ON SOME QUALITATIVE BEHAVIORS OF CERTAIN DIFFERENTIAL EQUATIONS OF FOURTH ORDER WITH MULTIPLE RETARDATIONS*

Erdal Korkmaz[1] and Cemil Tunc[2,†]

Abstract In this paper, we give sufficient conditions to guarantee the asymptotic stability and boundedness of solutions to a kind of fourth-order functional differential equations with multiple retardations. By using the Lyapunov-Krasovskii functional approach, we establish two new results on the stability and boundedness of solutions, which include and improve some related results in the literature.

Keywords Boundedness, stability, Lyapunov functional.

MSC(2010) 34D20, 34C11.

1. Introduction

Differential equations of higher order with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. (see [5, 9, 11–13, 22, 24, 29, 34, 37]).

In particular, problems concerning qualitative behaviors of linear and non-linear differential equations of the fourth order with and without delay have received the attention of many authors. Many criteria have been found, most of which based on the non-linearization of the famous Routh –Hurwitz criteria. Among numerous papers dealing with stability, boundedness and existence of periodic solutions, we refer in particular to the book of Reissig et al. [35] as a survey and the papers of Abou-El-Ela et al. [1] , Abou-El-Ela et al. [2,3], Adesina &Ogundare [4], Bereketoglu [6], Burganskaja [7], Cartwright [8],Chukwu [10], Ezeilo [14–16], Ezeilo &Tejumola [17], Harrow [18], Hu [19], Kang & Si [20], Kaufman & Harrow [21], Korkmaz & Tunc [23], Lalli & Skrapek [25–27], Lin et al. [28], Ogundare & Okecha [30], Ogurcov [31,32], Okoronkwo [33], Sadek [36], Sinha [38], Skidmore [39], Skrapek & Lalli [40], Tejumola [41], Tejumola & Tchegnani [42], Tunc [43–55], Wu & Xiong [56], Yang [57] and the references cited therein.

In the literature, the first attempt for the following linear and non-linear differ-

---

[†]The corresponding author. Email address:cemtunc@yahoo.com(C. Tunc)
[1]Dep. of Mathematics, Mus Alparslan University, Mus, 49100 Mus, Turkey
[2]Dep. of Mathematics, Yuzuncu Yil University, Van, 65080, Turkey

ential equations of the fourth order

$$x^{(4)} + a_1 x''' + a_2 x'' + a_3 x' + a_4 x = 0,$$
$$x^{(4)} + a_1 x''' + a_2 x'' + a_3 x' + f(x) = 0$$

and

$$x^{(4)} + a_1 x''' + a_2 x'' + \psi(x) x' + f(x) = 0$$

was due to Cartwright [8], who investigated the asymptotic stability of zero solution of these equations by the Lyapunov's second method based on the use of the Routh –Hurwitz criteria. The key idea in [8] is to transform the each of the above differential equations into an equivalent system of first order differential equations and then to construct an appropriate the Lyapunov's function which guarantees the asymptotic stability of the solution. Throughout the above mentioned book and papers, the authors followed the approach as presented in [8] and the stability, boundedness and existence of periodic solutions for fourth order differential equations with and without delay were studied. It should be noted that a disadvantage of the Lyapunov's second method is that construction of a Lyapunov function, which gives meaningful results, remains as an open problem in the literature by this time.

This paper is concerned with the problems of stability and boundedness of solutions to the fourth order nonlinear differential equation with multiple variable delays

$$x^{(4)} + \phi(x'')x''' + h(x, x', x'')$$
$$+ \sum_{i=1}^{n} g_i(x'(t - r_i(t))) + \sum_{i=1}^{n} f_i(x(t - r_i(t))) = p(t, x, x', x'', x''') \qquad (1.1)$$

or its equivalent system

$$x' = y,$$
$$y' = z,$$
$$z' = u,$$
$$u' = -\phi(z)u - h(x, y, z) - \sum_{i=1}^{n} g_i(y) - \sum_{i=1}^{n} f_i(x) \qquad (1.2)$$
$$+ \sum_{i=1}^{n} \int_{t-r_i(t)}^{t} g_i'(y(s))z(s)ds + \sum_{i=1}^{n} \int_{t-r_i(t)}^{t} f_i'(x(s))y(s)ds$$
$$+ p(t, x(t), y(t), z(t), u(t)),$$

where ϕ, h, g_i, f_i and p depend only on the variables displayed explicitly and $r_i(t)$ are positive bounded delays with $r(t) = \max_{1 \le i \le n} r_i(t)$, $r'(t) \le \gamma$, $0 < \gamma < 1$. It is assumed as basic that the functions ϕ, h, g_i, f_i and p are continuous in their respective arguments and satisfy a Lipchitz condition in x, y, z and u; $h(x, y, 0) = g_i(0) = f_i(0) = 0$ and the derivatives $\frac{dg_i}{dy'} = g_i'(y)$ and $\frac{df_i}{dx} = f_i'(x)$ exist and are also continuous.

The motivation of this paper comes by the above mentioned results on the fourth order differential equations. This paper is the first attempt on the stability

and boundedness of solutions of fourth order differential equations with multiple variable delays. Our results complement and improve some recent ones in the literature. By this we mean that when we check the related literature, there are many papers on the stability and boundedness of the solutions to certain nonlinear differential equations of fourth order without and with delay (See the references of this paper). However, we cannot find any paper on the same topic for the differential equations with multiple delays. Here, we will get some results for certain nonlinear differential equations of fourth order with multiple delays. Besides, when $r_i(t) = 0$, $(i = 1, 2, ..., n)$ in Eq.(1.1), the considered equation and assumptions to be established here reduce that in the literature. However, we would not like top give details here. Therefore, the results of this paper include and improve some related ones in in the literature.

2. Preliminaries

We also consider the general autonomous delay differential system

$$\dot{x} = f(x_t), \quad x_t(\theta) = x(t + \theta), \quad -r \le \theta \le 0, \quad t \ge 0. \tag{2.1}$$

Lemma 2.1 (Sinha [38]). *Suppose $f(0) = 0$. Let V be a continuous functional defined on $C_H = C$ with $V(0) = 0$, and let $u(s)$ be a function, non-negative and continuous for $0 \le s < \infty$, $u(s) \to \infty$ as $s \to \infty$ with $u(0) = 0$. If for all $\phi \in C$, $u(|\phi(0)|) \le V(\phi)$, $V(\phi) \ge 0, \dot{V}(\phi) \le 0$, then the solution $x = 0$ of Eq.(2.1) is stable. If we define $Z = \left\{ \phi \in C_H : \dot{V}(\phi) = 0 \right\}$, then the solution $x = 0$ of Eq.(2.1) is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.*

3. The main results

For convenience, we shall introduce the notations:

$$\phi_1(z) = \begin{cases} \dfrac{1}{z} \displaystyle\int_0^z \phi(\tau)d\tau, & z \ne 0, \\ \phi(0), & z = 0, \end{cases}$$

$$h_1(x, y, z) = \begin{cases} \dfrac{h(x, y, z)}{z}, & z \ne 0, \\ h(x, y, z), & z = 0 \end{cases}$$

and

$$G(y) = \begin{cases} \displaystyle\sum_{i=1}^{n} \dfrac{g_i(y)}{y}, & y \ne 0, \\ \displaystyle\sum_{i=1}^{n} g_i'(0), & y = 0. \end{cases}$$

Our first main result is the following theorem.

Theorem 3.1. *Let $P(.) \equiv 0$. In addition to the basic assumptions imposed on*

ϕ, h, g_i and f_i , we assume that there are positive constants $a, b, c, d, \delta, \varepsilon$ and k_i, c_i $(i = 1, 2, ..., n)$ such that:

(i) $abc - c \sum_{i=1}^{n} k_i - ad\phi(z) \geq \delta > 0$, $g_i'(y) \leq k_i$, for all y and z;

(ii) $0 < d - \frac{a\delta}{4c} < \sum_{i=1}^{n} f_i'(x) \leq \sum_{i=1}^{n} c_i \leq d$, $0 < f_i'(x) \leq c_i$ for all x;

(iii) $0 \leq G(y) - c < \frac{\delta}{8c} \sqrt{\frac{d}{2ac}}$ for all y;

(iv) $0 \leq h_1(x,y,z) - b \leq \frac{c^3 \varepsilon}{2d^2}$, for all z $(z \neq 0)$ in which $0 < \varepsilon \leq \frac{\delta}{2acD}$, $D = ab + \frac{bc}{d}$;

(v) $\phi(z) \geq a$, $\phi_1(z) - \phi(z) < \frac{\delta}{2a^2 c}$, for all z;

(vi) $\left| \frac{\partial h(x,y,z)}{\partial x} \right| \leq \frac{\sqrt{a\varepsilon\delta}}{4\sqrt{2}}$, $\left| \frac{\partial h(x,y,z)}{\partial y} \right| \leq \frac{\delta}{16c}$.

If

$$r(t) = \max_{1 \leq i \leq n} r_i(t) < \min \left\{ \frac{\varepsilon c}{2(\beta d + \beta ab + 2\lambda)}, \frac{\delta}{8ac(ab + d + 2\mu)}, \frac{2a\varepsilon}{\alpha(ab + d)} \right\},$$

with $\alpha = \varepsilon + \frac{1}{a}$, $\beta = \varepsilon + \frac{d}{c}$, $\lambda = \frac{d(\alpha+\beta+1)}{2(1-\gamma)} > 0$, $\mu = \frac{ab(\alpha+\beta+1)}{2(1-\gamma)} > 0$, then the zero solution of system Eq.(1.2) is asymptotically stable.

Remark 3.1. Conditions (i) and (v) imply

$$\phi(z) < \frac{bc}{d}, \quad \sum_{i=1}^{n} k_i < ab, \ a\varepsilon \leq 1. \tag{3.1}$$

Theorem 3.2. Let $P(.) \neq 0$. Assume that all the assumptions of Theorem 3.1 and the following assumption hold;

$$|p(t,x,y,z,u)| \leq q(t),$$

where $\max q(t) < \infty$ and $q \in L^1(0,\infty)$, $L^1(0,\infty)$ is space of integrable Lebesgue functions. If

$$r(t) = \max_{1 \leq i \leq n} r_i(t) < \min \left\{ \frac{\varepsilon c}{2(\beta d + \beta ab + 2\lambda)}, \frac{\delta}{8ac(ab + d + 2\mu)}, \frac{2a\varepsilon}{\alpha(ab + d)} \right\},$$

with $\alpha = \varepsilon + \frac{1}{a}$, $\beta = \varepsilon + \frac{d}{c}$, $\lambda = \frac{d(\alpha+\beta+1)}{2(1-\gamma)} > 0$, $\mu = \frac{ab(\alpha+\beta+1)}{2(1-\gamma)} > 0$, then there exists a finite positive constant K such that the solution $x(t)$ of Eq.(1.1) defined by the initial function

$$x(t) = \psi(t), \ x'(t) = \psi'(t), \ x''(t) = \psi''(t), \ x'''(t) = \psi'''(t), \text{where } t \in [t_0 - r, t_0],$$

satisfies the inequalities

$$|x(t)| \leq K, \ |x'(t)| \leq K, \ |x''(t)| \leq K, \ |x'''(t)| \leq K,$$

for all $t \geq t_0$, where $\psi \in C^3([t_0 - r, t_0], R)$.

Proof of Theorem 3.1. To prove the theorem, we define a Lyapunov-Krasovskii $V = V(x, y, z, u)$ functional by

$$
\begin{aligned}
2V =& 2\beta \int_0^x \sum_{i=1}^n f_i(\zeta)d\zeta + b\beta y^2 - \alpha dy^2 + 2\int_0^y \sum_{i=1}^n g_i(\eta)d\eta \\
& + 2\alpha \int_0^z h(x, y, \tau)d\tau + 2\int_0^z \phi(\tau)\tau d\tau - \beta z^2 + \alpha u^2 \\
& + 2y \sum_{i=1}^n f_i(x) + 2\alpha z \sum_{i=1}^n f_i(x) + 2\alpha z \sum_{i=1}^n g_i(y) \\
& + 2\beta y \int_0^z \phi(\tau)d\tau + 2\beta yu + 2zu \\
& + 2\lambda \int_{-r(t)}^0 \int_{t+s}^t y^2(\theta)d\theta ds + 2\mu \int_{-r(t)}^0 \int_{t+s}^t z^2(\theta)d\theta ds, \qquad (3.2)
\end{aligned}
$$

where $\alpha = \varepsilon + \frac{1}{a}$, $\beta = \varepsilon + \frac{d}{c}$ and λ, μ are positive constants that will be determined later in the proof. It is clear that $V(0, 0, 0, 0) = 0$. We write the expression of $2V$ in the following form:

$$
2V = V_1 + V_2 + V_3 + V_4 + V_5, \qquad (3.3)
$$

where

$$
V_1 = 2\beta \int_0^x \sum_{i=1}^n f_i(\zeta)d\zeta - \left(\frac{1}{c}\right)\left[\sum_{i=1}^n f_i(x)\right]^2,
$$

$$
V_2 = \left[b\beta - \alpha d - \beta^2 \phi_1(z)\right] y^2 + 2\int_0^y \sum_{i=1}^n g_i(\eta)d\eta - cy^2,
$$

$$
V_3 = \left[2\alpha \int_0^z h(x, y, \tau)d\tau - (\beta + \alpha^2 c)z^2\right] + \left[2\int_0^z \phi(\tau)\tau d\tau - \phi_1(z)z^2\right],
$$

$$
V_4 = \left[\alpha - \frac{1}{\phi_1(z)}\right] u^2 + 2\alpha yz \left[G(y) - c\right],
$$

$$
V_5 = \frac{1}{c}\left[\sum_{i=1}^n f_i(x) + cy + \alpha cz\right]^2 + \frac{1}{\phi_1(z)}\left[u + \phi_1(z)z + \beta\phi_1(z)y\right]^2,
$$

$$
+ 2\lambda \int_{-r(t)}^0 \int_{t+s}^t y^2(\theta)d\theta ds + 2\mu \int_{-r(t)}^0 \int_{t+s}^t z^2(\theta)d\theta ds.
$$

By using the mean value theorem for derivatives and (ii), it follows that

$$
\begin{aligned}
V_1 &= 2\beta \int_0^x \sum_{i=1}^n f_i(\zeta)d\zeta - \left(\frac{1}{c}\right)\left[\sum_{i=1}^n f_i(x)\right]^2 \\
&= 2\left(\varepsilon + \frac{d}{c}\right)\int_0^x \sum_{i=1}^n f_i(\zeta)d\zeta - \left(\frac{1}{c}\right)\left[\sum_{i=1}^n f_i(x)\right]^2 \\
&= 2\int_0^x \left[\left(\varepsilon + \frac{d}{c}\right) - \left(\frac{1}{c}\right)\sum_{i=1}^n f_i'(\zeta)\right]\sum_{i=1}^n f_i(\zeta)d\zeta
\end{aligned}
$$

$$\geq 2 \int_0^x \left[\left(\varepsilon + \frac{d}{c} \right) - \frac{d}{c} \right] \sum_{i=1}^n f_i(\zeta) d\zeta$$

$$\geq 2\varepsilon \int_0^x \sum_{i=1}^n f_i(\zeta) d\zeta$$

$$\geq \varepsilon \left(d - \frac{a\delta}{4c} \right) x^2. \tag{3.4}$$

Because of the mean value theorem for derivatives , from condition (iii) and Eq.(3.1), we have $c < ab$. By using the mean value theorem for integrals, we have

$$\phi_1(z) = \frac{1}{z} \int_0^z \phi(s) ds = \phi(\gamma z), \qquad 0 \leq \gamma \leq 1. \tag{3.5}$$

In view of (i), (iii), Eq.(3.1) and Eq.(3.5) , it follows

$$V_2 = \left[b\beta - \alpha d - \beta^2 \phi_1(z) \right] y^2 + 2 \int_0^y \sum_{i=1}^n g_i(\eta) d\eta - cy^2$$

$$\geq \left[b\beta - \alpha d - \beta^2 \phi(\theta_1 z) \right] y^2,$$

$$\geq \left(\frac{\delta d}{2ac^2} \right) y^2, \qquad 0 \leq \theta_1 \leq 1. \tag{3.6}$$

By noting the estimate $\phi_1'(z) = - \left(\frac{1}{z^2} \right) \int_0^z \phi(\tau) d\tau + \frac{\phi(z)}{z}$, it can be written

$$z^2 \phi_1(z) = \int_0^z \phi_1(\tau) \tau d\tau + \int_0^z \phi(\tau) \tau d\tau. \tag{3.7}$$

From (i), (iii)-(v), Eq.(3.1) and Eq.(3.7), we get

$$V_3 = \left[2\alpha \int_0^z h(x, y, \tau) d\tau - (\beta + \alpha^2 c) z^2 \right] + \left[2 \int_0^z \phi(\tau) \tau d\tau - \phi_1(z) z^2 \right]$$

$$\geq \left[\alpha b - \beta - \alpha^2 c \right] z^2 + \int_0^z \left[\phi(\tau) - \phi_1(\tau) \right] \tau d\tau$$

$$\geq \left(\frac{\delta}{2a^2 c} \right) z^2. \tag{3.8}$$

It is also clear that

$$\left[\alpha - \frac{1}{\phi_1(z)} \right] u^2 = \left[\varepsilon + \frac{1}{a} - \frac{1}{\phi_1(z)} \right] u^2$$

$$\geq \left[\varepsilon + \frac{1}{a} - \frac{1}{a} \right] u^2 = \varepsilon u^2$$

and

$$|2\alpha yz \left[G(y) - c \right]| \leq \frac{4}{a} |yz \left[G(y) - c \right]|$$

$$\leq \frac{\delta}{2ac} \sqrt{\frac{d}{2ac}} yz$$

$$\leq \frac{\delta d}{4ac^2} y^2 + \frac{\delta}{8a^2 c} z^2.$$

Then

$$V_4 \geq \varepsilon u^2 - \frac{\delta d}{4ac^2} y^2 - \frac{\delta}{8a^2 c} z^2. \tag{3.9}$$

Combining estimates Eq.(3.4), Eq.(3.6), Eq.(3.8) and Eq.(3.9), we obtain

$$2V \geq \varepsilon \left(d - \frac{a\delta}{4c} \right) x^2 + \left(\frac{\delta d}{4ac^2} \right) y^2 + \varepsilon u^2 + \frac{\delta}{8a^2 c} z^2 + V_5. \tag{3.10}$$

Calculating the time derivative of V along solutions of system Eq.(1.2), we have

$$\frac{dV}{dt} = - \left[d - \sum_{i=1}^{n} f_i'(x) \right] \left(y + \frac{\alpha}{2} z \right)^2 - [h_1(x, y, z) - b] \left(z + \frac{\beta}{2} y \right)^2$$

$$- [\beta G(y) - d] y^2 + \frac{\beta^2}{4} [h_1(x, y, z) - b] y^2 - \left[b - \alpha \sum_{i=1}^{n} g_i'(y) - \beta \phi_1(z) \right] z^2$$

$$+ \frac{\alpha^2}{4} \left[d - \sum_{i=1}^{n} f_i'(x) \right] z^2 - [\alpha \phi(z) - 1] u^2 + \alpha y \int_{0}^{z} \frac{\partial h(x, y, \tau)}{\partial x} d\tau$$

$$+ \alpha z \int_{0}^{z} \frac{\partial h(x, y, \tau)}{\partial y} d\tau + (\alpha u + \beta y + z) \sum_{i=1}^{n} \int_{t - r_i(t)}^{t} g_i'(y(s)) z(s) ds$$

$$+ (\alpha u + \beta y + z) \sum_{i=1}^{n} \int_{t - r_i(t)}^{t} f_i'(x(s)) y(s) ds + \lambda y^2 r(t) + \mu z^2 r(t)$$

$$- \lambda(1 - r'(t)) \int_{t - r(t)}^{t} y^2(s) ds - \mu(1 - r'(t)) \int_{t - r(t)}^{t} z^2(s) ds. \tag{3.11}$$

By conditions (i)-(vi), Eq.(3.5) and $a\varepsilon < 1$, it is obvious that

$$\beta G(y) - d \geq c\varepsilon, \varepsilon \leq \frac{\delta}{2acD} < \frac{d}{c}, \frac{\beta^2}{4} [h_1(x, y, z) - b] \leq \frac{c\varepsilon}{2},$$

and

$$b - \alpha \sum_{i=1}^{n} g_i'(y) - \beta \phi_1(z) > \frac{1}{ac} \left[abc - c \sum_{i=1}^{n} g_i'(y) - ad\phi(\gamma z) \right] - ab\varepsilon - \varepsilon \phi(\gamma z)$$

$$\geq \frac{\delta}{2ac}, \tag{3.12}$$

and

$$\frac{\alpha^2}{4} \left[d - \sum_{i=1}^{n} f_i'(x) \right] < \frac{\delta}{4ac}, \qquad \alpha \phi(z) - 1 \geq a\varepsilon,$$

$$\alpha \left| \int_{0}^{z} \frac{\partial h(x, y, \tau)}{\partial x} y d\tau \right| \leq \frac{c\varepsilon}{4} y^2 + \frac{\delta}{8ac} z^2, \qquad \alpha \left| \int_{0}^{z} \frac{\partial h(x, y, \tau)}{\partial y} z d\tau \right| \leq \frac{\delta}{16ac} z^2.$$

Hence, we get

$$\frac{dV}{dt} \leq -\frac{c\varepsilon}{4}y^2 - \frac{\delta}{16ac}z^2 - a\varepsilon u^2$$

$$+ (\alpha u + \beta y + z)\sum_{i=1}^{n}\int_{t-r_i(t)}^{t} f_i'(x(s))y(s)ds$$

$$+ (\alpha u + \beta y + z)\sum_{i=1}^{n}\int_{t-r_i(t)}^{t} g_i'(y(s))z(s)ds + \lambda y^2 r(t) + \mu z^2 r(t)$$

$$- \lambda(1 - r'(t))\int_{t-r(t)}^{t} y^2(s)ds - \mu(1 - r'(t))\int_{t-r(t)}^{t} z^2(s)ds. \qquad (3.13)$$

Because of $\sum_{i=1}^{n} f_i'(x) \leq \sum_{i=1}^{n} c_i \leq d$, $\sum_{i=1}^{n} g_i'(y) \leq \sum_{i=1}^{n} k_i \leq ab$ and $2mn \leq m^2 + n^2$, we can write

$$\frac{dV}{dt} \leq -\frac{1}{2}\left[\frac{c\varepsilon}{2} - (ab\beta + \beta d + 2\lambda)r\right]y^2 - \left[a\varepsilon - \frac{\alpha}{2}(ab + d)r\right]u^2$$

$$-\frac{1}{2}\left[\frac{\delta}{8ac} - (ab + d + 2\mu)r\right]z^2$$

$$+ \left[\frac{d}{2}(\alpha + \beta + 1) - \lambda(1 - \gamma)\right]\int_{t-r(t)}^{t} y^2(s)ds$$

$$+ \left[\frac{ab}{2}(\alpha + \beta + 1) - \mu(1 - \gamma)\right]\int_{t-r(t)}^{t} z^2(s)ds. \qquad (3.14)$$

Let $\lambda = \frac{d(\alpha+\beta+1)}{2(1-\gamma)} > 0$ and $\mu = \frac{ab(\alpha+\beta+1)}{2(1-\gamma)} > 0$. Then

$$\frac{dV}{dt} \leq -\frac{1}{2}\left[\frac{c\varepsilon}{2} - (ab\beta + \beta d + 2\lambda)r\right]y^2 - \left[a\varepsilon - \frac{\alpha}{2}(ab + d)r\right]u^2$$

$$-\frac{1}{2}\left[\frac{\delta}{8ac} - (ab + d + 2\mu)r\right]z^2. \qquad (3.15)$$

Therefore, if

$$r(t) = \max_{1\leq i\leq n} r_i(t) < \min\left\{\frac{\varepsilon c}{2(\beta d + \beta ab + 2\lambda)}, \frac{\delta}{8ac(ab + d + 2\mu)}, \frac{2a\varepsilon}{\alpha(ab + d)}\right\},$$

then we have

$$\frac{dV}{dt} \leq -\sigma\left(y^2 + z^2 + u^2\right), \text{ for some constants } \sigma > 0. \qquad (3.16)$$

Using $\frac{dV}{dt} = 0$ and system Eq.(1.2), we can easily obtain $x = y = z = u = 0$. This means that the largest invariant set in Z is $Q = \{0\}$. Hence, all the conditions in Lemma 2.1 are satisfied, and so the zero solution of Eq.(1.1)is asymptotically stable. The proof of Theorem 3.1 is now complete. □

Proof of Theorem 3.2. From equality Eq.(3.10), we have

$$V \geq D_1 \left(x^2 + y^2 + z^2 + u^2 \right), \tag{3.17}$$

where $D_1 = \frac{1}{2} \min \left\{ \varepsilon \left(d - \frac{a\delta}{4c} \right), \frac{\delta d}{4ac^2}, \frac{\delta}{8a^2c}, \varepsilon u^2 \right\}$. Taking the total derivative of E-q.(3.2) with respect to t along the trajectory of system Eq.(1.2), we obtain

$$
\begin{aligned}
\frac{dV}{dt} &\leq -\sigma \left(y^2 + z^2 + u^2 \right) + |\alpha u + \beta y + z| \, |p(t, x(t), y(t), z(t), u(t))| \\
&\leq |\alpha u + \beta y + z| \, |p(t, x(t), y(t), z(t), u(t))| \\
&\leq D_2 \left(|y| + |z| + |u| \right) q(t),
\end{aligned}
$$

where $D_2 = \max \{\alpha, \beta, 1\}$. Now, making use of the estimate $|y| < 1 + y^2$, it is clear that

$$\frac{dV}{dt} \leq D_2 \left(3 + y^2 + z^2 + u^2 \right) q(t).$$

By Eq.(3.17), we also have

$$\left(y^2 + z^2 + u^2 \right) \leq \left(x^2 + y^2 + z^2 + u^2 \right) \leq D_1^{-1} V(x, y, z, u).$$

Hence

$$
\begin{aligned}
\frac{dV}{dt} &\leq D_2 \left(3 + D_1^{-1} V(x, y, z, u) \right) q(t) \\
&= 3 D_2 q(t) + D_2 D_1^{-1} V(x, y, z, u) q(t).
\end{aligned}
$$

Now, integrating the last inequality from 0 to t, using the assumption $q \in L^1(0, \infty)$ and Gronwall-Reid-Bellman inequality, we obtain

$$V(x, y, z, u) \leq V(x_0, y_0, z_0, u_0) + 3 D_2 A + D_2 D_1^{-1} \int_0^t V(x_s, y_s, z_s, u_s) q(s) ds$$

$$\leq \left(V(x_0, y_0, z_0, u_0) + 3 D_2 A \right) \exp \left(D_2 D_1^{-1} \int_0^t q(s) ds \right)$$

$$\leq \left(V(x_0, y_0, z_0, u_0) + 3 D_2 A \right) \exp \left(D_2 D_1^{-1} A \right) = K_1 < \infty, \tag{3.18}$$

where $K_1 > 0$ is constant, $\left(V(x_0, y_0, z_0, u_0) + 3 D_2 A \right) \exp \left(D_2 D_1^{-1} A \right) = K_1 < \infty$ and $A = \int_0^t q(s) ds$. Now, the inequalities Eq.(3.17) and Eq.(3.18) together yield that

$$x^2(t) + y^2(t) + z^2(t) + u^2(t) \leq D_1^{-1} V(x_t, y_t, z_t, u_t) \leq K^2,$$

where $K^2 = K_1 D_1^{-1}$. Thus, we can conclude that

$$|x(t)| \leq K, \ \ |y(t)| \leq K, \ \ |z(t)| \leq K, \ \ |u(t)| \leq K$$

for all $t \geq t_0$. That is,

$$|x(t)| \leq K, \ \ |x'(t)| \leq K, \ \ |x''(t)| \leq K, \ \ |x'''(t)| \leq K$$

for all $t \geq t_0$. The proof of Theorem 3.2 is completed. $\qquad\square$

Example 3.1. Consider for $n = 2$ in Eq.(1.1) the fourth order nonlinear differential equation with multiple variable delays

$$
\begin{aligned}
& x^{(4)} + (2 + \cos x'')x''' + ((50.4)z + 0.4 \sin x \cos x' \sin x'') + 4x'(t - r_1(t)) \\
& + 0.5 \sin x'(t - r_1(t)) + 7x'(t - r_2(t)) + 0.5 \sin x'(t - r_2(t)) \\
& + x(t - r_1(t)) + 0.9 \sin x(t - r_1(t)) + 2x(t - r_2(t)) + \sin x(t - r_2(t)) \\
& = \frac{2}{1 + t^2 + \exp(x) + (x')^4 + \sin^2 x'' + (x''')^2}
\end{aligned}
\tag{3.19}
$$

or its equivalent system

$$
\begin{aligned}
x' =& y, \\
y' =& z, \\
z' =& u, \\
u' =& -(2 + \cos z)u - ((50.4)z + 0.4 \sin x \cos y \sin z) - (4y + 0.5 \sin y) \\
& - (7y + 0.5 \sin y) - (x + 0.9 \sin x) - (2x + \sin x) + \int_{t - r_1(t)}^{t} (4 + 0.5 \cos y)z\,ds \\
& + \int_{t - r_2(t)}^{t} (7 + 0.5 \cos y)z\,ds + \int_{t - r_1(t)}^{t} (1 + 0.9 \cos x)y\,ds \\
& + \int_{t - r_2(t)}^{t} (2 + \cos x)y\,ds + \frac{2}{1 + t^2 + \exp(x) + y^4 + \sin^2 z + u^2},
\end{aligned}
$$

where $r_1(t)$, $r_2(t)$ are positive bounded delays with $r(t) = \max\{r_1(t),\ r_2(t)\}$, $r'(t) \leq \gamma$, $0 < \gamma < 1$.

$\phi(z) = 2 + \cos z$, $\phi(z) \geq 1$, $a = 1$, $\phi_1(z) - \phi(z) = \frac{1}{z} \int_0^z (2 + \cos \tau)d\tau - (2 + \cos z) < 2 < \frac{\delta}{2a^2 c}$, for all z, $g_1(y) = 4y + 0.5 \sin y$, $g_1(0) = 0$, $g_1'(y) < 4.5$, $k_1 = 4.5$, $g_2(y) = 7y + 0.5 \sin y$, $g_2(0) = 0$, $g_2'(y) < 7.5$, $k_2 = 7.5$, $f_1(x) = x + 0.9 \sin x$, $f_1(0) = 0$, $0 < f_1'(x) < 2$, $c_1 = 2$, $f_2(x) = 2x + \sin x$, $f_2(0) = 0$,
$1 < f_2'(x) < 3$, $c_2 = 3$, $d = 10$, $G(y) = \begin{cases} \frac{11y + \sin y}{y}, & y \neq 0 \\ 12, & y = 0. \end{cases}$, $0 \leq G(y) - 10 <$

$\frac{\delta}{8c}\sqrt{\frac{d}{2ac}}$ for all y, $c = 10$, $h(x, y, z) = (50.4)z + 0.4 \sin x \cos y \sin z$, $h(x, y, 0) = 0$,

$h_1(x, y, z) = \begin{cases} \frac{(50.4)z + 0.4 \sin x \cos y \sin z}{z}, & z \neq 0 \\ h(x, y, z), & z = 0 \end{cases}$, $50 \leq 50.4 + \frac{0.4 \sin x \cos y \sin z}{z} \leq 50.8$,

for all z ($z \neq 0$), $b = 50$, $\left| \frac{\partial h(x,y,z)}{\partial x} \right| = |0.4 \cos x \cos y \sin z| \leq \frac{\sqrt{a\varepsilon\delta}}{4\sqrt{2}}$, $\left| \frac{\partial h(x,y,z)}{\partial y} \right| = |-0.4 \sin x \sin y \sin z| \leq \frac{\delta}{16c}$, $\delta = 350$, $D = 100$, $\varepsilon = 0.17$,
$p(t, x, y, z, u) = \frac{2}{1 + t^2 + \exp(x) + y^4 + \sin^2 z + u^2} \leq \frac{2}{1 + t^2} = q(t)$, $\int_0^\infty q(s)ds = \int_0^\infty \frac{2}{1 + s^2}ds = \pi < \infty$ that is $q \in L^1(0, \infty)$. Thus, all the assumptions of Theorem 3.1 and Theorem 3.2 hold. That is, null solution of Eq.(3.19) is asymptotically stable and all solutions of the same equation are bounded, when $p \equiv 0$ and $p \neq 0$ in Eq.(3.19), respectively.

References

[1] A. M. A. Abou-El-Ela and A. I. Sadek, *A stability result for certain fourth order differential equations,* Ann. Differential Equations, 6(1990)(1), 1–9.

[2] A. M. A. Abou-El-Ela, A. I. Sadek and A. M. Mahmoud, *On the stability of solutions of certain fourth-order nonlinear non-autonomous delay differential equation*, Int. J. Appl. Math., 22(2009)(2), 245–258.

[3] A. M. A. Abou-El-Ela, A. I. Sadek and A. M. Mahmoud, *On the stability of solutions to a certain fourth-order vector delay differential equation*, Ann. Differential Equations, 28(2012)(1), 1–10.

[4] O. A. Adesina and B. S.Ogundare, *Some new stability and boundedness results on a certain fourth order nonlinear differential equation*, Nonlinear Stud., 19(2012)(3), 359–369.

[5] S. Ahmad and M. Rama Mohana Rao, *Theory of ordinary differential equations*, With applications in biology and engineering. Affiliated East-West Press Pvt. Ltd., New Delhi, 1999.

[6] H. Bereketoglu, *Asymptotic stability in a fourth order delay differential equation*, Dynam. Systems Appl., 7(1998)(1), 105–115.

[7] L. I. Burganskaja, *The stability in the large of the zero solution of certain nonlinear fourth order differential equations*, Differencial'nye Uravnenija, 6(1970), 1314–1317.

[8] M. L. Cartwright, *On the stability of solutions of certain differential equations of the fourth order*, Quart. J. Mech. Appl. Math., 9(1956), 185–194.

[9] K. E. Chlouverakis and J. C. Sprott, *Chaotic hyperjerk systems*, Chaos Solitons Fractals, 28(2006)(3), 739–746.

[10] E. N. Chukwu, *On the stability of a nonhomogeneous differential equation of the fourth order*, Ann. Mat. Pura Appl., 92(1972)(4), 1–11.

[11] J. Cronin–Scanlon, *Some mathematics of biological oscillations*, SIAM Rev., 19(1977)(1), 100–138.

[12] R. Eichhorn, S. J. Linz and P. Hänggi, *Transformations of nonlinear dynamical systems to jerky motion and its application to minimal chaotic flows*, Phys Rev E, 58(1998), 7151–7164.

[13] Z. Elhadj and J. C. Sprott, *Boundedness of certain forms of jerky dynamics*, Qual. Theory Dyn. Syst., 11(2012)(2), 199–213.

[14] J. O. C. Ezeilo, *A stability result for solutions of a certain fourth order differential equation*, J. London Math. Soc., 37(1962), 28–32.

[15] J. O. C. Ezeilo, *On the boundedness and the stability of solutions of some differential equations of the fourth order*, J. Math. Anal. Appl., 5(1962), 136–146.

[16] J. O. C. Ezeilo, *Stability results for the solutions of some third and fourth order differential equations*, Ann. Mat. Pura Appl., 66(1964)(4), 233–249.

[17] J. O. C. Ezeilo and H. O. Tejumola, *On the boundedness and the stability properties of solutions of certain fourth order differential equations*, Ann. Mat. Pura Appl., 95(1973)(4), 131–145.

[18] M. Harrow, *Further results on the boundedness and the stability of solutions of some differential equations of the fourth order*, SIAM J. Math. Anal., 1(1970), 189–194.

[19] Chao Yang Hu, *The stability in the large for certain fourth order differential equations*, Ann. Differential Equations, 8(1992)(4), 422–428.

[20] H. Kang and L. Si, *Stability of solutions to certain fourth order delay differential equations*, Ann. Differential Equations, 26(2010(4), 407–413.

[21] H. Kaufman and M. Harrow, *A stability result for solutions of certain fourth order differential equations*, Rend. Circ. Mat.Palermo, 20(1971)(2), 186–194.

[22] V. Kolmanovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*, Kluwer Academic Publishers, Dordrecht, 1999.

[23] E. Korkmaz and C. Tunc, *Stability and boundedness to certain differential equations of fourth order with multiple delays*, Filomat, 28(2014)(5), 1049-1058.

[24] N. N. Krasovskiĭ, *Stability of motion*, Applications of Liapunov 's second method to differential systems and equations with delay. Stanford, Calif.: Stanford University Press, 1963.

[25] B. S. Lalli and W. A. Skrapek, *On the boundedness and the stability of some differential equations of the fourth order*, SIAM J. Math. Anal., 2(1971), 221–225.

[26] B. S. Lalli and W. A. Skrapek, *Some further stability and boundedness results of some differential equations of the fourth order*, Ann. Mat. Pura Appl., 90(1971)(4), 167–179.

[27] B. S. Lalli and W. A. Skrapek, *Further stability and boundedness results for the solutions of some differential equations of the fourth order*, Ann. Mat. Pura Appl., 95(1973)(4), 293–301.

[28] Shi-zhong Lin, Zheng-rong Liu and Yuan-hong Yu, *Stability for certain fourth order nonlinear differential equations*, Demonstratio Math., 31(1998)(1), 87–96.

[29] S. J. Linz, *On hyperjerky systems*, Chaos Solitons Fractals, 37(2008)(3), 741–747.

[30] B. S. Ogundare and G. E. Okecha, *Boundedness and stability properties of solution to certain fourth order non-linear differential equation*, Nonlinear Stud., 15(2008)(1), 61–70.

[31] A. I. Ogurcov, *On the stability in the large of the solutions of third and fourth order non-linear differential equations*, Izv. Vysš. Učebn. Zaved. Matematika, 1(1958)(2), 124–129.

[32] A. I. Ogurcov, *On the stability of the solutions of certain non-linear differential equations of third and fourth order*, Izv. Vysš. Učebn. Zaved. Matematika, 3(1959)(10), 200–209.

[33] O. E. Okoronkwo, *On stability and boundedness of solutions of a certain fourth-order delay differential equation*, Internat. J. Math. Math. Sci., 12(1989)(3), 589–602.

[34] L. L. Rauch, *Oscillation of a third order nonlinear autonomous system*, Contributions to the Theory of Nonlinear Oscillations, Annals of Mathematics Studies, Princeton University Press, Princeton, N. J., (20)(1950), 39–88..

[35] R. Reissig, G. Sansone and R. Conti, *Non-linear differential equations of higher order*, Translated from the German. Noordhoff International Publishing, Leyden, 1974.

[36] A. I. Sadek, *On the stability of solutions of certain fourth order delay differential equations*, Appl. Math. Comput., 148(2004)(2), 587–597.

[37] H. Smith, *An introduction to delay differential equations with applications to the life sciences*, Texts in Applied Mathematics, 57. Springer, New York, 2011.

[38] A. S. C. Sinha, *On stability of solutions of some third and fourth order delay-differential equations*, Information and Control, 23(1973), 165–172.

[39] A. S. Skidmore, *On the stability of solutions of a differential equation of fourth order*, J. London Math. Soc., 41(1966), 649–661.

[40] W. A. Skrapek and B. S. Lalli, *On the boundedness and stability of a fourth order differential equation*, Ann. Mat. Pura Appl., 123(1980)(4), 1–9.

[41] H. O. Tejumola, *Further results on the boundedness and the stability of certain fourth order differential equations*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., 52(1972)(8), 16–23

[42] H. O. Tejumola and B. Tchegnani, *Stability, boundedness and existence of periodic solutions of some third and fourth order nonlinear delay differential equations*, J. Nigerian Math. Soc., 19(2000), 9–19.

[43] C. Tunc, *On the stability and the boundedness properties of solutions of certain fourth order differential equations*, Istanbul Univ. Fen Fak. Mat. Derg., 54(1995), 161–173.

[44] C. Tunc, *A note on the stability and boundedness results of solutions of certain fourth order differential equations*, Appl. Math. Comput., 155(2004)(3), 837–843.

[45] C. Tunc, *Some stability and boundedness results for the solutions of certain fourth order differential equations*, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math., 44(2005), 161–171.

[46] C. Tunc, *Stability and boundedness of solutions to certain fourth order differential equations*, Electron. J. Differential Equations, 35(2006), 10.

[47] C. Tunc, *On stability of solutions of certain fourth order delay differential equations*, Chinese translation appears in Appl. Math. Mech., 27(2006)(8), 994–1000. Appl. Math. Mech. (English Ed.), 27(2006)(8), 1141–1148.

[48] C. Tunc, *About stability and boundedness of solutions of certain fourth order differential equations*, Nonlinear Phenom. Complex Syst., 9(2006)(4), 380–387.

[49] C. Tunc, *Some remarks on the stability and boundedness of solutions of certain differential equations of fourth-order*, Comput. Appl. Math., 26(2007)(1), 1–17.

[50] C. Tunc, *Some stability results for the solutions of certain fourth order delay differential equations*, Differential equations and applications. Vol. 4, 133–140, Nova Sci. Publ., New York, 2007.

[51] C. Tunc, *A boundedness criterion for fourth order nonlinear ordinary differential equations with delay*, Int. J. Nonlinear Sci., 6(2008)(3),195–201.

[52] C. Tunc, *On the stability of solutions to a certain fourth order delay differential equation*, Nonlinear Dynam., 51(2008)(1-2), 71–81.

[53] C. Tunc, *On the stability of solutions of non-autonomous differential equations of fourth order with delay*, Funct. Differ. Equ., 17(2010)(1-2), 195–212.

[54] C. Tunc, *On the stability and boundedness of solutions in a class of nonlinear differential equations of fourth order with constant delay*, Vietnam J. Math., 38(2010)(4), 453–466.

[55] C. Tunc, *The boundedness to nonlinear differential equations of fourth order with delay*, Nonlinear Stud., 17(2010)(1), 47–56.

[56] X. Wu and K. Xiong, *Remarks on stability results for the solutions of certain fourth order autonomous differential equations*, Internat. J. Control, 69(1998)(2), 353–360.

[57] J. Yang, *Stability and bifurcation analysis of a certain fourth-order delay differential equation*, Math. Appl., 26(2013)(3), 526–537.