}{c} \to 0$) of system (1.1) and (1.2) is the following classical Euler system for compressible isentropic fluids:

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ (\rho v)_t + (\rho v^2 + p)_x = 0, \end{cases} \tag{1.5}$$

In general, a solution to systems (1.1) or (1.5) strongly depends on the state equation $p = p(\rho)$, which expresses how the pressure p depends on the density ρ. For a polytropic gas, it can be expressed as

$$p(\rho) = k^2 \rho^\gamma (\gamma > 1, k < c), \tag{1.6}$$

for which systems (1.1) and (1.5) have extensively been studied by Chang and Hsiao [2], Chen etal. [3, 4], etc. For a Chaplygin gas, Brenier [1] firstly studied the 1-D Riemann problem and obtained solutions with concentration when initial data belong to a certain domain in the phase plane. Furthermore, Guo, Sheng, and Zhang [8] abandoned this constrain and constructively obtained the global solutions to the 1-D Riemann problem, in which the δ-shock developed. Moreover, they also systematically studied the 2-D Riemann problem for isentropic Chaplygin gas equations. For the 2-D case, we can also refer to [15] in which D. Serre studied the interaction of the pressure waves for the 2-D isentropic irrotational Chaplygin gas and constructively proved the existence of transonic solutions for two cases, saddle and vortex of 2-D Riemann problem. Recently, Wang and Zhang [21] studied the Riemann problem with delta initial data for the one-dimensional Chaplygin gas equations. However, it is noticed that few literatures contribute to system (1.1) for a Chaplygin gas so far. Recently, Cheng and Yang [6] proved the existence and uniqueness of delta shock solutions of Riemann problem for the relativistic Chaplygin Euler equations (1.1) and (1.2). In particular, the delta shock waves appear in the Riemann solutions of (1.1) and (1.2). From the mathematical point of view, a delta shock wave is more compressive than an ordinary shock wave in the sense that more characteristics enter the discontinuity line of the delta shock wave. From the physical point of view, a delta shock represents the process of concentration of the mass. As for delta shock waves, we refer readers to [5–12, 14, 16–18, 20–25] and the references cited therein for more details.

In the present paper, we consider the Riemann problem (1.1) and (1.2) with initial data

$$(\rho, v)(t = 0, x) = \begin{cases} (\rho_-, v_-), & x < 0, \\ (m_0\delta, v_0), & x = 0, \\ (\rho_+, v_+), & x > 0, \end{cases} \tag{1.7}$$

where δ is the standard Dirac delta function, and m_0, v_0, $\rho_\pm$ and $v_\pm$ are arbitrary constants. Because the delta shocks appear in Riemann solutions of (1.1) and

(1.2), it is natural to consider system (1.1) and (1.2) with initial data (1.7) which contains Dirac delta functions. This kind of Riemann problem, which is also called the Randon measure initial data problem, was studied in [12, 13, 21, 23, 25] for the zero-pressure flow in gas dynamics and other related equations.

In our paper, we will solve the Riemann problem (1.1), (1.2) and (1.7). Under the generalized Rankine-Hugoniot conditions and suitable entropy condition, we constructively obtain the global existence and uniqueness of generalized solutions including delta shocks that explicitly exhibit four kinds of different structure. However, much more different from [12, 23, 25], the δ-entropy condition is not enough to guarantee the uniqueness of generalized solutions. As in [21], we construct our solution on the basis of the stability theory of generalized solutions. Especially, when $m_0 = 0$, $v_0 = 0$, our results are consistent with those in [6].

The paper is organized as follows. In Section 2, we first present some preliminary knowledge about system (1.1) and (1.2); then display the Riemann solution of (1.1) and (1.2) with constant initial data. In Section 3, we construct the Riemann solution of (1.1) and (1.2) with delta initial data case by case.

2. Riemann problem with constant initial data

In this section, we briefly review the Riemann solution of (1.1) and (1.2) with initial data

$$(\rho, v)(0, x) = (\rho_\pm, v_\pm), \qquad \pm x > 0, \tag{2.1}$$

which is in physically relevant region

$$V = \{(\rho, v) : \rho > \frac{1}{c}, |v| < c\}, \tag{2.2}$$

where $\rho_\pm > 0$, the detailed study of which can be found in [6]. For more details about the study of the Riemann solution of (1.1) can be found in [4, 19].

The relativistic Euler equations for Chaplygin pressure (1.1) and (1.2) have two eigenvalues

$$\lambda_1 = \frac{c^2(v - \sqrt{p'(\rho)})}{c^2 - v\sqrt{p'(\rho)}}, \qquad \lambda_2 = \frac{c^2(v + \sqrt{p'(\rho)})}{c^2 + v\sqrt{p'(\rho)}},$$

with corresponding right eigenvectors

$$r_j(v)//\left(\frac{(-1)^j}{c^2 - v^2}, \frac{\sqrt{p'(\rho)}}{\rho c^2 + p}\right)^T, \quad j = 1, 2.$$

By a straightforward calculation, we obtain $\triangledown\lambda_i \cdot \overrightarrow{r_i} = 0 (i = 1, 2)$, which means system (1.1) and (1.2) is strictly hyperbolic and full linear degenerate.

For convenience in the next, we note

$$\lambda_1(\rho_-, v_-) = \frac{c^2(v_- - \frac{1}{\rho_-})}{c^2 - \frac{v_-}{\rho_-}}, \qquad \lambda_2(\rho_+, v_+) = \frac{c^2(v_+ + \frac{1}{\rho_+})}{c^2 + \frac{v_+}{\rho_+}}.$$

As usual, we seek the self-similar solution,

$$(\rho, v)(t, x) = (\rho, v)(\xi), \quad \xi = \frac{x}{t}.$$

Then the Riemann problem (1.1), (1.2) and (2.1) can be reduced to

$$\begin{cases} -\xi\Big((-\frac{1}{\rho}+\rho c^2)\frac{v^2}{c^2(c^2-v^2)}+\rho\Big)_\xi+\Big((-\frac{1}{\rho}+\rho c^2)\frac{v}{c^2-v^2}\Big)_\xi=0, \\ -\xi\Big((-\frac{1}{\rho}+\rho c^2)\frac{v}{c^2-v^2}\Big)_\xi+\Big((-\frac{1}{\rho}+\rho c^2)\frac{v^2}{c^2-v^2}-\frac{1}{\rho}\Big)_\xi=0, \end{cases} \tag{2.4}$$

with $(\rho, v)(\pm\infty) = (\rho_\pm, v_\pm)$.

For any smooth solution, system (2.4) can be written as

$$\begin{pmatrix} \frac{\left(\frac{1}{\rho^2}+c^2\right)c^2v-\xi\left(\frac{v^2}{\rho^2}+c^4\right)}{c^2(c^2-v^2)} & \frac{\left(-\frac{1}{\rho}+\rho c^2\right)(c^2+v^2-2v\xi)}{(c^2-v^2)^2} \\ \frac{\left(\frac{1}{\rho^2}+v^2\right)c^2-\xi v\left(\frac{1}{\rho^2}+c^2\right)}{c^2-v^2} & \frac{\left(-\frac{1}{\rho}+\rho c^2\right)(2vc^2-\xi c^2-\xi v^2)}{(c^2-v^2)^2} \end{pmatrix}\begin{pmatrix} \rho_\xi \\ v_\xi \end{pmatrix}=0, \tag{2.5}$$

which provides either general solutions (constant states)

$$(\rho, v)(\xi) = constant, \quad (\rho > \frac{1}{c})$$

or singular solutions

$$\xi = \lambda_1 = \lambda_1(\rho_-, v_-), \quad \xi = \lambda_2 = \lambda_2(\rho_-, v_-). \tag{2.6}$$

For a bounded discontinuity at $\xi = \sigma$, the Rankine-Hugoniot conditions holds

$$\begin{cases} -\sigma\Big[\big((-\frac{1}{\rho}+\rho c^2)\frac{v^2}{c^2(c^2-v^2)}+\rho\big)\Big]+\Big[\big(-\frac{1}{\rho}+\rho c^2\big)\frac{v}{c^2-v^2}\Big]=0, \\ -\sigma\Big[\big(-\frac{1}{\rho}+\rho c^2\big)\frac{v}{c^2-v^2}\Big]+\Big[\big(-\frac{1}{\rho}+\rho c^2\big)\frac{v^2}{c^2-v^2}-\frac{1}{\rho}\Big]=0, \end{cases} \tag{2.7}$$

where $[\rho] = \rho - \rho_-$, and σ is the velocity of the discontinuity. By solving (2.7), we have

$$\sigma = \lambda_1 = \lambda_1(\rho_-, v_-), \quad \sigma = \lambda_2 = \lambda_2(\rho_-, v_-). \tag{2.8}$$

From (2.6) and (2.8), we find that the rarefaction waves and the shock waves are coincident in the phase plane, which correspond to contact discontinuities:

$$J_1 : \xi = \frac{c^2(v-\frac{1}{\rho})}{c^2-\frac{v}{\rho}} = \frac{c^2(v_- - \frac{1}{\rho_-})}{c^2-\frac{v_-}{\rho_-}}, \tag{2.9}$$

$$J_2 : \xi = \frac{c^2(v+\frac{1}{\rho})}{c^2+\frac{v}{\rho}} = \frac{c^2(v_- + \frac{1}{\rho_-})}{c^2+\frac{v_-}{\rho_-}}. \tag{2.10}$$

In the phase plane, through the point (ρ_-, v_-), we draw a branch of curve (2.9) for $\rho > \frac{1}{c}$, which has the asymptotic line $v = \lambda_1(\rho_-, v_-)$, denote by J_1. Through the point (ρ_-, v_-), we draw a branch of curve (2.10) for $\rho > \frac{1}{c}$, which has the asymptotic line $v = \lambda_2(\rho_-, v_-)$, denote by J_2. Through the point $\left(\rho_-, \frac{\rho_- v_- c^2 - 2c^2 + \frac{v_-}{\rho_-}}{\rho_- c^2 - 2v_- + \frac{1}{\rho_-}}\right)$, we draw the contact discontinuity curve (2.10). Now, the phase is divided into five regions denoted by I, II, III, IV, and V.

For any given right state (ρ_+, v_+), we can construct Riemann solutions of (1.1) and (2.1). when $(\rho_+, v_+) \in I \cup II \cup III \cup IV$, the Riemann solution contains a 1-contact discontinuity, a 2-contact discontinuity, a nonvacuum intermediate constant state (ρ_*, v_*), where

$$\begin{cases} \frac{c^2(v_*+\frac{1}{\rho_*})}{c^2+\frac{v_*}{\rho_*}} = \frac{c^2(v_++\frac{1}{\rho_+})}{c^2+\frac{v_+}{\rho_+}}, \\ \frac{c^2(v_*-\frac{1}{\rho_*})}{c^2-\frac{v_*}{\rho_*}} = \frac{c^2(v_--\frac{1}{\rho_-})}{c^2-\frac{v_-}{\rho_-}}, \end{cases} \tag{2.11}$$

when $(\rho_+, v_+) \in V, i.e, \lambda_1(\rho_-, v_-) > \lambda_2(\rho_+, v_+)$, the characteristics originating from the origin $\Omega = \{(x,t) : \lambda_2(\rho_+, v_+) < \frac{x}{t} < \lambda_1(\rho_-, v_-)\}$ will overlap. So, singularity must happen in Ω. It is easy to know that the singularity is impossible to be a jump with finite amplitude because the Rankine-Hugoniot condition is not satisfied on the bounded jump. In other words, there is no solution which is piecewise smooth and bounded. Motivated by [18], we seek solutions with delta distribution at the jump. In fact, the appearance of delta shock wave is due to the overlap of linear degenerate characteristic lines.

For system (1.1) and (1.2), the definition of solution in the sense of distributions can be given as follows.

Definition 2.1. A pair (ρ, v) constitutes a solution of (1.1) and (1.2) in the sense of distributions if it satisfies

$$\begin{cases} \int_0^{+\infty}\int_{-\infty}^{+\infty}\Big(\big((-\frac{1}{\rho}+\rho c^2)\frac{v^2}{c^2(c^2-v^2)}+\rho\big)\varphi_t + \big((-\frac{1}{\rho}+\rho c^2)\frac{v}{c^2-v^2}\big)\varphi_x\Big)dxdt = 0, \\ \int_0^{+\infty}\int_{-\infty}^{+\infty}\Big(\big((-\frac{1}{\rho}+\rho c^2)\frac{v}{c^2-v^2}\big)\varphi_t + \big((-\frac{1}{\rho}+\rho c^2)\frac{v^2}{c^2-v^2}-\frac{1}{\rho}\big)\varphi_x\Big)dxdt = 0, \end{cases} \tag{2.12}$$

for all test functions $\varphi \in C_0^\infty(R^+ \times R^1)$.

Moreover, we define a two-dimensional weighted delta function in the following way.

Definition 2.2. A two-dimensional weighted delta function $w(s)\delta_L$ supported on a smooth curve $L = \{(t(s), x(s)) : a < s < b\}$ is defined by

$$\langle w(s)\delta_L, \varphi\rangle = \int_a^b w(s)\varphi(t(s), x(s))ds \tag{2.10}$$

for all test functions $\varphi \in C_0^\infty(R^2)$.

Let us consider a solution of (1.1) and (1.4) of the form

$$(\rho, v)(t,x) = \begin{cases} (\rho_-, v_-), & x < \sigma t, \\ (w(t)\delta(x-\sigma t), \sigma), & x = \sigma t, \\ (\rho_+, v_+), & x > \sigma t, \end{cases} \tag{2.14}$$

where $w(t)$ and σ are weight and velocity of Dirac delta wave respectively, satisfying the generalized Rankine-Hugoniot conditions

$$\begin{cases} \frac{d}{dt}\big(\frac{w(t)c^2}{c^2-v_\delta^2(t)}\big) = -v_\delta(t)\big[(-\frac{1}{\rho}+\rho c^2)\frac{v^2}{c^2(c^2-v^2)}+\rho\big] + \big[(-\frac{1}{\rho}+\rho c^2)\frac{v}{c^2-v^2}\big], \\ \frac{d}{dt}\big(\frac{w(t)c^2 v_\delta(t)}{c^2-v_\delta^2(t)}\big) = -v_\delta(t)\big[(-\frac{1}{\rho}+\rho c^2)\frac{v}{c^2-v^2}\big] + \big[(-\frac{1}{\rho}+\rho c^2)\frac{v^2}{c^2-v^2}-\frac{1}{\rho}\big], \end{cases} \tag{2.15}$$

where $[\rho]=\rho_- - \rho_+$, with initial data

$$(x,w)(0)=(0,0). \tag{2.16}$$

By solving (2.15), we can get

$$(x,w,v_\delta)(t)=\begin{cases}\big(\frac{G}{2F}t, F\big(1-(\frac{G}{2Fc})^2\big)t, \frac{G}{2F}\big), & E=0,\\ \big(\frac{F-\sqrt{F^2-EG}}{E}t, \sqrt{F^2-EG}\big(1-(\frac{F-\sqrt{F^2-EG}}{E})^2\big)t, \frac{F-\sqrt{F^2-EG}}{E}\big), & E\neq 0,\end{cases}$$

where $E=\big[(-\frac{1}{\rho}+\rho c^2)\frac{v^2}{c^2(c^2-v^2)}+\rho\big]$, $F=\big[(-\frac{1}{\rho}+\rho c^2)\frac{v}{c^2-v^2}\big]$, $G=\big[(-\frac{1}{\rho}+\rho c^2)\frac{v^2}{c^2-v^2}-\frac{1}{\rho}\big]$.

We also can justify the delta shock wave satisfies the entropy condition:

$$\lambda_2(\rho_+,v_+)<\sigma<\lambda_1(\rho_-,v_-).$$

which means that all the characteristics on both sides of the delta shock are incoming.

Thus, we obtain the global solution to the one-dimensional Riemann problem for the relativistic Chaplygin Euler equations (1.1) and (1.2).

3. Riemann problem with delta initial data

In this section, we construct Riemann solution of system (1.1) and (1.2) with initial data (1.7). According to the relations among $\lambda_1(\rho_-,v_-)$, v_0 and $\lambda_2(\rho_+,v_+)$, we discuss the Riemann problem case by case.

Case 3.1. $\lambda_1(\rho_-,v_-)\leq v_0\leq\lambda_2(\rho_+,v_+)$.

According to the value of v_0, we divide our discussion into the following three subcases.

Subcase 3.1.1. $\lambda_1(\rho_-,v_-)<v_0<\lambda_2(\rho_+,v_+)$.

To construct the solution of (1.1), (1.2) and (1.7), here we first consider the initial value problem (1.1) and (1.2) with the following initial data:

$$(\rho,v)(t=0,x)=\begin{cases}(\rho_-,v_-), & x<-\varepsilon,\\ (\frac{m_0}{2\varepsilon},v_0), & -\varepsilon<x<\varepsilon,\\ (\rho_+,v_+), & x>\varepsilon,\end{cases} \tag{3.1}$$

where $\varepsilon>0$ is sufficiently small. On the basis of the stability theory of weak solutions, if we obtain a solution of (1.1), (1.2) and (3.1), then by letting $\varepsilon\to 0$, we can get a solution of (1.1), (1.2) and (1.7).

Because $\lambda_1(\rho_-,v_-)<v_0<\lambda_2(\frac{m_0}{2\varepsilon},v_0)$, $\lambda_1(\frac{m_0}{2\varepsilon},v_0)<v_0<\lambda_2(\rho_+,v_+)$, when t is small, the solution of the initial value problem (1.1), (1.2) and (3.1) can be expressed as

$$(\rho_-,v_-)+\widehat{J}_1^-+(\widehat{\rho}_1,\widehat{v}_1)+\widehat{J}_2^-+(\frac{m_0}{2\varepsilon},v_0)+\widehat{J}_1^++(\widehat{\rho}_2,\widehat{v}_2)+\widehat{J}_2^++(\rho_+,v_+),$$

where "+" means "followed by", Moreover, from (2.11), we have

$$\begin{cases} \frac{c^2\left(\widehat{v}_1-\frac{1}{\widehat{\rho}_1}\right)}{c^2-\frac{\widehat{v}_1}{\widehat{\rho}_1}}=\frac{c^2\left(v_--\frac{1}{\rho_-}\right)}{c^2-\frac{v_-}{\rho_-}}, \\ \frac{c^2\left(\widehat{v}_1+\frac{1}{\widehat{\rho}_1}\right)}{c^2+\frac{\widehat{v}_1}{\widehat{\rho}_1}}=\frac{c^2\left(v_0+\frac{2\varepsilon}{m_0}\right)}{c^2+\frac{2\varepsilon v_0}{m_0}}, \end{cases}$$

and

$$\begin{cases} \frac{c^2\left(\widehat{v}_2-\frac{1}{\widehat{\rho}_2}\right)}{c^2-\frac{\widehat{v}_2}{\widehat{\rho}_2}}=\frac{c^2\left(v_0-\frac{2\varepsilon}{m_0}\right)}{c^2-\frac{2\varepsilon v_0}{m_0}}, \\ \frac{c^2\left(\widehat{v}_2+\frac{1}{\widehat{\rho}_2}\right)}{c^2+\frac{\widehat{v}_2}{\widehat{\rho}_2}}=\frac{c^2\left(v_++\frac{1}{\rho_+}\right)}{c^2+\frac{v_+}{\rho_+}}, \end{cases}$$

respectively. The propagation speed of $\widehat{J}_1^+$ is $\lambda_1(\frac{m_0}{2\varepsilon},v_0)$, and that of $\widehat{J}_2^-$ is $\lambda_2(\frac{m_0}{2\varepsilon},v_0)$. Because $\lambda_2(\frac{m_0}{2\varepsilon},v_0)>v_0>\lambda_1(\frac{m_0}{2\varepsilon},v_0)$, the contact discontinuity $\widehat{J}_2^-$ will overtake the contact discontinuity $\widehat{J}_1^+$ in a finite time. The intersection point (x_0,t_0) is determined by

$$\begin{cases} x_0+\varepsilon=\frac{c^2\left(v_0+\frac{2\varepsilon}{m_0}\right)}{c^2+\frac{2\varepsilon v_0}{m_0}}t_0, \\ x_0-\varepsilon=\frac{c^2\left(v_0-\frac{2\varepsilon}{m_0}\right)}{c^2-\frac{2\varepsilon v_0}{m_0}}t_0. \end{cases}$$

A simple calculation leads to

$$(x_0,t_0)=\left(\frac{v_0m_0\left(c^2-\frac{4\varepsilon^2}{m_0^2}\right)}{2(c^2-v_0^2)},\frac{m_0\left(c^4-\frac{4\varepsilon^2v_0^2}{m_0^2}\right)}{2c^2(c^2-v_0^2)}\right).$$

It is clear that a new Riemann problem is formed when two elementary waves intersect at a finite time. At the time $t=t_0$, we again have a Riemann problem with initial data:

$$(\rho,v)(x,t_0)=\begin{cases}(\widehat{\rho}_1,\widehat{v}_1), & x<x_0,\\ (\widehat{\rho}_2,\widehat{v}_2), & x>x_0.\end{cases} \tag{3.2}$$

Since $\lambda_1(\widehat{\rho}_1,\widehat{v}_1)=\lambda_1(\rho_-,v_-)<\lambda_2(\widehat{\rho}_2,\widehat{v}_2)=\lambda_2(\rho_+,v_+)$, the Riemann solution contains a 1-contact discontinuity $\widehat{J}_1$, a 2-contact discontinuity $\widehat{J}_2$ and an intermediate state $(\widehat{\rho}_3,\widehat{v}_3)$, where

$$\begin{cases} \frac{c^2\left(\widehat{v}_3-\frac{1}{\widehat{\rho}_3}\right)}{c^2-\frac{\widehat{v}_3}{\widehat{\rho}_3}}=\frac{c^2\left(v_--\frac{1}{\rho_-}\right)}{c^2-\frac{v_-}{\rho_-}}, \\ \frac{c^2\left(\widehat{v}_3+\frac{1}{\widehat{\rho}_3}\right)}{c^2+\frac{\widehat{v}_3}{\widehat{\rho}_3}}=\frac{c^2\left(v_++\frac{1}{\rho_+}\right)}{c^2+\frac{v_+}{\rho_+}}. \end{cases}$$

Therefore, when $t>t_0$, the solution of (1.1), (1.2) and (3.1) can be expressed as

$$(\rho_-,v_-)+\widehat{J}_1^-+(\widehat{\rho}_1,\widehat{v}_1)+\widehat{J}_1+(\widehat{\rho}_3,\widehat{v}_3)+\widehat{J}_2+(\widehat{\rho}_2,\widehat{v}_2)+\widehat{J}_2^++(\rho_+,v_+).$$

So far, we have completely constructed a solution of (1.1), (1.2) and (3.1). Letting $\varepsilon\to 0$, we obtain a solution of (1.1), (1.2) and (1.7), and we have

$$\begin{cases} \frac{c^2\left(\widehat{v}_1-\frac{1}{\widehat{\rho}_1}\right)}{c^2-\frac{\widehat{v}_1}{\widehat{\rho}_1}}=\frac{c^2\left(v_--\frac{1}{\rho_-}\right)}{c^2-\frac{v_-}{\rho_-}}, \\ \frac{c^2\left(\widehat{v}_1+\frac{1}{\widehat{\rho}_1}\right)}{c^2+\frac{\widehat{v}_1}{\widehat{\rho}_1}}=v_0, \end{cases}$$

$$\begin{cases} \frac{c^2\left(\widehat{v}_2-\frac{1}{\widehat{\rho}_2}\right)}{c^2-\frac{\widehat{v}_2}{\widehat{\rho}_2}} = v_0, \\ \frac{c^2\left(\widehat{v}_2+\frac{1}{\widehat{\rho}_2}\right)}{c^2+\frac{\widehat{v}_2}{\widehat{\rho}_2}} = \frac{c^2\left(v_++\frac{1}{\rho_+}\right)}{c^2+\frac{v_+}{\rho_+}}, \end{cases}$$

$$\begin{cases} \frac{c^2\left(\widehat{v}_3-\frac{1}{\widehat{\rho}_3}\right)}{c^2-\frac{\widehat{v}_3}{\widehat{\rho}_3}} = \frac{c^2\left(v_--\frac{1}{\rho_-}\right)}{c^2-\frac{v_-}{\rho_-}}, \\ \frac{c^2\left(\widehat{v}_3+\frac{1}{\widehat{\rho}_3}\right)}{c^2+\frac{\widehat{v}_3}{\widehat{\rho}_3}} = \frac{c^2\left(v_++\frac{1}{\rho_+}\right)}{c^2+\frac{v_+}{\rho_+}}. \end{cases}$$

$$(x_0, t_0) = \left(\frac{v_0 c^2 m_0}{2(c^2 - v_0^2)}, \frac{m_0 c^2}{2(c^2 - v_0^2)} \right),$$

and a δ-shock wave δS with

$$x(t) = v_0 t, \quad w(t) = m_0 - \frac{2c^2 - 2v_0^2}{c^2} t, \quad v_\delta(t) = v_0, \ \ for \ \ 0 \le t \le \frac{c^2 m_0}{2c^2 - 2v_0^2},$$

where $x(t)$, $w(t)$ and $v_\delta(t)$ respectively denote the location, weight and propagation speed of the δ-shock.

It is easy to check that δS satisfies (2.15), where $[\rho] = \widehat{\rho}_1 - \widehat{\rho}_2$, with initial data

$$(x, w, v_\delta)(0) = (0, m_0, v_0).$$

Subcase 3.1.2. $\lambda_1(\rho_-, v_-) = v_0 < \lambda_2(\rho_+, v_+)$.

Similar to subcase 3.1.1, we have the Riemann solution of (1.1), (1.2) and (1.7), where (ρ_*, v_*) is given by

$$\begin{cases} \frac{c^2(v_*-\frac{1}{\rho_*})}{c^2-\frac{v_*}{\rho_*}} = v_0, \\ \frac{c^2(v_*+\frac{1}{\rho_*})}{c^2+\frac{v_*}{\rho_*}} = \frac{c^2(v_++\frac{1}{\rho_+})}{c^2+\frac{v_+}{\rho_+}}, \end{cases}$$

and the δ-shock wave δS has the following expression:

$$x(t) = v_0 t, \quad w(t) = m_0, \quad v_\delta(t) = v_0, \quad \text{for} \quad t \ge 0.$$

Subcase 3.1.3. $\lambda_1(\rho_-, v_-) < v_0 = \lambda_2(\rho_+, v_+)$.

Similar to subcase 3.1.2, (ρ_*, v_*) is given by

$$\begin{cases} \frac{c^2(v_*-\frac{1}{\rho_*})}{c^2-\frac{v_*}{\rho_*}} = \frac{c^2(v_--\frac{1}{\rho_-})}{c^2-\frac{v_-}{\rho_-}}, \\ \frac{c^2(v_*+\frac{1}{\rho_*})}{c^2+\frac{v_*}{\rho_*}} = v_0, \end{cases}$$

and the δ-shock wave δS has the following expression:

$$x(t) = v_0 t, \quad w(t) = m_0, \quad v_\delta(t) = v_0, \quad \text{for} \quad t \ge 0.$$

Case 3.2. $v_0 < \lambda_1(\rho_-, v_-) < \lambda_2(\rho_+, v_+)$. (If $\lambda_1(\rho_-, v_-) < \lambda_2(\rho_+, v_+) < v_0$, then the structure of the solution is similar.)

It is seen that the particles $x_0 < 0$ collide with the particles $x_0 = 0$ at the start, while the particles $x_0 \le 0$ never collide with the particles $x_0 > 0$. Thus the solution can be expressed as

$$(\rho, v)(t,x) = \begin{cases} (\rho_-, v_-), & x < x(t), \\ (w(t)\delta(x - x(t)), v_\delta(t)), & x = x(t), \\ (\overline{\rho}, \overline{v})(t,x), & x(t) < x < \lambda_2(\rho_+, v_+)t, \\ (\rho_+, v_+), & x > \lambda_2(\rho_+, v_+)t, \end{cases} \tag{3.3}$$

where $(\overline{\rho}, \overline{v})(t,x) = (\rho_*, v_*)(\overline{t})$ along the straight line $\lambda_2(\rho_+, v_+)t - x = \lambda_2(\rho_+, v_+)\overline{t} - x(\overline{t})$, for $\overline{t} \geq 0$.

Here, $(\rho_*, v_*)(t)$ is the right state of the $\delta-$ shock wave δS difined by

$$\begin{cases} \frac{c^2\left(v_*(t) - \frac{1}{\rho_*(t)}\right)}{c^2 - \frac{v_*(t)}{\rho_*(t)}} = v_\delta(t), \\ \frac{c^2\left(v_*(t) + \frac{1}{\rho_*(t)}\right)}{c^2 + \frac{v_*(t)}{\rho_*(t)}} = \frac{c^2(v_+ + \frac{1}{\rho_+})}{c^2 + \frac{v_+}{\rho_+}}. \end{cases} \tag{3.4}$$

and δS satisfies the following generalized Rankine-Hugoniot conditions:

$$\begin{cases} \frac{dx(t)}{dt} = v_\delta(t), \\ \frac{d}{dt}\left(\frac{w(t)c^2}{c^2 - v_\delta^2(t)}\right) = v_\delta(t)E - F, \\ \frac{d}{dt}\left(\frac{w(t)c^2 v_\delta(t)}{c^2 - v_\delta^2(t)}\right) = v_\delta(t)F - G, \end{cases} \tag{3.5}$$

where $[\rho] = \rho_*(t) - \rho_-$, with initial data

$$(x, w, v_\delta)(0) = (0, m_0, v_0),$$

since $\frac{c^2\left(v_*(t) - \frac{1}{\rho_*(t)}\right)}{c^2 - \frac{v_*(t)}{\rho_*(t)}} = v_\delta(t)$, we have

$$v_*(t) - \frac{c^2 - v_*^2(t)}{c^2\rho_*(t) - v_*(t)} = v_\delta(t). \tag{3.7}$$

Next, We only need to solve the initial value problem (3.5) and (3.6).

From $(3.5)_2$, (3.7) and (3.4), we have

$$\begin{aligned} \frac{d}{dt}\left(\frac{w(t)c^2}{c^2 - v_\delta^2(t)}\right) &= \frac{(v_* - c^2\rho_*)(v_* - v_\delta(t))}{c^2 - v_*^2} - \frac{-\frac{v_-^2}{\rho_- c^2} + \rho_- c^2}{c^2 - v_-^2} v_\delta(t) + \frac{\left(-\frac{1}{\rho_-} + \rho_- c^2\right)v_-}{c^2 - v_-^2} \\ &= -1 - \frac{-\frac{v_-^2}{\rho_- c^2} + \rho_- c^2}{c^2 - v_-^2} v_\delta(t) + \frac{-\frac{v_-^2}{\rho_-} + \rho_- c^4}{c^2(c^2 - v_-^2)} \cdot \frac{c^2(v_- - \frac{1}{\rho_-})}{c^2 - \frac{v_-}{\rho_-}} + 1 \\ &= \frac{-\frac{v_-^2}{\rho_-} + \rho_- c^4}{c^2(c^2 - v_-^2)}\left(\frac{c^2(v_- - \frac{1}{\rho_-})}{c^2 - \frac{v_-}{\rho_-}} - v_\delta(t)\right). \end{aligned} \tag{3.8}$$

From $(3.5)_3$, (3.7) and (3.4), we have

$$
\begin{aligned}
&\tfrac{d}{dt}\Big(\tfrac{w(t)c^2v_\delta(t)}{c^2-v_\delta^2(t)}\Big)\\
&=\left(\tfrac{-\frac{v_*}{\rho_*}+c^2\rho_*v_*}{c^2-v_*^2}\right)-\tfrac{-\frac{v_-}{\rho_-}+c^2\rho_-v_-}{c^2-v_-^2}\left(v_\delta(t)-\left(\tfrac{-\frac{c^2}{\rho_*}+\rho_*c^2v_*^2}{c^2-v_*^2}\right)-\tfrac{-\frac{c^2}{\rho_-}+\rho_-c^2v_-^2}{c^2-v_-^2}\right)\\
&=\tfrac{\frac{1}{\rho_*}(c^2-v_*v_\delta t)+c^2\rho_*v_*(v_\delta(t)-v_*)}{c^2-v_*^2}-\tfrac{-\frac{v_-}{\rho_-}+c^2\rho_-v_-}{c^2-v_-^2}v_\delta(t)+\tfrac{-\frac{c^2}{\rho_-}+\rho_-c^2v_-^2}{c^2-v_-^2}\\
&=\left(-1-\tfrac{-\frac{v_-}{\rho_-}+c^2\rho_-v_-}{c^2-v_-^2}\right)v_\delta(t)+\tfrac{-\frac{c^2}{\rho_-}+\rho_-c^2v_-^2}{c^2-v_-^2}\\
&=\tfrac{c^2(v_-+\frac{1}{\rho_-})}{c^2+\frac{v_-}{\rho_-}}\cdot\tfrac{-\frac{v_-^2}{\rho_-}+\rho_-c^4}{c^2(c^2-v_-^2)}\left(\tfrac{c^2(v_--\frac{1}{\rho_-})}{c^2-\frac{v_-}{\rho_-}}-v_\delta(t)\right).
\end{aligned}
\tag{3.9}
$$

Combining (3.8) and (3.9), we have

$$
\frac{d}{dt}\Big(\frac{w(t)c^2v_\delta(t)}{c^2-v_\delta^2(t)}\Big)=\lambda_2(\rho_-,v_-)\frac{d}{dt}\Big(\frac{w(t)c^2}{c^2-v_\delta^2(t)}\Big). \tag{3.10}
$$

Integrating (3.10) from 0 to t, we have

$$
\begin{aligned}
\tfrac{w(t)c^2}{c^2-v_\delta^2(t)}(\lambda_2(\rho_-,v_-)-v_\delta(t))&=\tfrac{m_0c^2}{c^2-v_0^2}(\lambda_2(\rho_-,v_-)-v_0)\\
&>\tfrac{m_0c^2}{c^2-v_0^2}(\lambda_1(\rho_-,v_-)-v_0)>0.
\end{aligned}
\tag{3.11}
$$

Combining (3.11) and (3.8), we obtain

$$
\frac{d}{dt}\Big(\frac{w(t)c^2}{c^2-v_\delta^2(t)}\Big)=\frac{A-\frac{2w(t)c^2}{c^2-v_\delta^2(t)}}{\frac{w(t)c^2}{c^2-v_\delta^2(t)}}, \tag{3.12}
$$

where

$$
A=\frac{-\frac{v_-^2}{\rho_-}+\rho_-c^4}{c^2(c^2-v_-^2)}\cdot\frac{m_0c^2}{c^2-v_0^2}(\lambda_2(\rho_-,v_-)-v_0). \tag{3.13}
$$

In addition, the delta shock wave should satisfy the entropy condition:

$$
v_0<v_\delta(t)<\lambda_1(\rho_-,v_-).
$$

This, together with (3.8), (3.11) implies

$$
\frac{d}{dt}\Big(\frac{w(t)c^2}{c^2-v_\delta^2(t)}\Big)>0. \tag{3.14}
$$

and

$$
\frac{w(t)c^2}{c^2-v_\delta^2(t)}>0 \tag{3.15}
$$

Combining (3.14) and (3.15), we have

$$
0<\frac{w(t)c^2}{c^2-v_\delta^2(t)}<\frac{A}{2}. \tag{3.16}
$$

Setting
$$\frac{w(t)c^2}{c^2-v_\delta^2(t)}=H(t). \tag{3.17}$$
Solving (3.12) with initial data $H(0)=H_0=\frac{m_0c^2}{c^2-v_0^2}$, we have
$$H_0-H+\frac{1}{2}A(\ln(A-2H_0)-\ln(A-2H))=2t. \tag{3.18}$$
Letting $f(H)=H_0-H+\frac{1}{2}A(\ln(A-2H_0)-\ln(A-2H))$, then, from (3.16) we have
$$f'(H)=\frac{2H}{A-2H}>0. \tag{3.19}$$
Thus, from (3.11), there exists a unique inverse function $f^{-1}(2t)$, such that $H=H(t)=f^{-1}(2t)$. Then from (3.18), we have
$$w(t)=\frac{f^{-1}(2t)(c^2-v_\delta^2(t))}{c^2}.$$
From (3.11), we obtain
$$v_\delta(t)=\lambda_2(\rho_-,v_-)-\frac{\frac{m_0c^2}{c^2-v_0^2}(\lambda_2(\rho_-,v_-)-v_0)}{f^{-1}(2t)}. \tag{3.20}$$
Furthermore, we have
$$x(t)=\int_0^t v_\delta(\tau)d\tau.$$

Remark 3.1. From (3.18), we have $\lim\limits_{t\to+\infty}H(t)=A$. Then from (3.20), we have $\lim\limits_{t\to+\infty}v_\delta(t)=\lambda_1(\rho_-,v_-)<\lambda_2(\rho_+,v_+)$. This implies that the delta shock wave δS will never overtake those 2-contact discontinuities.

Case 3.3. $\lambda_2(\rho_+,v_+)<v_0<\lambda_1(\rho_-,v_-)$.

This is a typical case, a delta shock wave emits from the origin. We seek the solution in the following form
$$(\rho,v)(t,x)=\begin{cases}(\rho_-,v_-), & x<x(t),\\ (w(t)\delta(x-x(t)),v_\delta(t)), & x=x(t),\\ (\rho_+,v_+), & x>x(t),\end{cases} \tag{3.21}$$
which satisfies (2.15), where $[\rho]=\rho_--\rho_+$, with initial data
$$(x,w,v_\delta)(0)=(0,m_0,v_0). \tag{3.22}$$

Now, we are going to solve the initial value problem (3.21) and (3.22).

Integrating (2.15) from 0 to t with initial data (3.22), we have
$$\begin{cases}\frac{w(t)c^2}{c^2-v_\delta^2(t)}-\frac{m_0c^2}{c^2-v_0^2}=-xE+Ft,\\ \frac{w(t)c^2v_\delta(t)}{c^2-v_\delta^2(t)}-\frac{m_0c^2v_0}{c^2-v_0^2}=-xF+Gt.\end{cases} \tag{3.23}$$

Cancelling $w(t)$ in (3.23), we have

$$\frac{m_0c^2v_\delta(t)}{c^2-v_0^2}-\frac{m_0c^2v_0}{c^2-v_0^2}=Exv_\delta(t)-Fv_\delta(t)t-Fx+Gt$$

or

$$\frac{d}{dt}\left\{\frac{Ex^2}{2}-\big(\frac{m_0c^2}{c^2-v_0^2}+Ft\big)x+\frac{1}{2}Gt^2+\frac{m_0c^2v_0}{c^2-v_0^2}t\right\}=0. \tag{3.24}$$

Integrating (3.24) from 0 to t, we obtain

$$\frac{Ex^2}{2}-\left(\frac{m_0c^2}{c^2-v_0^2}+Ft\right)x+\frac{1}{2}Gt^2+\frac{m_0c^2v_0}{c^2-v_0^2}t=0. \tag{3.25}$$

From $\lambda_2(\rho_+,v_+)<v_0<\lambda_1(\rho_-,v_-)$, we know that

$$\frac{c^2(v_+-\frac{1}{\rho_+})}{c^2-\frac{v_+}{\rho_+}}<\frac{c^2(v_++\frac{1}{\rho_+})}{c^2+\frac{v_+}{\rho_+}}<v_0<\frac{c^2(v_--\frac{1}{\rho_-})}{c^2-\frac{v_-}{\rho_-}}<\frac{c^2(v_-+\frac{1}{\rho_-})}{c^2+\frac{v_-}{\rho_-}}.$$

So one can get that

$$A_1:=-c^2v_+\rho_+\rho_-+v_+\rho_+v_-+c^2\rho_--v_-+c^2v_-\rho_+\rho_--v_+v_-\rho_--c^2\rho_++v_+>0,$$

$$A_2:=-c^2v_+\rho_+\rho_--v_+\rho_+v_--c^2\rho_--v_-+c^2v_-\rho_+\rho_-+v_+v_-\rho_-+c^2\rho_++v_+>0,$$

$$A_3:=-c^2v_+\rho_+\rho_-+v_+\rho_+v_--c^2\rho_-+v_-+c^2v_-\rho_+\rho_-+v_+v_-\rho_--c^2\rho_+-v_+\geq 0,$$

Thus,

$$F^2-EG=\frac{A_1A_2}{\rho_+\rho_-(c-v_-)(c+v_-)(c-v_+)(c+v_+)}>0. \tag{3.26}$$

$$\begin{aligned}F-v_0E=&\left(\frac{\big(-\frac{1}{\rho_-}+c^2\rho_-\big)v_-}{c^2-v_-^2}-\frac{\big(-\frac{1}{\rho_+}+c^2\rho_+\big)v_+}{c^2-v_+^2}\right)\\&-v_0\left(\frac{\big(-\frac{1}{\rho_-}+c^2\rho_-\big)v_-^2}{c^2(c^2-v_-^2)}+\rho_--\frac{\big(-\frac{1}{\rho_+}+c^2\rho_+\big)v_+^2}{c^2(c^2-v_+^2)}-\rho_+\right)\\=&\frac{\frac{v_+(c^2-v_+v_0)}{\rho_+}+c^4\rho_+(v_0-v_+)}{c^2(c^2-v_+^2)}+\frac{\big(-\frac{1}{\rho_-}+c^2\rho_-\big)v_-}{c^2-v_-^2}-\frac{-\frac{v_-^2}{\rho_-}+\rho_-c^4}{c^2(c^2-v_-^2)}v_0>0.\end{aligned} \tag{3.27}$$

If $E\neq 0$, we have

$$\begin{aligned}\triangle=&\left(\frac{m_0c^2}{c^2-v_0^2}+Ft\right)^2-2E\left(\frac{1}{2}Gt^2+\frac{m_0c^2v_0}{c^2-v_0^2}t\right)\\=&\left(\frac{m_0c^2}{c^2-v_0^2}\right)^2+\frac{2m_0c^2}{c^2-v_0^2}(F-v_0E)+(F^2-EG)t^2>0,\end{aligned}$$

we solve Eq. (3.25) to obtain

$$x(t)=\frac{Ft+\frac{m_0c^2}{c^2-v_0^2}\pm\sqrt{\triangle}}{E}. \tag{3.28}$$

From (3.28), we have

$$v_\delta(t) = \frac{F \pm \sqrt{\triangle\left(\frac{m_0 c^2}{c^2 - v_0^2}(F - v_0 E) + t(F^2 - EG)\right)}}{E}.$$

Thus,

$$\lim_{t\to+\infty} v_\delta(t) = \frac{F \pm \sqrt{F^2 - EG}}{E}.$$

Substituting (3.28) into $(3.23)_1$, we have

$$w(t) = \mp \frac{c^2 - v_\delta^2(t)}{c^2}\sqrt{\triangle}.$$

In addition, to guarantee uniqueness, the delta shock wave should satisfy the entropy condition:

$$\lambda_2(\rho_+, v_+) < \lim_{t\to\infty} v_\delta(t) < \lambda_1(\rho_-, v_-).$$

So, we have

$$w(t) = \frac{c^2 - v_\delta^2(t)}{c^2}\sqrt{\triangle}.$$

Then, we obtain a unique solution

$$\begin{cases} x(t) = \frac{Ft + \frac{m_0 c^2}{c^2 - v_0^2} - \sqrt{\triangle}}{E}, \\ v_\delta(t) = \frac{F - \sqrt{\triangle\left(\frac{m_0 c^2}{c^2 - v_0^2}(F - v_0 E) + t(F^2 - EG)\right)}}{E}, \\ w(t) = \frac{c^2 - v_\delta^2(t)}{c^2}\sqrt{\triangle}. \end{cases} \tag{3.29}$$

If $E = 0$, solving (3.25), we have

$$x(t) = \frac{\frac{m_0 c^2 v_0}{c^2 - v_0^2} t + \frac{1}{2} G t^2}{\frac{m_0 c^2}{c^2 - v_0^2} + Ft}.$$

Then, we have

$$w(t) = \frac{c^2 - v_\delta^2(t)}{c^2}\left(\frac{m_0 c^2}{c^2 - v_0^2} + Ft\right),$$

and

$$v_\delta(t) = \frac{\frac{m_0^2 c^4 v_0}{(c^2 - v_0^2)^2} + \frac{1}{2} F G t^2 + \frac{m_0 c^2}{c^2 - v_0^2} G t}{\left(\frac{m_0 c^2}{c^2 - v_0^2} + Ft\right)^2}.$$

Remark 3.2. It is seen that

$$\lim_{t\to\infty} v_\delta(t) = \begin{cases} \frac{G}{2F}, & E = 0, \\ \frac{F - \sqrt{F^2 - EG}}{E}, & E \neq 0. \end{cases}$$

So, the delta-shock satisfies the entropy condition

$$\lambda_2(\rho_+, v_+) < \lim_{t\to\infty} v_\delta(t) < \lambda_1(\rho_-, v_-),$$

which means that all the characteristics on both sides of the delta shock are incoming.

Remark 3.3. If $m_0 = 0$, $v_0 = 0$, then

$$(x, w, v_\delta)(t) = \begin{cases} \Big(\frac{G}{2F}t, F\big(1 - (\frac{G}{2Fc})^2\big)t, \frac{G}{2F}\Big), & E = 0, \\ \Big(1 - (\frac{F-\sqrt{F^2-EG}}{E})^2\big)t, \frac{F-\sqrt{F^2-EG}}{E}\Big), & E \neq 0. \end{cases}$$

This is consistent with the results in Cheng and Yang [6]. It implies that the solution constructed here is stable under some perturbations.

Case 3.4. $v_0 < \lambda_2(\rho_+, v_+) < \lambda_1(\rho_-, v_-)$. (If $\lambda_2(\rho_+, v_+) < \lambda_1(\rho_-, v_-) < v_0$, then the structure of the solution is similar.)

Similar to the analysis in Case 3.2, we know that, in this case, when t is small enough, the solution is the same as that in case 3.2. From (3.15), (3.17) and (3.20), we have

$$v_\delta'(t) = \frac{\frac{m_0 c^2}{c^2 - v_0^2}(\lambda_2(\rho_-, v_-) - v_0)}{H^2}\frac{dH}{dt} > 0, \qquad \text{for} \quad t > 0, \tag{3.30}$$

which shows that $v_\delta(t)$ is a strictly monotonic increasing function of t for $t \in [0, +\infty)$. On the other hand, $v_\delta(0) = v_0$, $\lim\limits_{t\to+\infty} v_\delta(t) = \frac{c^2(v_- - \frac{1}{\rho_-})}{c^2 - \frac{v_-}{\rho_-}} = \lambda_1(\rho_-, v_-)$ and $v_0 < \lambda_2(\rho_+, v_+) < \lambda_1(\rho_-, v_-)$. Thus we can apply the intermediate value theorem in mathematical analysis, and conclude that there exists a unique $t^* \in [0, +\infty)$ such that $v_\delta(t^*) = \lambda_2(\rho_+, v_+)$. When $0 \le t \le t^*$, the solution is the same as that in case 3.2, which can be expressed as

$$(\rho, v)(t, x) = \begin{cases} (\rho_-, v_-), & x < x(t), \\ (w(t)\delta(x - x(t)), v_\delta(t)), & x = x(t), \\ (\overline{\rho}, \overline{v})(t, x), & x(t) < x < \lambda_2(\rho_+, v_+)t, \\ (\rho_+, v_+), & x > \lambda_2(\rho_+, v_+)t, \end{cases} \tag{3.31}$$

where $x(t)$, $w(t)$ and $v_\delta(t)$ are the same as those in case 3.2. When $t > t^*$, the delta shock wave will overtake all the 2-contact discontinuities and penetrate them in finite time. Suppose that the penetration ends at time $t = t^\#$.

When $t^* \le t < t^\#$, the solution can be written in the following form

$$(\rho, v)(t, x) = \begin{cases} (\rho_-, v_-), & x < x^1(t), \\ (w^1(t)\delta(x - x^1(t)), v_\delta^1(t)), & x = x^1(t), \\ (\overline{\rho}, \overline{v})(t, x), & x^1(t) < x < \lambda_2(\rho_+, v_+)t, \\ (\rho_+, v_+), & x^1(t) > \lambda_2(\rho_+, v_+)t. \end{cases} \tag{3.32}$$

Moreover, for any point $(x^1(t), t)$ on the delta shock wave δS_1, there exists a unique point $(x(t_1), t_1)$ $(0 \le t_1 \le t^*)$ on the delta shock wave δS, such that

$$\lambda_2(\rho_+, v_+)t - x^1(t) = \lambda_2(\rho_+, v_+)t_1 - x(t_1). \tag{3.33}$$

Let $(x, w, v_\delta)(t^*) = (x^*, w^*, v_\delta^*)$, then $v_\delta^* = \lambda_2(\rho_+, v_+)$, and the δ-shock wave δS_1 satisfies the following generalized Rankine-Hugoniot conditions:

$$\begin{cases} \frac{dx^1(t)}{dt} = v_\delta^1(t), \\ \frac{d}{dt}\left(\frac{w^1(t)c^2}{c^2-\left(v_\delta^1(t)\right)^2}\right) = v_\delta^1(t)\big[(-\frac{1}{\rho} + \rho c^2)\frac{v^2}{c^2(c^2-v^2)} + \rho\big] - \big[(-\frac{1}{\rho} + \rho c^2)\frac{v}{c^2-v^2}\big], \\ \frac{d}{dt}\left(\frac{w^1(t)c^2v_\delta^1(t)}{c^2-\left(v_\delta^1(t)\right)^2}\right) = v_\delta^1(t)\big[(-\frac{1}{\rho} + \rho c^2)\frac{v}{c^2-v^2}\big] - \big[(-\frac{1}{\rho} + \rho c^2)\frac{v^2}{c^2-v^2} - \frac{1}{\rho}\big], \end{cases} \tag{3.34}$$

where$[\rho] = \rho_*(t_1) - \rho_-$, with initial data

$$(x^1, w^1, v_\delta^1)(t^*) = (x^*, w^*, v_\delta^*). \tag{3.35}$$

Here, $(\rho_*, v_*)(t_1)$ is the right state of the δ-shock wave δS_1 defined by

$$\begin{cases} \frac{c^2\left(v_*(t_1) - \frac{1}{\rho_*(t_1)}\right)}{c^2 - \frac{v_*(t_1)}{\rho_*(t_1)}} = v_\delta^1(t_1), \\ \frac{c^2\left(v_*(t_1) + \frac{1}{\rho_*(t_1)}\right)}{c^2 + \frac{v_*(t_1)}{\rho_*(t_1)}} = \frac{c^2(v_+ + \frac{1}{\rho_+})}{c^2 + \frac{v_+}{\rho_+}}. \end{cases} \tag{3.36}$$

When $t^\# \le t < +\infty$, the solution can be expressed as

$$(\rho, v)(t, x) = \begin{cases} (\rho_-, v_-), & x < x^2(t), \\ (w^2(t)\delta(x - x^2(t)), v_\delta^2(t)), & x = x^2(t), \\ (\rho_+, v_+), & x > x^2(t). \end{cases} \tag{3.37}$$

It is easy to know that ρ_* is a function of t_1 . Next, our aim is to express ρ_* as a function of t . Integrating (3.8) from 0 to t_1, we have

$$\frac{w(t_1)c^2}{c^2 - v_\delta^2(t_1)} - \frac{m_0c^2}{c^2 - v_0^2} = \frac{-\frac{v_-^2}{\rho_-} + c^4\rho_-}{c^2(c^2 - v_-^2)}(\lambda_2(\rho_-, v_-)t_1 - x(t_1)) - 2t_1. \tag{3.38}$$

From (3.18), we have

$$\frac{w(t_1)c^2}{c^2 - v_\delta^2(t_1)} - \frac{m_0c^2}{c^2 - v_0^2} + \frac{1}{2}A\Bigg(\ln\left(A - \frac{2w(t_1)c^2}{c^2 - v_\delta^2(t_1)}\right) - \ln\left(A - \frac{2m_0c^2}{c^2 - v_0^2}\right)\Bigg) = -2t_1. \tag{3.39}$$

Letting $a = \frac{-\frac{v_-^2}{\rho_-} + c^4\rho_-}{c^2(c^2 - v_-^2)}(\lambda_2(\rho_-, v_-) - \lambda_2(\rho_+, v_+))$, then calculating (3.39)$\times(\frac{a}{2} - 1)$+(3.38), we have

$$\frac{w(t_1)c^2}{c^2 - v_\delta^2(t_1)}\cdot\frac{a}{2} - \frac{m_0c^2}{c^2 - v_0^2}\cdot\frac{a}{2} + \frac{1}{2}(\frac{a}{2} - 1)A\Bigg(\ln\left(A - \frac{2w(t_1)c^2}{c^2 - v_\delta^2(t_1)}\right) - \ln\left(A - \frac{2m_0c^2}{c^2 - v_0^2}\right)\Bigg)$$

$$= \frac{-\frac{v_-^2}{\rho_-} + c^4\rho_-}{c^2(c^2 - v_-^2)}(\lambda_2(\rho_+, v_+)t_1 - x(t_1)). \tag{3.40}$$

Substituting (3.33) into (3.40), we have

$$\frac{w(t_1)c^2}{c^2 - v_\delta^2(t_1)} \cdot \frac{a}{2} - \frac{m_0c^2}{c^2 - v_0^2} \cdot \frac{a}{2} + \frac{1}{2}(\frac{a}{2} - 1)A\Big(\ln\Big(A - \frac{2w(t_1)c^2}{c^2 - v_\delta^2(t_1)}\Big) - \ln\Big(A - \frac{2m_0c^2}{c^2 - v_0^2}\Big)\Big)$$

$$= \frac{-\frac{v_-^2}{\rho_-} + c^4\rho_-}{c^2(c^2 - v_-^2)}(\lambda_2(\rho_+, v_+)t - x^1(t)). \tag{3.41}$$

By straightforward calculation, we have

$$\frac{-\frac{v^2}{\rho} + c^4\rho}{c^2(c^2 - v^2)} \cdot \frac{\frac{c^4}{\rho} - \frac{c^2v^2}{\rho}}{(c^2 + \frac{v}{\rho})(c^2 - \frac{v}{\rho})} = 1. \tag{3.42}$$

From (3.4), (3.11) and (3.42), we have

$$\begin{aligned}
\frac{w(t_1)c^2}{c^2 - v_\delta^2(t_1)} &= \frac{\frac{m_0c^2}{c^2 - v_0^2}(\lambda_2(\rho_-, v_-) - v_0)}{\frac{c^2(v_- + \frac{1}{\rho_-})}{c^2 + \frac{v_-}{\rho_-}} - v_\delta(t_1)} = \frac{A}{B_-\Big(\frac{c^2(v_- + \frac{1}{\rho_-})}{c^2 + \frac{v_-}{\rho_-}} - \frac{c^2(v_* - \frac{1}{\rho_*})}{c^2 - \frac{v_*}{\rho_*}}\Big)} \\
&= \frac{A}{B_-\Big(\frac{c^2(v_- + \frac{1}{\rho_-})}{c^2 + \frac{v_-}{\rho_-}} - \frac{c^2(v_* + \frac{1}{\rho_*})}{c^2 + \frac{v_*}{\rho_*}} + \frac{2\big(\frac{c^4}{\rho_*} - \frac{c^2v_*^2}{\rho_*}\big)}{(c^2 + \frac{v_*}{\rho_*})(c^2 - \frac{v_*}{\rho_*})}\Big)} \\
&= \frac{A}{B_-\Big(\frac{c^2(v_- + \frac{1}{\rho_-})}{c^2 + \frac{v_-}{\rho_-}} - \frac{c^2(v_+ + \frac{1}{\rho_+})}{c^2 + \frac{v_+}{\rho_+}} + \frac{2}{B_*}\Big)},
\end{aligned} \tag{3.43}$$

where

$$B = \frac{-\frac{v^2}{\rho} + c^4\rho}{c^2(c^2 - v^2)}, \quad B_- = \frac{-\frac{v_-^2}{\rho_-} + c^4\rho_-}{c^2(c^2 - v_-^2)}, \quad B_* = \frac{-\frac{v_*^2}{\rho_*} + c^4\rho_*}{c^2(c^2 - v_*^2)}. \tag{3.44}$$

Substituting (3.43) into (3.41), we have

$$F(\frac{1}{B_*}) = \lambda_2(\rho_+, v_+)t - x^1(t), \tag{3.45}$$

where

$$\begin{aligned}
F(s) = {} & \tfrac{a}{2B_-} \cdot \tfrac{A}{B_-(\lambda_2(\rho_-, v_-) - \lambda_2(\rho_+, v_+) + 2s)} \\
& + \tfrac{1}{2B_-}(\tfrac{a}{2} - 1)A\ln\big(A - \tfrac{2A}{B_-(\lambda_2(\rho_-, v_-) - \lambda_2(\rho_+, v_+) + 2s)}\big) \\
& - \tfrac{m_0c^2}{B_-(c^2 - v_0^2)} \cdot \tfrac{a}{2} - \tfrac{1}{2B_-}(\tfrac{a}{2} - 1)A\ln\big(A - \tfrac{2m_0c^2}{c^2 - v_0^2}\big).
\end{aligned}$$

For $s > 0$,

$$F'(s) = -\frac{4AB_-s}{y^2(s)(y(s) - 2)} < 0, \tag{3.46}$$

where $y(s) = B_-(\lambda_2(\rho_-, v_-) - \lambda_2(\rho_+, v_+) + 2s)$, which shows that $F(s)$ is a strictly monotonic decreasing function of s for $s \in [0, +\infty)$.

And from (3.45) together with (3.46), we have

$$\frac{1}{B_*} = G(\lambda_2(\rho_+, v_+)t - x^1(t)), \tag{3.47}$$

where $G = F^{-1}$ and $\frac{1}{G}$ is integrable.

From (3.34), (3.36), (3.42) and (3.44), we have

$$\begin{aligned}
\frac{d}{dt}\Big(\frac{w^1c^2}{c^2-(v_\delta^1)^2}\Big) &= v_\delta^1\left(\frac{-\frac{v_*^2}{\rho_*}+c^4\rho_*}{c^2(c^2-v_*^2)} - \frac{-\frac{v_-^2}{\rho_-}+c^4\rho_-}{c^2(c^2-v_-^2)}\right) - \left(\frac{-\frac{v_*}{\rho_*}+c^2\rho_*v_*}{c^2-v_*^2} - \frac{-\frac{v_-}{\rho_-}+c^2\rho_-v_-}{c^2-v_-^2}\right)\\
&= \frac{-\frac{v_*^2}{\rho_*}+c^4\rho_*}{c^2(c^2-v_*^2)}(v_\delta^1 - \lambda_2(\rho_+, v_+)) - \frac{-\frac{v_-^2}{\rho_-}+c^4\rho_-}{c^2(c^2-v_-^2)}v_\delta^1 + \frac{-\frac{v_-^2}{\rho_-}+c^4\rho_-}{c^2(c^2-v_-^2)}\cdot\lambda_2(\rho_-, v_-)\\
&= B_*(v_\delta^1 - \lambda_2(\rho_+, v_+)) - B_-v_\delta^1 + B_-\lambda_2(\rho_-, v_-)
\end{aligned} \tag{3.48}$$

and

$$\begin{aligned}
&\frac{d}{dt}\Big(\frac{w^1c^2v_\delta^1}{c^2-(v_\delta^1)^2}\Big)\\
&= v_\delta^1\left(\frac{-\frac{v_*}{\rho_*}+c^2\rho_*v_*}{c^2-v_*^2} - \frac{-\frac{v_-}{\rho_-}+c^2\rho_-v_-}{c^2-v_-^2}\right) - \left(\frac{-\frac{c^2}{\rho_*}+\rho_*c^2v_*^2}{c^2-v_*^2} - \frac{-\frac{c^2}{\rho_-}+\rho_-c^2v_-^2}{c^2-v_-^2}\right)\\
&= v_\delta^1\left(\frac{-\frac{v_*}{\rho_*}+c^2\rho_*v_*}{c^2-v_*^2} - \frac{-\frac{v_-}{\rho_-}+c^2\rho_-v_-}{c^2-v_-^2}\right) - \left(\frac{-\frac{v_*}{\rho_*}+c^2\rho_*v_*}{c^2-v_*^2}\cdot\frac{c^2(v_*+\frac{1}{\rho_*})}{c^2+\frac{v_*}{\rho_*}}\right.\\
&\quad\left.-\frac{c^2(v_*+\frac{1}{\rho_*})}{c^2+\frac{v_*}{\rho_*}} - \frac{-\frac{v_-}{\rho_-}+c^2\rho_-v_-}{c^2-v_-^2}\cdot\frac{c^2(v_-+\frac{1}{\rho_-})}{c^2+\frac{v_-}{\rho_-}} + \frac{c^2(v_-+\frac{1}{\rho_-})}{c^2+\frac{v_-}{\rho_-}}\right)\\
&= \left(B_*\cdot\frac{c^2(v_*+\frac{1}{\rho_*})}{c^2+\frac{v_*}{\rho_*}} - 1\right)\left(v_\delta^1 - \frac{c^2(v_*+\frac{1}{\rho_*})}{c^2+\frac{v_*}{\rho_*}}\right) + \frac{c^2(v_*+\frac{1}{\rho_*})}{c^2+\frac{v_*}{\rho_*}} - v_\delta^1\left(B_-\cdot\frac{c^2(v_-+\frac{1}{\rho_-})}{c^2+\frac{v_-}{\rho_-}} - 1\right)\\
&\quad+\left(B_-\cdot\frac{c^2(v_-+\frac{1}{\rho_-})}{c^2+\frac{v_-}{\rho_-}} - 1\right)\frac{c^2(v_-+\frac{1}{\rho_-})}{c^2+\frac{v_-}{\rho_-}} - \frac{c^2(v_-+\frac{1}{\rho_-})}{c^2+\frac{v_-}{\rho_-}}\\
&= B_*\lambda_2(\rho_+, v_+)(v_\delta^1 - \lambda_2(\rho_+, v_+)) - B_-v_\delta^1\lambda_2(\rho_-, v_-) + B_-(\lambda_2(\rho_-, v_-))^2\\
&\quad+2\lambda_2(\rho_+, v_+) - 2\lambda_2(\rho_-, v_-).
\end{aligned} \tag{3.49}$$

Substituting (3.47) into (3.48) and (3.49) respectively, and integrating from t^* to t, we have

$$\begin{aligned}
&\frac{w^1c^2}{c^2-(v_\delta^1)^2} - \frac{w^*c^2}{c^2-(v_\delta^*)^2}\\
&= \int_{t^*}^t \frac{v_\delta^1(\tau)-\lambda_2(\rho_+,v_+)}{G(\lambda_2(\rho_+,v_+)\tau-x^1(\tau))}d\tau - B_-(x^1 - x^*) + B_-\lambda_2(\rho_-, v_-)(t - t^*),
\end{aligned} \tag{3.50}$$

and

$$\begin{aligned}
&\frac{w^1c^2v_\delta^1}{c^2-(v_\delta^1)^2} - \frac{w^*c^2v_\delta^*}{c^2-(v_\delta^*)^2}\\
&= \lambda_2(\rho_+, v_+)\int_{t^*}^t \frac{v_\delta^1(\tau)-\lambda_2(\rho_+,v_+)}{G(\lambda_2(\rho_+,v_+)\tau-x^1(\tau))}d\tau - B_-\lambda_2(\rho_-, v_-)(x^1 - x^*)\\
&\quad+(2\lambda_2(\rho_+, v_+) - 2\lambda_2(\rho_-, v_-) + B_-\lambda_2^2(\rho_-, v_-))(t - t^*).
\end{aligned} \tag{3.51}$$

Calculating (3.51)-(3.50)$\times v_\delta^1$, and noting the fact that $v_\delta^* = \lambda_2(\rho_+, v_+)$, we have

$$(\lambda_2(\rho_+, v_+) - v_\delta^1)\int_{t^*}^t \frac{v_\delta^1(\tau) - \lambda_2(\rho_+, v_+)}{G(\lambda_2(\rho_+, v_+)\tau - x^1(\tau))}d\tau - B_-\lambda_2(\rho_-, v_-)(x^1 - x^* + v_\delta^1(t - t^*))$$

$$+B_-v_\delta^1(x^1-x^*)+(2\lambda_2(\rho_+,v_+)-2\lambda_2(\rho_-,v_-)+B_-\lambda_2^2(\rho_-,v_-))(t-t^*)-\frac{w^*c^2}{c^2-(v_\delta^*)^2}(v_\delta^1-\lambda_2(\rho_+,v_+))=0. \tag{3.52}$$

Integrating (3.52) from t^* to t, we have

$$\begin{aligned}&\textstyle\int_{t^*}^t(\lambda_2(\rho_+,v_+)-v_\delta^1(s))\int_{t^*}^s\frac{v_\delta^1(\tau)-\lambda_2(\rho_+,v_+)}{G(\lambda_2(\rho_+,v_+)\tau-x^1(\tau))}d\tau ds-B_-\lambda_2(\rho_-,v_-)(x^1-x^*)(t-t^*)\\&+\tfrac{1}{2}B_-(x^1-x^*)^2+\tfrac{1}{2}(2\lambda_2(\rho_+,v_+)-2\lambda_2(\rho_-,v_-)+B_-\lambda_2^2(\rho_-,v_-))(t-t^*)^2\\&-\tfrac{w^*c^2}{c^2-(v_\delta^*)^2}(x^1-x^*-\lambda_2(\rho_+,v_+)(t-t^*))=0.\end{aligned} \tag{3.53}$$

Letting $Y=\lambda_2(\rho_+,v_+)\tau-x^1(\tau)$, $Z=\lambda_2(\rho_+,v_+)s-x^1(s)$, then the first term on the left-hand side of (3.53) equals to

$$-\int_{\lambda_2(\rho_+,v_+)t^*-x^*}^{\lambda_2(\rho_+,v_+)t-x^1}\int_{\lambda_2(\rho_+,v_+)t^*-x^*}^{Z}\frac{1}{G(Y)}dYdZ. \tag{3.54}$$

So, (3.53) can be written as

$$H(x^1,t)=0, \tag{3.55}$$

where

$$\begin{aligned}&H(x^1,t)\\=&-\int_{\lambda_2(\rho_+,v_+)t^*-x^*}^{\lambda_2(\rho_+,v_+)t-x^1}\int_{\lambda_2(\rho_+,v_+)t^*-x^*}^{Z}\frac{1}{G(Y)}dYdZ-B_-\lambda_2(\rho_-,v_-)(x^1-x^*)(t-t^*)\\&+\frac{1}{2}B_-(x^1-x^*)^2+\frac{1}{2}(2\lambda_2(\rho_+,v_+)-2\lambda_2(\rho_-,v_-)+B_-\lambda_2^2(\rho_-,v_-))(t-t^*)^2\\&-\frac{w^*c^2}{c^2-(v_\delta^*)^2}(x^1-x^*-\lambda_2(\rho_+,v_+)(t-t^*)).\end{aligned}$$

When $t^*<t<t^\#$, we have

$$\begin{aligned}&H|_{x^1=x^*+\lambda_2(\rho_+,v_+)(t-t^*)}\\=&-B_-\lambda_2(\rho_-,v_-)\lambda_2(\rho_+,v_+)(t-t^*)^2+\frac{1}{2}B_-\lambda_2^2(\rho_+,v_+)(t-t^*)^2\\&+\frac{1}{2}(2\lambda_2(\rho_+,v_+)-2\lambda_2(\rho_-,v_-)+B_-\lambda_2^2(\rho_-,v_-))(t-t^*)^2\\=&\frac{1}{2}B_-(\lambda_2(\rho_-,v_-)-\lambda_2(\rho_+,v_+))\big(\lambda_2(\rho_-,v_-)-\lambda_2(\rho_+,v_+)+\frac{2}{B_-}\big)(t-t^*)^2.\end{aligned}$$

From (3.42), we have

$$\begin{aligned}&H|_{x^1=x^*+\lambda_2(\rho_+,v_+)(t-t^*)}\\=&\frac{1}{2}B_-(\lambda_2(\rho_-,v_-)-\lambda_2(\rho_+,v_+))(\lambda_1(\rho_-,v_-)-\lambda_2(\rho_+,v_+))(t-t^*)^2>0.\end{aligned} \tag{3.56}$$

and

$$\begin{aligned}&H|_{x^1=x^*+\lambda_2(\rho_-,v_-)(t-t^*)}\\\le&-B_-\lambda_2^2(\rho_-,v_-)(t-t^*)^2+\tfrac{1}{2}B_-\lambda_2^2(\rho_-,v_-))(t-t^*)^2\\&+\tfrac{1}{2}(2\lambda_2(\rho_+,v_+)-2\lambda_2(\rho_-,v_-)+B_-\lambda_2^2(\rho_-,v_-))(t-t^*)^2\\&-\tfrac{w^*c^2}{c^2-(v_\delta^*)^2}(\lambda_2(\rho_-,v_-)-\lambda_2(\rho_+,v_+))(t-t^*)-2\lambda_2(\rho_-,v_-)+B_-\lambda_2^2(\rho_-,v_-))(t-t^*)^2\end{aligned}$$

$$
\begin{aligned}
&-\tfrac{w^*c^2}{c^2-(v_\delta^*)^2}(\lambda_2(\rho_-,v_-)-\lambda_2(\rho_+,v_+))(t-t^*)\\
&=-(\lambda_2(\rho_-,v_-)-\lambda_2(\rho_+,v_+))(t-t^*)^2-\tfrac{w^*c^2}{c^2-(v_\delta^*)^2}(\lambda_2(\rho_-,v_-)-\lambda_2(\rho_+,v_+))(t-t^*)\\
&<0.
\end{aligned}
\tag{3.57}
$$

Moreover, for $x^*+\lambda_2(\rho_+,v_+)(t-t^*)<x^1<x^*+\lambda_2(\rho_-,v_-)(t-t^*)$, we have

$$
\frac{\partial H}{\partial x^1}=\int_{\lambda_2(\rho_+,v_+)t^*-x^*}^{\lambda_2(\rho_+,v_+)t-x^1}\frac{1}{G(Y)}dY+B_-(x^1-x^*-\lambda_2(\rho_-,v_-)(t-t^*))-\frac{w^*c^2}{c^2-(v_\delta^*)^2}<0.
\tag{3.58}
$$

On account of (3.56), (3.57) and (3.58), there exists a unique function $x^1=x^1(t)\in(x^*+\lambda_2(\rho_+,v_+)(t-t^*),x^*+\lambda_2(\rho_-,v_-)(t-t^*))$, such that $H(x^1,t)=0$ for $t\in(t^*,t^{\#})$. Futhermore , we have $v_\delta^1(t)=\frac{dx^1(t)}{dt}$. From (3.50), we have

$$
\begin{aligned}
w^1(t)=&\tfrac{c^2-(v_\delta^1)^2}{c^2}\int_{t^*}^{t}\tfrac{v_\delta^1(\tau)-\lambda_2(\rho_+,v_+)}{G(\lambda_2(\rho_+,v_+)\tau-x^1(\tau))}d\tau-\tfrac{c^2-(v_\delta^1)^2}{c^2}B_-(x^1-x^*)\\
&+\tfrac{c^2-(v_\delta^1)^2}{c^2}B_-\lambda_2(\rho_-,v_-)(t-t^*)+\tfrac{c^2-(v_\delta^1)^2}{c^2}\tfrac{w^*c^2}{c^2-(v_\delta^*)^2}.
\end{aligned}
\tag{3.59}
$$

When $t^{\#}\le t<+\infty$, where $t^{\#}$ is determined by $x^1(t^{\#})=\lambda_2(\rho_+,v_+)t^{\#}$, the solution (3.37) is similar to that in case 3.3, which is determined by the Rankine-Hugoniot condition (2.15) with initial data

$$
(x^2,w^2,v_\delta^2)(t^{\#})=(x^1(t^{\#}),w^1(t^{\#}),v_\delta^1(t^{\#})).
$$

The details are omitted.

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