STABILITY IN TOTALLY NONLINEAR DELAY DIFFERENCE EQUATIONS

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Abstract In this paper we use fixed point method to prove asymptotic stability results of the zero solution of a nonlinear delay difference equation. An asymptotic stability theorem with a sufficient condition is proved, which improves and generalizes some results due to Raffoul (2006) [23], Yankson (2009) [27], Jin and Luo (2009) [17] and Chen (2013) [9].

Keywords Fixed point, stability, delay difference equations.

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1. Introduction

Certainly, the Lyapunov direct method has been, for more than 100 years, the efficient tool for the study of stability properties of ordinary, functional, partial differential and difference equations. Nevertheless, the application of this method to problems of stability in differential and difference equations with delay has encountered serious difficulties if the delay is unbounded or if the equation has unbounded terms ([5, 6, 11–13, 15, 25]). Recently, Burton, Furumochi, Zhang, Raffoul, Islam, Yankson and others have noticed that some of these difficulties vanish or might be overcome by means of fixed point theory (see [1, 5, 6, 9, 16, 17, 23, 24, 27–29]). The fixed point theory does not only solve the problem on stability but has a significant advantage over Lyapunov's direct method. The conditions of the former are often averages but those of the latter are usually pointwise (see [5]). Yet the stability theory of difference equations with/without delay has been considered by many authors without the application of Lyapunov and fixed point methods, see the papers [3, 4, 7, 8, 14, 19–22, 30].

Let $a, a_j : \mathbb{Z}^+ \to \mathbb{R}$ and $\tau, \tau_j : \mathbb{Z}^+ \to \mathbb{Z}^+$ with $n - \tau(n) \to \infty$ and $n - \tau_j(n) \to \infty$ as $n \to \infty$. Here $\triangle$ denotes the forward difference operator $\triangle x(t) = x(n+1) - x(n)$ for any sequence $\{x(n), \ n \in \mathbb{Z}^+\}$.

In [23], Raffoul studied the equation

$$\triangle x(n) = -a(n) x(n - \tau(n)), \tag{1.1}$$

and proved the following theorem.

Theorem A (Raffoul [23]). *Suppose that* $\tau(n) = r$ *and* $a(n+r) \neq 1$ *and there*

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exists a constant $\alpha < 1$ such that

$$\sum_{s=n-r}^{n-1} |a\,(s+r)| + \sum_{s=0}^{n-1} \left(|a\,(s+r)| \left| \prod_{k=s+1}^{n-1} [1 - a\,(k+r)] \right| \sum_{u=s-r}^{s-1} |a\,(u+r)| \right) \le \alpha, \tag{1.2}$$

for all $n \in \mathbb{Z}^+$ and $\prod_{s=0}^{n-1} [1 - a\,(s+r)] \to 0$ as $n \to \infty$. Then, for every small initial sequence $\psi : [-r, 0] \cap \mathbb{Z} \to \mathbb{R}$, the solution $x\,(n) = x\,(n, 0, \psi)$ of (1.1) is bounded and tends to zero as $n \to \infty$.

In [27], Yankson studied the generalization of (1.1) as follows

$$\triangle x\,(n) = -\sum_{j=1}^{N} a_j\,(n)\, x\,(n - \tau_j\,(n)) \tag{1.3}$$

and obtained the following theorem.

Theorem B (Yankson [27]). *Suppose that $Q\,(n) \ne 0$ for all $n \in [n_0, \infty) \cap \mathbb{Z}$, the inverse sequence g_j of $n - \tau_j\,(n)$ exists and there exists a constant $\alpha \in (0, 1)$ for all $n \in [n_0, \infty) \cap \mathbb{Z}$ such that*

$$\sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} |a_j\,(g_j\,(s))| + \sum_{s=n_0}^{n-1} \left(|1 - Q\,(s)| \left| \prod_{k=s+1}^{n-1} Q\,(k) \right| \sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} |a_j\,(g_j\,(u))| \right)$$

$$\le \alpha, \tag{1.4}$$

where $Q\,(n) = 1 - \sum_{j=1}^{N} a_j\,(g_j\,(n))$. Then the zero solution of (1.3) is asymptotically stable if $\prod_{s=n_0}^{n-1} Q\,(s) \to 0$ as $n \to \infty$.

Obviously, Theorem B improves and generalizes Theorem A. On other hand, Jin and Luo in [17] and Chen in [9] considered the generalized form of *(1.1)*,

$$\triangle x\,(n) = -a\,(n)\, f\,(x\,(n - \tau\,(n))) \tag{1.5}$$

and obtained the following theorems.

Theorem C (Jin and Luo [17]). *Suppose that $\tau\,(n) = r$. Let f be odd, increasing on $[0, l]$, satisfy a Lipschitz condition, and let $x - f\,(x)$ be nondecreasing on $[0, l]$. Suppose that $|a\,(n)| < 1$ and for each $l_1 \in (0, l]$ we have*

$$|l_1 - f\,(l_1)| \sup_{n \in \mathbb{Z}^+} \sum_{s=0}^{n-1} |a\,(s+r)| \prod_{k=s+1}^{n-1} [1 - a\,(k+r)]$$

$$+ f\,(l_1) \sup_{n \in \mathbb{Z}^+} \sum_{s=0}^{n-1} |a\,(s+r)| \prod_{k=s+1}^{n-1} [1 - a\,(k+r)] \sum_{u=s-r}^{n-1} |a\,(u+r)|$$

$$+ f\,(l_1) \sup_{n \in \mathbb{Z}^+} \sum_{s=n-r}^{n-1} |a\,(s+r)|$$

$$\le \alpha l_1. \tag{1.6}$$

Then the zero solution of (1.5) is stable.

Theorem D (Chen [9]). *Suppose that the following conditions are satisfied*

(i) *the function f is odd, increasing on $[0, l]$,*

(ii) *$f(x)$ and $x - f(x)$ satisfy a Lipschitz condition with constant K on an interval $[-l, l]$, and $x - f(x)$ is nondecreasing on $[0, l]$,*

(iii) *the inverse function $g(n)$ of $n - \tau(n)$ exists and $|a(g(n))| < 1$,*

(iv) *there exists a constant $\alpha \in (0, 1)$ for all $n \in \mathbb{Z}^+$ such that*

$$\sum_{s=0}^{n-1} |a(g(s))| \prod_{k=s+1}^{n-1} [1 - a(g(k))] + \sum_{s=n-\tau(n)}^{n-1} |a(g(s))|$$

$$+ \sum_{s=0}^{n-1} |a(g(s))| \prod_{k=s+1}^{n-1} [1 - a(g(k))] \sum_{u=s-\tau(s)}^{n-1} |a(g(u))|$$

$$\leq \alpha. \tag{1.7}$$

Then the zero solution of (1.5) is asymptotically stable if $\displaystyle\prod_{k=0}^{n-1} [1 - a(g(k))] \to 0$ as $n \to \infty$.

Obviously, Theorem D improves Theorem C.

Recently, in the continuous case, the authors [2] have studied the linear delay differential system with time varying coefficients

$$x_i'(t) = -\sum_{j=1}^{N} \sum_{k=1}^{r_{ij}} a_{ij}^k(t) x_j\left(h_{ij}^k(t)\right), \quad i = 1 \dots N$$

and obtained the uniform exponential stability results by using Bohl–Perron theorem.

In this paper, we consider the generalization of a nonlinear delay difference equation (1.5) of the form

$$\triangle x(n) = -\sum_{j=1}^{N} a_j(n) f_j(x(n - \tau_j(n))) \tag{1.8}$$

with the initial condition

$$x(n) = \psi(n) \text{ for } n \in [m(n_0), n_0] \cap \mathbb{Z}, \tag{1.9}$$

where $\psi : [m(n_0), n_0] \cap \mathbb{Z} \to \mathbb{R}$ is a bounded sequence and for $n_0 \geq 0$,

$$m_j(n_0) = \inf\{n - \tau_j(n), \ n \geq n_0\}, \ m(n_0) = \min\{m_j(n_0), \ 1 \leq j \leq N\}.$$

Note that (1.8) includes (1.1), (1.3) and (1.5) as special cases.

Our purpose here is to improve Theorems A–D and extend it to investigate a wide class of nonlinear delay difference equation presented in (1.8). Our results are obtained with no need of further assumptions on the inverse of sequence $n - \tau_j(n)$, so that for a given bounded initial sequence ψ a mapping P for (1.8) is constructed in

such a way to map a, carefully chosen, complete metric space S_ψ^ϵ into itself on which P is a contraction mapping possessing a fixed point. This procedure will enable us to establish and prove by means of the contraction mapping theorem an asymptotic stability theorem for the zero solution of (1.8) with a less restrictive conditions. For details on contraction mapping principle we refer the reader to [26] and for more on the calculus of difference equations, we refer the reader to [10] and [18]. The results presented in this paper improve and generalize the main results in [9, 17, 23, 27].

2. Main results

For a fixed n_0, we denote $D(n_0)$ the set of bounded sequences $\psi : [m(n_0), n_0] \cap \mathbb{Z} \to \mathbb{R}$ with the norm $|\psi|_0 = \max\{|\psi(n)| : n \in [m(n_0), n_0] \cap \mathbb{Z}\}$. For each $(n_0, \psi) \in \mathbb{Z}^+ \times D(n_0)$, a solution of (1.8) through (n_0, ψ) is a sequence $x : [m(n_0), \infty) \cap \mathbb{Z} \to \mathbb{R}$ such that x satisfies (1.8) on $[n_0, \infty) \cap \mathbb{Z}$ and $x = \psi$ on $[m(n_0), n_0] \cap \mathbb{Z}$. We denote such a solution by $x(n) = x(n, n_0, \psi)$. For each $(n_0, \psi) \in \mathbb{Z}^+ \times D(n_0)$, there exists a unique solution $x(n) = x(n, n_0, \psi)$ of (1.8) defined on $[m(n_0), \infty) \cap \mathbb{Z}$.

Let $h_j : [m(n_0), \infty) \cap \mathbb{Z} \to \mathbb{R}$ be an arbitrary sequence. Rewrite (1.8) as

$$\triangle x(n) = -\sum_{j=1}^{N} h_j(n) f_j(x(n)) + \triangle_n \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} h_j(s) f_j(x(s))$$

$$+ \sum_{j=1}^{N} \{h_j(n - \tau_j(n)) - a_j(n)\} f_j(x(n - \tau_j(n)))$$

$$= -\sum_{j=1}^{N} h_j(n) x(n) + \sum_{j=1}^{N} h_j(n) [x(n) - f_j(x(n))]$$

$$+ \triangle_n \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} h_j(s) f_j(x(s))$$

$$+ \sum_{j=1}^{N} \{h_j(n - \tau_j(n)) - a_j(n)\} f_j(x(n - \tau_j(n))), \qquad (2.1)$$

where $\triangle_n$ represents that the difference is with respect to n. If we let $H(n) = 1 - \sum_{j=1}^{N} h_j(n)$ then (2.1) is equivalent to

$$x(n+1) = H(n) x(n) + \sum_{j=1}^{N} h_j(n) [x(n) - f_j(x(n))]$$

$$+ \triangle_n \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} h_j(s) f_j(x(s))$$

$$+ \sum_{j=1}^{N} \{h_j(n - \tau_j(n)) - a_j(n)\} f_j(x(n - \tau_j(n))). \qquad (2.2)$$

In the process, For any sequence x, we denote

$$\sum_{k=a}^{b} x(k) = 0 \text{ and } \prod_{k=a}^{b} x(k) = 1 \text{ for any } a > b.$$

Lemma 2.1. *Suppose that $H(n) \neq 0$ for all $n \in [n_0, \infty) \cap \mathbb{Z}$. Then x is a solution of equation (1.8) if and only if*

$$
x(n) = \left\{ x(n_0) - \sum_{j=1}^{N} \sum_{s=n_0-\tau_j(n_0)}^{n_0-1} h_j(s) f_j(x(s)) \right\} \prod_{u=n_0}^{n-1} H(u)
$$

$$
+ \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} h_j(s) \prod_{u=s+1}^{n-1} H(u) \left[x(s) - f_j(x(s)) \right]
$$

$$
+ \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} h_j(s) f_j(x(s))
$$

$$
- \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} \{1 - H(s)\} \prod_{u=s+1}^{n-1} H(u) \sum_{v=s-\tau_j(s)}^{s-1} h_j(v) f_j(x(v))
$$

$$
+ \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} H(u) \{h_j(s - \tau_j(s)) - a_j(s)\} f_j(x(s - \tau_j(s))). \quad (2.3)
$$

Proof.   Let x be a solution of (1.8). By multiplying both sides of (2.2) by $\prod\limits_{u=n_0}^{n} [H(u)]^{-1}$ and by summing from n_0 to $n-1$ we obtain

$$
\sum_{s=n_0}^{n-1} \triangle \left[\prod_{u=n_0}^{s-1} [H(u)]^{-1} x(s) \right]
$$

$$
= \sum_{s=n_0}^{n-1} \prod_{u=n_0}^{s} [H(u)]^{-1} \sum_{j=1}^{N} h_j(s) \left[x(s) - f_j(x(s)) \right]
$$

$$
+ \sum_{s=n_0}^{n-1} \prod_{u=n_0}^{s} [H(u)]^{-1} \triangle_s \sum_{j=1}^{N} \sum_{v=s-\tau_j(s)}^{s-1} h_j(v) f_j(x(v))
$$

$$
+ \sum_{s=n_0}^{n-1} \prod_{u=n_0}^{s} [H(u)]^{-1} \sum_{j=1}^{N} \{h_j(s - \tau_j(s)) - a_j(s)\} f_j(x(s - \tau_j(s))).
$$

As a consequence, we arrive at

$$
\prod_{u=n_0}^{n-1} [H(u)]^{-1} x(n) - \prod_{u=n_0}^{n_0-1} [H(u)]^{-1} x(n_0)
$$

$$
= \sum_{s=n_0}^{n-1} \prod_{u=n_0}^{s} [H(u)]^{-1} \sum_{j=1}^{N} h_j(s) \left[x(s) - f_j(x(s)) \right]
$$

$$
+ \sum_{s=n_0}^{n-1} \prod_{u=n_0}^{s} [H(u)]^{-1} \triangle_s \sum_{j=1}^{N} \sum_{v=s-\tau_j(s)}^{s-1} h_j(v) f_j(x(v))
$$

$$
+ \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} \prod_{u=n_0}^{s} [H(u)]^{-1} \{h_j(s - \tau_j(s)) - a_j(s)\} f_j(x(s - \tau_j(s))).
$$

By dividing both sides of the above expression by $\prod\limits_{u=n_0}^{n-1} [H(u)]^{-1}$ we get

$$
\begin{aligned}
x\left(n\right) =& x\left(n_0\right) \prod_{u=n_0}^{n-1} H\left(u\right) + \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} H\left(u\right) \sum_{j=1}^{N} h_j\left(s\right) \left[x\left(s\right) - f_j\left(x\left(s\right)\right)\right] \\
&+ \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} H\left(u\right) \triangle_s \sum_{v=s-\tau_j(s)}^{s-1} h_j\left(v\right) f_j\left(x\left(v\right)\right) \\
&+ \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} H\left(u\right) \left\{h_j\left(s - \tau_j\left(s\right)\right) - a_j\left(s\right)\right\} f_j\left(x\left(s - \tau_j\left(s\right)\right)\right). \quad (2.4)
\end{aligned}
$$

By performing a summation by parts, we have

$$
\begin{aligned}
&\sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} H\left(u\right) \triangle_s \sum_{v=s-\tau_j(s)}^{s-1} h_j\left(v\right) f_j\left(x\left(v\right)\right) \\
=& \sum_{s=n-\tau_j(n)}^{n-1} h_j\left(s\right) f_j\left(x\left(s\right)\right) - \prod_{u=n_0}^{n-1} H\left(u\right) \sum_{s=n_0-\tau_j(n_0)}^{n_0-1} h_j\left(s\right) f_j\left(x\left(s\right)\right) \\
&- \sum_{s=n_0}^{n-1} \left\{1 - H\left(s\right)\right\} \prod_{u=s+1}^{n-1} H\left(u\right) \sum_{v=s-\tau_j(s)}^{s-1} h_j\left(v\right) f_j\left(x\left(v\right)\right). \quad (2.5)
\end{aligned}
$$

Finally, substituting (2.5) into (2.4) completes the proof. $\qquad\square$

From equation (2.3) we shall derive a fixed point mapping P for (1.8). But the challenge here is to choose a suitable metric space of sequences on which the map P can be defined. Below a weighted metric on a specific space is defined. Let C be the Banach space of real bounded sequences $\varphi : [m\left(n_0\right), \infty) \cap \mathbb{Z} \to \mathbb{R}$ with the supremum norm $\|.\|$, that is, for $\varphi \in C$,

$$
\|\varphi\| = \sup\left\{|\varphi\left(n\right)| : n \in [m\left(n_0\right), \infty) \cap \mathbb{Z}\right\}.
$$

In other words, we carry out investigations in the complete metric space (C, d) where d denotes the supremum metric $d\left(\varphi_1, \varphi_2\right) = \|\varphi_1 - \varphi_2\|$ for $\varphi_1, \varphi_2 \in C$. For a given initial sequence $\psi : [m\left(n_0\right), n_0] \cap \mathbb{Z} \to [-l, l]$ with $l > 0$, define the set

$$
S_\psi^l = \left\{\varphi \in C,\ \varphi\left(n\right) = \psi\left(n\right)\ \text{for}\ n \in [m\left(n_0\right), n_0] \cap \mathbb{Z},\ |\varphi\left(n\right)| \leq l\right\}.
$$

Since S_ψ^l is a closed subset of C, the metric space $\left(S_\psi^l, d\right)$ is complete.

Definition 2.1. The zero solution of (1.8) is Lyapunov stable if for any $\epsilon > 0$ and any integer $n_0 \geq 0$ there exists a $\delta > 0$ such that $|\psi\left(n\right)| \leq \delta$ for $n \in [m\left(n_0\right), n_0] \cap \mathbb{Z}$ implies $|x\left(n, n_0, \psi\right)| \leq \epsilon$ for $n \in [n_0, \infty) \cap \mathbb{Z}$.

Theorem 2.1. *Define a mapping P on S_ψ^l as follows, for $\varphi \in S_\psi^l$ $(P\varphi)\left(n\right) = \psi\left(n\right)$*

if $n \in [m(n_0), n_0] \cap \mathbb{Z}$, while, for $n \in [n_0, \infty) \cap \mathbb{Z}$

$$(P\varphi)(n) = \left\{ \psi(n_0) - \sum_{j=1}^{N} \sum_{s=n_0-\tau_j(n_0)}^{n_0-1} h_j(s) f_j(\psi(s)) \right\} \prod_{u=n_0}^{n-1} H(u)$$

$$+ \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} h_j(s) \prod_{u=s+1}^{n-1} H(u) [\varphi(s) - f_j(\varphi(s))]$$

$$+ \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} h_j(s) f_j(\varphi(s)) \tag{2.6}$$

$$- \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} \{1 - H(s)\} \prod_{u=s+1}^{n-1} H(u) \sum_{v=s-\tau_j(s)}^{s-1} h_j(v) f_j(\varphi(v))$$

$$+ \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} H(u) \{h_j(s - \tau_j(s)) - a_j(s)\} f_j(\varphi(s - \tau_j(s))).$$

Suppose that the following conditions are satisfied,

(i) *the function f_j is odd, increasing on $[0, l]$,*

(ii) *$f_j(x)$ and $x - f_j(x)$ satisfy a Lipschitz condition with constant K_j on an interval $[-l, l]$, and $x - f_j(x)$ is nondecreasing on $[0, l]$,*

(iii) *$\sum_{j=1}^{N} |h_j(n)| < 1$ for $n \in [m(n_0), \infty) \cap \mathbb{Z}$ and $|h_j(s - \tau_j(s)) - a_j(s)| < 1$ for $n \in [n_0, \infty) \cap \mathbb{Z}$, $j = 1, 2, ..., N$,*

(iv) *there exist constants $\alpha_j, \alpha \in (0, 1)$ for all $n \in [n_0, \infty) \cap \mathbb{Z}$ such that*

$$\sum_{s=n_0}^{n-1} |h_j(s)| \prod_{u=s+1}^{n-1} H(u) + \sum_{s=n-\tau_j(n)}^{n-1} |h_j(s)|$$

$$+ \sum_{s=n_0}^{n-1} |1 - H(s)| \prod_{u=s+1}^{n-1} H(u) \sum_{v=s-\tau_j(s)}^{s-1} |h_j(v)|$$

$$+ \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} H(u) |h_j(s - \tau_j(s)) - a_j(s)| \leq \alpha_j$$

and

$$\alpha = \sum_{j=1}^{N} \alpha_j.$$

Then there exists $\delta > 0$ such that for any $\psi : [m(n_0), n_0] \cap \mathbb{Z} \to (-\delta, \delta)$, we have that $P : S_\psi^l \to S_\psi^l$ and P is a contraction mapping with respect to the metric defined on S_ψ^l.

Proof. Since f_j is odd and satisfies the Lipshitz condition on $[-l, l]$, $f_j(0) = 0$ and f_j is uniformly continuous on $[-l, l]$. So we can choose a δ that satisfies

$$\delta \left(1 + \sum_{j=1}^{N} K_j \sum_{s=n_0-\tau_j(n_0)}^{n_0-1} |h_j(s)| \right) \leq (1 - \alpha) l. \tag{2.7}$$

Let $\psi \in D(n_0)$ such that $|\psi(n)| \leq \delta$ for $n \in [m(n_0), n_0] \cap \mathbb{Z}$. Note that (2.7) implies $\delta < l$ since $f_j(l) \leq l$ by condition (ii). Thus, $|\psi(n)| \leq l$ for $n \in [m(n_0), n_0] \cap \mathbb{Z}$. Now we show that for such a ψ the mapping $P : S_\psi^l \to S_\psi^l$. Indeed, consider (2.6). For an arbitrary $\varphi \in S_\psi^l$, if follows from conditions (i) and (ii) that

$$
\begin{aligned}
|(P\varphi)(n)| \leq & \left\{ \|\psi\| + \sum_{j=1}^{N} \sum_{s=n_0-\tau_j(n_0)}^{n_0-1} |h_j(s)| \|f_j(\psi)\| \right\} \prod_{u=n_0}^{n-1} H(u) \\
& + \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} |h_j(s)| \prod_{u=s+1}^{n-1} H(u) |\varphi(s) - f_j(\varphi(s))| \\
& + \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} |h_j(s)| |f_j(\varphi(s))| \\
& + \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} |1 - H(s)| \prod_{u=s+1}^{n-1} H(u) \sum_{v=s-\tau_j(s)}^{s-1} |h_j(v)| |f_j(\varphi(v))| \\
& + \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} H(u) |h_j(s - \tau_j(s)) - a_j(s)| |f_j(\varphi(s - \tau_j(s)))| \\
\leq & \left(\delta + \delta \sum_{j=1}^{N} K_j \sum_{s=n_0-\tau_j(n_0)}^{n_0-1} |h_j(s)| \right) \\
& + \sum_{j=1}^{N} (l - f_j(l)) \sum_{s=n_0}^{n-1} |h_j(s)| \prod_{u=s+1}^{n-1} H(u) + \sum_{j=1}^{N} f_j(l) \sum_{s=n-\tau_j(n)}^{n-1} |h_j(s)| \\
& + \sum_{j=1}^{N} f_j(l) \sum_{s=n_0}^{n-1} |1 - H(s)| \prod_{u=s+1}^{n-1} H(u) \sum_{v=s-\tau_j(s)}^{s-1} |h_j(v)| \\
& + \sum_{j=1}^{N} f_j(l) \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} H(u) |h_j(s - \tau_j(s)) - a_j(s)|,
\end{aligned}
$$

for $n \in [n_0, \infty) \cap \mathbb{Z}$. By applying (iv) and (2.7), we see that

$$
\begin{aligned}
|(P\varphi)(n)| \leq & \delta \left(1 + \sum_{j=1}^{N} K_j \sum_{s=n_0-\tau_j(n_0)}^{n_0-1} |h_j(s)| \right) \\
& + \sum_{j=1}^{N} (l - f_j(l)) \alpha_j + \sum_{j=1}^{N} f_j(l) \alpha_j \\
\leq & (1 - \alpha) l + \sum_{j=1}^{N} [(l - f_j(l)) + f_j(l)] \alpha_j = l.
\end{aligned}
$$

Hence, $|(P\varphi)(n)| \leq l$ for $n \in [m(n_0), \infty) \cap \mathbb{Z}$ because $|(P\varphi)(n)| = |\psi(n)| \leq l$ for $n \in [m(n_0), n_0] \cap \mathbb{Z}$. Therefore, $P\varphi \in S_\psi^l$.

Suppose that $\rho > \max \left\{ 4, 1/\sum_{j=1}^{N} K_j \right\}$. If we define a metric on S_ψ^l as follows,

$$
|\varphi - \eta|_\rho := \sup_{n \in [n_0, \infty) \cap \mathbb{Z}} \prod_{u=n_0}^{n-1} \frac{\prod_{k=1}^{N} [1 - |h_k(u)|] [1 - |h_k(s - \tau_k(u)) - a_k(u)|]}{\rho \sum_{i=1}^{N} K_i \prod_{k=1}^{N} [1 + |h_k(u)|] [1 + |h_k(u - \tau_k(u)) - a_k(u)|]}
$$
$$
\times |\varphi(n) - \eta(n)|, \tag{2.8}
$$

then $\left(S_\psi^l, |.|_k \right)$ is a complete metric space.

Next, we show that P is a contraction mapping on S_ψ^l with respect to the metric (2.8). For $\varphi, \eta \in S_\psi^l$, we have

$$
|(P\varphi)(n) - (P\eta)(n)|
$$
$$
\leq \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} |h_j(s)| \prod_{u=s+1}^{n-1} H(u) |\varphi(s) - f_j(\varphi(s)) - \eta(s) + f_j(\eta(s))|
$$
$$
+ \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} |h_j(s)| |f_j(\varphi(s)) - f_j(\eta(s))|
$$
$$
+ \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} |1 - H(s)| \prod_{u=s+1}^{n-1} H(u) \sum_{v=s-\tau_j(s)}^{s-1} |h_j(v)| |f_j(\varphi(v)) - f_j(\eta(v))|
$$
$$
+ \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} H(u) |h_j(s - \tau_j(s)) - a_j(s)| |f_j(\varphi(s - \tau_j(s))) - f_j(\eta(s - \tau_j(s)))|.
$$
$$
\tag{2.9}
$$

Let $F_j(x) = x - f_j(x)$, then $F_j(x)$ satisfies a Lipschitz condition with constant $K_j > 0$ on an interval $[-l, l]$. If we multiply both sides of (2.9) by

$$
\prod_{u=n_0}^{n-1} \frac{\prod_{k=1}^{N} [1 - |h_k(u)|] [1 - |h_k(u - \tau_k(u)) - a_k(u)|]}{\rho \sum_{i=1}^{N} K_i \prod_{k=1}^{N} [1 + |h_k(u)|] [1 + |h_k(u - \tau_k(u)) - a_k(u)|]},
$$

then the first term on the right-hand side of (2.9) becomes

$$
\prod_{u=n_0}^{n-1} \frac{\prod_{k=1}^{N} [1 - |h_k(u)|] [1 - |h_k(u - \tau_k(u)) - a_k(u)|]}{\rho \sum_{i=1}^{N} K_i \prod_{k=1}^{N} [1 + |h_k(u)|] [1 + |h_k(u - \tau_k(u)) - a_k(u)|]}
$$
$$
\times \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} |h_j(s)| \prod_{u=s+1}^{n-1} H(u) |F_j(\varphi(s)) - F_j(\eta(s))|
$$

$$\leq \sum_{j=1}^{N} K_j \sum_{s=n_0}^{n-1} \frac{|h_j(s)| \prod\limits_{k=1}^{N} [1-|h_k(s)|] [1-|h_k(s-\tau_k(s))-a_k(s)|]}{\rho \sum\limits_{i=1}^{N} K_i \prod\limits_{k=1}^{N} [1+|h_k(s)|] [1+|h_k(s-\tau_k(s))-a_k(s)|]}$$

$$\times \prod_{u=n_0}^{s-1} \frac{\prod\limits_{k=1}^{N} [1-|h_k(u)|] [1-|h_k(u-\tau_k(u))-a_k(u)|]}{\rho \sum\limits_{i=1}^{N} K_i \prod\limits_{k=1}^{N} [1+|h_k(u)|] [1+|h_k(u-\tau_k(u))-a_k(u)|]} |\varphi(s)-\eta(s)|$$

$$\times \prod_{u=s+1}^{n-1} \frac{H(u) \prod\limits_{k=1}^{N} [1-|h_k(u)|] [1-|h_k(u-\tau_k(u))-a_k(u)|]}{\rho \sum\limits_{i=1}^{N} K_i \prod\limits_{k=1}^{N} [1+|h_k(u)|] [1+|h_k(u-\tau_k(u))-a_k(u)|]}$$

$$\leq \sum_{j=1}^{N} K_j \frac{1}{\rho \sum\limits_{i=1}^{N} K_i} |\varphi-\eta|_\rho \sum_{s=n_0}^{n-1} |h_j(s)| \prod_{u=s+1}^{n-1} [1-|h_j(u)|]$$

$$\leq \frac{1}{\rho} |\varphi-\eta|_\rho.$$

Similarly, we have

$$\prod_{u=n_0}^{n-1} \frac{\prod\limits_{k=1}^{N} [1-|h_k(u)|] [1-|h_k(u-\tau_k(u))-a_k(u)|]}{\rho \sum\limits_{i=1}^{N} K_i \prod\limits_{k=1}^{N} [1+|h_k(u)|] [1+|h_k(u-\tau_k(u))-a_k(u)|]}$$

$$\times \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} |h_j(s)| |f_j(\varphi(s))-f_j(\eta(s))|$$

$$\leq \sum_{j=1}^{N} K_j \sum_{s=n_0}^{n-1} \frac{|h_j(s)| \prod\limits_{k=1}^{N} [1-|h_k(s)|] [1-|h_k(s-\tau_k(s))-a_k(s)|]}{\rho \sum\limits_{i=1}^{N} K_i \prod\limits_{k=1}^{N} [1+|h_k(s)|] [1+|h_k(s-\tau_k(s))-a_k(s)|]}$$

$$\times \prod_{u=n_0}^{s-1} \frac{\prod\limits_{k=1}^{N} [1-|h_k(u)|] [1-|h_k(u-\tau_k(u))-a_k(u)|]}{\rho \sum\limits_{i=1}^{N} K_i \prod\limits_{k=1}^{N} [1+|h_k(u)|] [1+|h_k(u-\tau_k(u))-a_k(u)|]} |\varphi(s)-\eta(s)|$$

$$\times \prod_{u=s+1}^{n-1} \frac{\prod\limits_{k=1}^{N} \left[1 - |h_k(u)|\right] \left[1 - |h_k(u - \tau_k(u)) - a_k(u)|\right]}{\rho \sum\limits_{i=1}^{N} K_i \prod\limits_{k=1}^{N} \left[1 + |h_k(u)|\right] \left[1 + |h_k(u - \tau_k(u)) - a_k(u)|\right]}$$

$$\leq \sum_{j=1}^{N} K_j \frac{1}{\rho \sum\limits_{i=1}^{N} K_i} |\varphi - \eta|_\rho \sum_{s=n_0}^{n-1} |h_j(s)| \prod_{u=s+1}^{n-1} \left[1 - |h_j(u)|\right]$$

$$\leq \frac{1}{\rho} |\varphi - \eta|_\rho,$$

$$\prod_{u=n_0}^{n-1} \frac{\prod\limits_{k=1}^{N} \left[1 - |h_k(u)|\right] \left[1 - |h_k(u - \tau_k(u)) - a_k(u)|\right]}{\rho \sum\limits_{i=1}^{N} K_i \prod\limits_{k=1}^{N} \left[1 + |h_k(u)|\right] \left[1 + |h_k(u - \tau_k(u)) - a_k(u)|\right]}$$

$$\times \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} |1 - H(s)| \prod_{u=s+1}^{n-1} H(u) \sum_{v=s-\tau_j(s)}^{s-1} |h_j(v)| |f_j(\varphi(v)) - f_j(\eta(v))|$$

$$\leq \sum_{j=1}^{N} K_j \sum_{s=n_0}^{n-1} \frac{|1 - H(s)| \prod\limits_{k=1}^{N} \left[1 - |h_k(s)|\right] \left[1 - |h_k(s - \tau_k(s)) - a_k(s)|\right]}{\rho \sum\limits_{i=1}^{N} K_i \prod\limits_{k=1}^{N} \left[1 + |h_k(s)|\right] \left[1 + |h_k(s - \tau_k(s)) - a_k(s)|\right]}$$

$$\times \prod_{u=s+1}^{n-1} \frac{H(u) \prod\limits_{k=1}^{N} \left[1 - |h_k(u)|\right] \left[1 - |h_k(u - \tau_k(u)) - a_k(u)|\right]}{\rho \sum\limits_{i=1}^{N} K_i \prod\limits_{k=1}^{N} \left[1 + |h_k(u)|\right] \left[1 + |h_k(u - \tau_k(u)) - a_k(u)|\right]}$$

$$\times \sum_{v=s-\tau_j(s)}^{s-1} |h_j(v)| \prod_{u=n_0}^{v-1} \frac{\prod\limits_{k=1}^{N} \left[1 - |h_k(v)|\right] \left[1 - |h_k(v - \tau_k(v)) - a_k(v)|\right]}{\rho \sum\limits_{i=1}^{N} K_i \prod\limits_{k=1}^{N} \left[1 + |h_k(v)|\right] \left[1 + |h_k(v - \tau_k(v)) - a_k(v)|\right]}$$

$$\times |\varphi(v) - \eta(v)|$$

$$\times \prod_{u=v}^{s-1} \frac{\prod\limits_{k=1}^{N} \left[1 - |h_k(v)|\right] \left[1 - |h_k(v - \tau_k(v)) - a_k(v)|\right]}{\rho \sum\limits_{i=1}^{N} K_i \prod\limits_{k=1}^{N} \left[1 + |h_k(v)|\right] \left[1 + |h_k(v - \tau_k(v)) - a_k(v)|\right]}$$

$$\leq \sum_{j=1}^{N} K_j \frac{1}{\rho \sum\limits_{i=1}^{N} K_i} |\varphi - \eta|_\rho \sum_{s=n_0}^{n-1} |h_j(s)| \prod_{u=s+1}^{n-1} \left[1 - |h_j(u)|\right]$$

$$\times \sum_{v=s-\tau_j(s)}^{s-1} |h_j(v)| \prod_{u=v}^{s-1} [1-|h_j(v)|]$$

$$\leq \frac{1}{\rho} |\varphi - \eta|_\rho$$

and

$$\prod_{u=n_0}^{n-1} \frac{\prod\limits_{k=1}^{N} [1-|h_k(u)|] [1-|h_k(u-\tau_k(u))-a_k(u)|]}{\rho \sum\limits_{i=1}^{N} K_i \prod\limits_{k=1}^{N} [1+|h_k(u)|] [1+|h_k(u-\tau_k(u))-a_k(u)|]}$$

$$\times \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} H(u) |h_j(s-\tau_j(s))-a_j(s)| |f_j(\varphi(s-\tau_j(s)))-f_j(\eta(s-\tau_j(s)))|$$

$$\leq \sum_{j=1}^{N} K_j \sum_{s=n_0}^{n-1} \frac{|h_j(s-\tau_j(s))-a_j(s)| \prod\limits_{k=1}^{N} [1-|h_k(s)|] [1-|h_k(s-\tau_k(s))-a_k(s)|]}{\rho \sum\limits_{i=1}^{N} K_i \prod\limits_{k=1}^{N} [1+|h_k(s)|] [1+|h_k(s-\tau_k(s))-a_k(s)|]}$$

$$\times \prod_{u=n_0}^{s-1} \frac{\prod\limits_{k=1}^{N} [1-|h_k(u)|] [1-|h_k(u-\tau_k(u))-a_k(u)|]}{\rho \sum\limits_{i=1}^{N} K_i \prod\limits_{k=1}^{N} [1+|h_k(u)|] [1+|h_k(u-\tau_k(u))-a_k(u)|]} |\varphi(s)-\eta(s)|$$

$$\times \prod_{u=s+1}^{n-1} \frac{H(u) \prod\limits_{k=1}^{N} [1-|h_k(u)|] [1-|h_k(u-\tau_k(u))-a_k(u)|]}{\rho \sum\limits_{i=1}^{N} K_i \prod\limits_{k=1}^{N} [1+|h_k(u)|] [1+|h_k(u-\tau_k(u))-a_k(u)|]}$$

$$\leq \sum_{j=1}^{N} K_j \frac{1}{\rho \sum\limits_{i=1}^{N} K_i} |\varphi - \eta|_\rho \sum_{s=n_0}^{n-1} |h_j(s-\tau_j(s))-a_j(s)|$$

$$\times \prod_{u=s+1}^{n-1} [1-|h_k(u-\tau_k(u))-a_k(u)|]$$

$$\leq \frac{1}{\rho} |\varphi - \eta|_\rho .$$

Hence, $|P\varphi - P\eta|_\rho \leq \dfrac{4}{\rho} |\varphi - \eta|_\rho$, since $\rho > 4$, we have that P is a contraction mapping on S_ψ^l. $\qquad \square$

Theorem 2.2. *Assume that the hypotheses of Theorem 2.1 hold. Then the zero solution of (1.8) is stable.*

Proof. Let P be defined as in Theorem 2.1. By the contraction mapping principle

([26], p. 2), P has a unique fixed point in S_ψ^l, which is a solution of (1.8) with $x = \psi$ on $[m(n_0), n_0] \cap \mathbb{Z}$.

To prove stability at $n = n_0$, let $\epsilon > 0$ be given, then we choose $m > 0$ so that $m < \min\{l, \epsilon\}$. By considering S_ψ^m, we obtain there is a $\delta > 0$ such that $\|\psi\| < \delta$ implies that the unique solution of (1.8) with $x = \psi$ on $[m(n_0), n_0] \cap \mathbb{Z}$ satisfies $|x(n)| \leq m < \epsilon$ for all $n \in [m(n_0), \infty) \cap \mathbb{Z}$. This proves that the zero solution of (1.8) is stable. $\square$

Definition 2.2. The zero solution of (1.8) is asymptotically stable if it is Lyapunov stable and if for any integer $n_0 \geq 0$ there exists a $\delta > 0$ such that $|\psi(n)| \leq \delta$ for $n \in [m(n_0), n_0] \cap \mathbb{Z}$ implies $x(n, n_0, \psi) \to 0$ as $n \to \infty$.

Theorem 2.3. *Assume that the hypotheses of Theorem 2.1 hold. Also assume that*

$$\prod_{u=n_0}^{n-1} H(u) \to 0 \text{ as } n \to \infty. \tag{2.10}$$

Then the zero solution of (1.8) is asymptotically stable.

Proof.   From Theorem 2.2, the zero solution of (1.8) is stable. For a given $\epsilon > 0$ let $\psi \in D(n_0)$ such that $|\psi(n)| \leq \delta$ for $n \in [m(n_0), n_0] \cap \mathbb{Z}$ where $\delta > 0$ and define

$$S_\psi^\epsilon = \big\{ \varphi \in C, \ \varphi(n) = \psi(n) \text{ for } n \in [m(n_0), n_0] \cap \mathbb{Z},$$
$$\|\varphi\| \leq \epsilon \text{ and } \varphi(n) \to 0 \text{ as } n \to \infty \big\}.$$

Then S_ψ^ϵ is a complete metric space with respect to the metric (2.8). Define $P : S_\psi^\epsilon \to S_\psi^\epsilon$ by (2.6). From the proof of Theorem 2.1, the mapping P is a contraction and for every $\varphi \in S_\psi^\epsilon$, $\|P\varphi\| \leq \epsilon$.

We next show that $(P\varphi)(n) \to 0$ as $n \to \infty$. There are five terms on the right hand side in (2.6). Denote them, respectively, by I_k, $k = 1, 2, ..., 5$. It is obvious that the first term I_1 tends to zero as $t \to \infty$, by condition (2.10). Therefore, the second term I_2 in (2.6) satisfies

$$|I_2| = \left| \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} h_j(s) \prod_{u=s+1}^{n-1} H(u) [\varphi(s) - f_j(\varphi(s))] \right|$$
$$\leq \sum_{j=1}^{N} K_j \sum_{s=n_0}^{n-1} |h_j(s)| \prod_{u=s+1}^{n-1} H(u) |\varphi(s)|$$
$$\leq \epsilon \sum_{j=1}^{N} K_j \alpha_j < \epsilon \sum_{j=1}^{N} K_j.$$

Thus, $I_2 \to 0$ as $n \to \infty$. Also, due to the facts that $\varphi(n) \to 0$ and $n - \tau_j(n) \to \infty$ for $j = 1, 2, ..., N$ as $n \to \infty$, the third term I_3 tends to zero, as $n \to \infty$.

Now, for a given $\epsilon_1 \in (0, \epsilon)$, there exists a $N_1 > n_0$ such that $s \geq N_1$ implies $|\varphi(s - \tau_j(s))| < \epsilon_1$ for $j = 1, 2, ..., N$. Thus, for $n \geq N_1$, the term I_4 in (2.6)

satisfies

$$|I_4| = \left| \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} \{1 - H(s)\} \prod_{u=s+1}^{n-1} H(u) \sum_{v=s-\tau_j(s)}^{s-1} h_j(v) f_j(\varphi(v)) \right|$$

$$\leq \sum_{j=1}^{N} \sum_{s=n_0}^{N_1-1} |1 - H(s)| \prod_{u=s+1}^{n-1} H(u) \sum_{v=s-\tau_j(s)}^{s-1} |h_j(v)| |f_j(\varphi(v))|$$

$$+ \sum_{j=1}^{N} \sum_{s=N_1}^{n-1} |1 - H(s)| \prod_{u=s+1}^{n-1} H(u) \sum_{v=s-\tau_j(s)}^{s-1} |h_j(v)| |f_j(\varphi(v))|$$

$$\leq \sup_{\sigma \geq m(n_0)} |\varphi(\sigma)| \sum_{j=1}^{N} K_j \sum_{s=n_0}^{N_1-1} |1 - H(s)| \prod_{u=s+1}^{n-1} H(u) \sum_{v=s-\tau_j(s)}^{s-1} |h_j(v)|$$

$$+ \epsilon_1 \sum_{j=1}^{N} K_j \sum_{s=N_1}^{n-1} |1 - H(s)| \prod_{u=s+1}^{n-1} H(u) \sum_{v=s-\tau_j(s)}^{s-1} |h_j(v)|.$$

By (2.10), we can find $N_2 > N_1$ such that $n \geq N_2$ implies

$$\sup_{\sigma \geq m(n_0)} |\varphi(\sigma)| \sum_{j=1}^{N} K_j \sum_{s=n_0}^{N_1-1} |1 - H(s)| \prod_{u=s+1}^{n-1} H(u) \sum_{v=s-\tau_j(s)}^{s-1} |h_j(v)|$$

$$= \sup_{\sigma \geq m(n_0)} |\varphi(\sigma)| \sum_{j=1}^{N} K_j \prod_{u=N_2}^{n-1} H(u) \sum_{s=n_0}^{N_1-1} |1 - H(s)| \prod_{u=s+1}^{N_2-1} H(u) \sum_{v=s-\tau_j(s)}^{s-1} |h_j(v)|$$

$$< \epsilon_1 \sum_{j=1}^{N} K_j.$$

Now, apply condition (iv) to have $|I_4| < \epsilon_1 \sum_{j=1}^{N} K_j + \epsilon_1 \sum_{j=1}^{N} K_j \alpha_j < 2\epsilon_1 \sum_{j=1}^{N} K_j$. Thus, $I_4 \to 0$ as $n \to \infty$. Similarly, by using (2.10), then, if $n \geq N_2$ then term I_5 in (2.6) satisfies

$$|I_4| = \left| \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} H(u) \{h_j(s - \tau_j(s)) - a_j(s)\} f_j(\varphi(s - \tau_j(s))) \right|$$

$$\leq \sum_{j=1}^{N} \sum_{s=n_0}^{N_1-1} \prod_{u=s+1}^{n-1} H(u) |h_j(s - \tau_j(s)) - a_j(s)| |f_j(\varphi(s - \tau_j(s)))|$$

$$+ \sum_{j=1}^{N} \sum_{s=N_1}^{n-1} \prod_{u=s+1}^{n-1} H(u) |h_j(s - \tau_j(s)) - a_j(s)| |f_j(\varphi(s - \tau_j(s)))|$$

$$\leq \sup_{\sigma \geq m(n_0)} |\varphi(\sigma)| \sum_{j=1}^{N} K_j \prod_{u=N_2}^{n-1} H(u) \sum_{s=n_0}^{N_1-1} \prod_{u=s+1}^{N_2-1} H(u) |h_j(s - \tau_j(s)) - a_j(s)|$$

$$+ \epsilon_1 \sum_{j=1}^{N} K_j \sum_{s=N_1}^{n-1} \prod_{u=s+1}^{n-1} H(u) |h_j(s - \tau_j(s)) - a_j(s)|$$

$$\leq \epsilon_1 \sum_{j=1}^{N} K_j + \epsilon_1 \sum_{j=1}^{N} K_j \alpha_j < 2\epsilon_1 \sum_{j=1}^{N} K_j.$$

Thus, $I_5 \to 0$ as $n \to \infty$. In conclusion $(P\varphi)(n) \to 0$ as $n \to \infty$, as required. Hence P maps S_ψ^ϵ into S_ψ^ϵ.

By the contraction mapping principle, P has a unique fixed point $x \in S_\psi^\epsilon$ which solves (1.8). Therefore, the zero solution of (1.8) is asymptotically stable. $\square$

Letting $N = 1$, $\tau_1 = \tau$, $f_1 = f$, we have

Corollary 2.1. *Let* $h : [m(n_0), \infty) \cap \mathbb{Z} \to \mathbb{R}$ *be an arbitrary sequence. Suppose that the following conditions are satisfied,*

  (i) the function f is odd, increasing on $[0, l]$,

  (ii) $f(x)$ and $x - f(x)$ satisfy a Lipschitz condition with constant K on an interval $[-l, l]$, and $x - f(x)$ is nondecreasing on $[0, l]$,

 (iii) $|h(n)| < 1$ for $n \in [m(n_0), \infty) \cap \mathbb{Z}$ and $|h(n - \tau(n)) - a(n)| < 1$ for $n \in [n_0, \infty) \cap \mathbb{Z}$,

 (iv) there exist constants $\alpha \in (0, 1)$ for all $n \in [n_0, \infty) \cap \mathbb{Z}$ such that

$$\sum_{s=n_0}^{n-1} |h(s)| \prod_{u=s+1}^{n-1} [1 - h(u)] + \sum_{s=n-\tau(n)}^{n-1} |h(s)|$$

$$+ \sum_{s=n_0}^{n-1} |h(s)| \prod_{u=s+1}^{n-1} [1 - h(u)] \sum_{v=s-\tau(s)}^{s-1} |h(v)|$$

$$+ \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} [1 - h(u)] |h(s - \tau(s)) - a(s)|$$

$$\leq \alpha.$$

Then the zero solution of (1.5) is asymptotically stable if

$$\prod_{u=n_0}^{n-1} [1 - h(u)] \to 0 \ \text{as } n \to \infty.$$

Remark 2.1. When $h(s) = a(g(s))$, where $g(s)$ is the inverse function of $s - \tau(s)$, Corollary 2.1 reduces to Theorem D. Thus Theorem 2.3 generalizes and improves Theorem D.

For the special case $f_j(x) = x$, we can get

Corollary 2.2. *Suppose that $H(n) \neq 0$ for all $n \in [n_0, \infty) \cap \mathbb{Z}$ and there exists a constant $\alpha \in (0, 1)$ such that for $n \in [n_0, \infty) \cap \mathbb{Z}$*

$$\sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} |h_j(s)| + \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} |1 - H(s)| \prod_{u=s+1}^{n-1} |H(u)| \sum_{v=s-\tau_j(s)}^{s-1} |h_j(v)|$$

$$+ \sum_{j=1}^{N} \sum_{s=n_0}^{n-1} \prod_{u=s+1}^{n-1} |H(u)| |h_j(s - \tau_j(s)) - a_j(s)|$$

$$\leq \alpha.$$

Then the zero solution of (1.3) is asymptotically stable if

$$\prod_{u=n_0}^{n-1} H(u) \to 0 \ as \ n \to \infty.$$

Remark 2.2. When $h_j(s) = a_j(g_j(s))$, where $g_j(s)$ is the inverse function of $s - \tau_j(s)$, for $j = 1, 2, ..., N$, Corollary 2.2 reduces to Theorem B. Thus Theorem 2.3 improves Theorem B.

Remark 2.3. The method in this paper can be extended to the following nonlinear delay difference systems with several variable delays

$$\triangle x_i(n) = -\sum_{j=1}^{N}\sum_{k=1}^{r_{ij}} a_{ij}^k(n) f_{ij}^k \left(x_j \left(n - \tau_{ij}^k(n) \right) \right), \ i = 1, ..., N.$$

Acknowledgements

References

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