

CONVEX SOLUTIONS OF THE POLYNOMIAL-LIKE ITERATIVE EQUATION WITH VARIABLE COEFFICIENTS*

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Abstract In this paper, by applying the Schauder's fixed point theorem we prove the existence of increasing and decreasing solutions of the polynomial-like iterative equation with variable coefficients and further completely investigate increasing convex (or concave) solutions and decreasing convex (or concave) solutions of this equation. The uniqueness and continuous dependence of those solutions are also discussed.

Keywords Iteration, functional equation, convex (concave) solutions, variable coefficients, divided difference.

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1. Introduction

The polynomial-like iterative equation ([10])

$$\lambda_1 f(x) + \lambda_2 f^2(x) + \cdots + \lambda_n f^n(x) = F(x), \quad x \in S, \quad (1.1)$$

where S is a subset of a linear space over \mathbb{R} , $F : S \rightarrow S$ is a given function, λ_i ($i = 1, \dots, n$) are real constants, $f : S \rightarrow S$ is the unknown function and f^i is the i th iterate of f , i.e., $f^i(x) = f(f^{i-1}(x))$ and $f^0(x) = x$ for all $x \in S$, is one of important forms of functional equation ([1, 7, 17, 22]) since it is the basic form of iterative functional equation, and the problem of iterative roots and the problem of invariant curves can be reduced to the kind of equations. For $S \subset \mathbb{R}$, many works (e.g. [6, 9, 11, 13, 21, 23, 24]) were contributed to the existence of solutions for Eq.(1.1). Some efforts were also devoted to Eq.(1.1) in high-dimensional spaces (e.g. [3, 5, 8, 16, 25]). One of generalizations for Eq.(1.1) is the following equation

$$\lambda_1(x)f(x) + \lambda_2(x)f^2(x) + \cdots + \lambda_n(x)f^n(x) = F(x), \quad x \in S, \quad (1.2)$$

where $\lambda_i : S \rightarrow \mathbb{R}$ ($i = 1, 2, \dots, n$) is a mapping. Eq.(1.2) is called polynomial-like iterative equations with variable coefficients, which was investigated in 1-dimensional space in [14, 19, 26]. More concretely, continuous solutions, differentiable solutions and analytic solutions for Eq.(1.2) were discussed respectively in [26], [14] and [19].

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Also in 1-dimensional space, the more general functional equation of Eq.(1.2), an iterative functional series equation with variable coefficients was studied and the existence and uniqueness of the solution were proved in [12].

In our paper, we consider convex solutions of Eq.(1.2). The study of convex solutions for the polynomial-like iterative Eq.(1.1) in 1-dimensional space can be found from [18, 20, 27] and in high dimensional spaces one can refer to [3, 5]. In [27], convex solutions and concave ones of Eq.(1.1) were discussed under the normalization condition: $\sum_{j=1}^n \lambda_j = 1$ on a compact interval, and continued the work of [27], increasing convex (or concave) solutions and decreasing convex (or concave) solutions of Eq.(1.1) were completely investigated with no normalization condition and no requirement of uniform sign of coefficients on a compact interval in [20]. In [18], nondecreasing convex solutions for Eq.(1.1) on open intervals (possibly unbounded) were discussed. In [3], a partial order was introduced by an order cone and the monotonicity and convexity depending on this order were considered. The existence and continuous dependence of increasing convex (concave) solutions for Eq.(1.1) in the ordered real Banach spaces were proved. In [5], monotone solutions and convex solutions of the Eq.(1.1) on an open set (possibly unbounded) in \mathbb{R}^N were discussed. Furthermore, convexity of multi-valued solutions ([2]) for Eq.(1.1) and convex solutions for generalized Eq.(1.1) ([4]) were also discussed. Up to now, there are no further results on convexity of solutions for the polynomial-like iterative equations with variable coefficients.

In this paper, we discuss convex (or concave) solutions of Eq.(1.2). Using the idea of [20], we first discuss the monotonicity and convexity of the product of two functions by divided difference and prove the existence of increasing and decreasing solutions for this equation. Then we completely investigate increasing convex (or concave) solutions and decreasing convex (or concave) solutions. The uniqueness and continuous dependence of those solutions are also discussed.

2. Some Lemmas

In this paper, we discuss Eq.(1.2) on $[a, b]$. As in [26], we may assume that $a = 0, b = 1$, and $I := [0, 1]$. Let $C(I)$ denote the real Banach space consisting of all continuous maps of I into \mathbb{R} with respect to the uniform norm $\|f\| = \max_{x \in I} |f(x)|$. As in [15], the first-order divided difference of f in $C(I)$ is denoted by

$$f[x_1, x_2] := \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

for distinct two points $x_1, x_2 \in I$, and the second-order divided difference of f is denoted by

$$f[x_1, x_2, x_3] := \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$$

for distinct three points $x_1, x_2, x_3 \in I$. Obviously, f is increasing (resp. decreasing) if $f[x_1, x_2] \geq 0$ (resp. ≤ 0) and f is convex (resp. concave) if $f[x_1, x_2, x_3] \geq 0$ (resp. ≤ 0) for all possible three points x_1, x_2, x_3 .

Lemma 2.1. *Let $f_1, f_2 \in C(I)$ and $x_1, x_2, x_3 \in I$ with $x_1 \neq x_2 \neq x_3$. Then*

$$\begin{aligned}(f_1 f_2)[x_1, x_2] &= f_1(x_2)f_2[x_1, x_2] + f_2(x_1)f_1[x_1, x_2] \\ &= f_2(x_2)f_1[x_1, x_2] + f_1(x_1)f_2[x_1, x_2], \\ (f_1 f_2)[x_1, x_2, x_3] &= f_2(x_3)f_1[x_1, x_2, x_3] + f_1(x_2)f_2[x_1, x_2, x_3] + f_1[x_1, x_2]f_2[x_1, x_3].\end{aligned}$$

Proof. For two points $x_1, x_2 \in I$ with $x_1 \neq x_2$, by definition of divided difference,

$$\begin{aligned}(f_1 f_2)[x_1, x_2] &= \frac{(f_1 f_2)(x_2) - (f_1 f_2)(x_1)}{x_2 - x_1} \\ &= \frac{f_1(x_2)f_2(x_2) - f_1(x_1)f_2(x_1)}{x_2 - x_1} \\ &= \frac{f_1(x_2)(f_2(x_2) - f_2(x_1)) + f_2(x_1)(f_1(x_2) - f_1(x_1))}{x_2 - x_1} \\ &= f_1(x_2)f_2[x_1, x_2] + f_2(x_1)f_1[x_1, x_2].\end{aligned}$$

Similarly,

$$(f_1 f_2)[x_1, x_2] = f_2(x_2)f_1[x_1, x_2] + f_1(x_1)f_2[x_1, x_2].$$

For three points $x_1, x_2, x_3 \in I$ with $x_1 \neq x_2 \neq x_3$,

$$\begin{aligned}&(f_1 f_2)[x_1, x_2, x_3] \\ &= \frac{(f_1 f_2)[x_2, x_3] - (f_1 f_2)[x_1, x_2]}{x_3 - x_1} \\ &= \frac{(f_2(x_3)f_1[x_2, x_3] + f_1(x_2)f_2[x_2, x_3]) - (f_1(x_2)f_2[x_1, x_2] + f_2(x_1)f_1[x_1, x_2])}{x_3 - x_1} \\ &= \frac{f_2(x_3)(f_1[x_2, x_3] - f_1[x_1, x_2]) + f_2(x_3)f_1[x_1, x_2]}{x_3 - x_1} \\ &\quad + \frac{f_1(x_2)(f_2[x_2, x_3] - f_2[x_1, x_2]) - f_2(x_1)f_1[x_1, x_2]}{x_3 - x_1} \\ &= f_2(x_3)f_1[x_1, x_2, x_3] + f_1(x_2)f_2[x_1, x_2, x_3] + f_1[x_1, x_2]f_2[x_1, x_3].\end{aligned}$$

The proof is completed. □

Let $J := [c, d]$. Similar to [20], define the following function classes.

$$\begin{aligned}C(I, J) &:= \{f \in C(I) : f(I) \subset J\}, \\ C_+(I, J) &:= \{f \in C(I, J) : f(0) = c, f(1) = d\}, \\ C_-(I, J) &:= \{f \in C(I, J) : f(0) = d, f(1) = c\}.\end{aligned}$$

For $-\infty \leq m \leq M \leq +\infty$, define

$$C(I, J; m, M) := \{f \in C(I, J) : m \leq f[x_1, x_2] \leq M, \forall x_1 \neq x_2\}.$$

Moreover, for $-\infty \leq m \leq M \leq +\infty$ and $-\infty \leq k \leq K \leq +\infty$, define

$$C(I, J; m, M, k, K) := \{f \in C(I, J; m, M) : k \leq f[x_1, x_2, x_3] \leq K, \forall x_1 \neq x_2 \neq x_3\}.$$

Let

$$\begin{aligned} C_+(I, J; m, M) &:= C(I, J; m, M) \cap C_+(I, J), \\ C_-(I, J; m, M) &:= C(I, J; m, M) \cap C_-(I, J), \\ C_+(I, J; m, M, k, K) &:= C(I, J; m, M, k, K) \cap C_+(I, J), \\ C_-(I, J; m, M, k, K) &:= C(I, J; m, M, k, K) \cap C_-(I, J). \end{aligned}$$

As shown in [20], $C_+(I, I; m, M)$, $C_-(I, I; m, M)$, $C_+(I, I; m, M, k, K)$, $C_-(I, I; m, M, k, K)$ are compact convex subsets of $C(I)$.

For any $c \in \mathbb{R}$, let $c^+ := \max\{c, 0\}$ and $c^- := \max\{-c, 0\}$. Both c^+ and c^- are nonnegative. Let $J^* = [-c^-, d^+]$.

Lemma 2.2. *Suppose that $f_1 \in C(I, J; m_1, M_1, k_1, K_1)$, $f_2 \in C(I, I; m_2, M_2, k_2, K_2)$. Then*

- (i) $f_1 f_2 \in C(I, J^*; m_1 + c^+ m_2 - c^- M_2, M_1 - d^- m_2 + d^+ M_2, k_1 + c^+ k_2 - c^- K_2 + m_1 M_2, K_1 - d^- k_2 + d^+ K_2 + M_1 M_2)$ if $m_1 \leq 0 \leq M_1, 0 \leq m_2 \leq M_2, k_1 \leq 0 \leq K_1, 0 \leq k_2 \leq K_2$;
- (ii) $f_1 f_2 \in C(I, J^*; m_1 + c^+ m_2 - c^- M_2, M_1 - d^- m_2 + d^+ M_2, k_1 + d^+ k_2 - d^- K_2 + m_1 M_2, K_1 - c^- k_2 + c^+ K_2 + M_1 M_2)$ if $m_1 \leq 0 \leq M_1, 0 \leq m_2 \leq M_2, k_1 \leq 0 \leq K_1, k_2 \leq K_2 \leq 0$;
- (iii) $f_1 f_2 \in C(I, J^*; m_1 + c^+ m_2 - c^- M_2, M_1 - d^- m_2 + d^+ M_2, k_1 + d^+ k_2 - c^- K_2 + m_1 M_2, K_1 - c^- k_2 + d^+ K_2 + M_1 M_2)$ if $m_1 \leq 0 \leq M_1, 0 \leq m_2 \leq M_2, k_i \leq 0 \leq K_i, i = 1, 2, cd \geq 0$;
- (iv) $f_1 f_2 \in C(I, J^*; m_1 + d^+ m_2 - d^- M_2, M_1 - c^- m_2 + c^+ M_2, k_1 + d^+ k_2 - c^- K_2 + m_2 M_1, K_1 - c^- k_2 + d^+ K_2 + m_1 m_2)$ if $m_1 \leq 0 \leq M_1, m_2 \leq M_2 \leq 0, k_i \leq 0 \leq K_i, i = 1, 2, cd \geq 0$.

Proof. The proofs of results (ii)-(iv) are similar to (i), so we only prove (i) in details. It is obviously $-c^- \leq f_1(x)f_2(x) \leq d^+$ because $c \leq f_1(x) \leq d, 0 \leq f_2(x) \leq 1, \forall x \in I$. Hence $f_1 f_2 \in C(I, J^*)$. Since $c \leq f_1(x) \leq d, \forall x \in I$ and $0 \leq m_2 \leq f_2[x_1, x_2] \leq M_2$, we have

$$cf_2[x_1, x_2] \leq f_1(x_2)f_2[x_1, x_2] \leq df_2[x_1, x_2]. \quad (2.1)$$

Note that

$$c^+ m_2 - c^- M_2 \leq cf_2[x_1, x_2] \leq c^+ M_2 - c^- m_2$$

and

$$d^+ m_2 - d^- M_2 \leq df_2[x_1, x_2] \leq d^+ M_2 - d^- m_2.$$

By (2.1), we get

$$c^+ m_2 - c^- M_2 \leq f_1(x_2)f_2[x_1, x_2] \leq d^+ M_2 - d^- m_2. \quad (2.2)$$

Similarly, $m_1 \leq f_1[x_1, x_2] \leq M_1$ and $0 \leq f_2(x) \leq 1, \forall x \in I$ imply that

$$m_1 f_2(x_1) \leq f_2(x_1)f_1[x_1, x_2] \leq M_1 f_2(x_1).$$

So

$$m_1 \leq f_2(x_1)f_1[x_1, x_2] \leq M_1 \quad (2.3)$$

because $m_1 \leq 0 \leq M_1$. By Lemma 2.1 and summarizing (2.2) and (2.3) we get

$$m_1 + c^+ m_2 - c^- M_2 \leq (f_1 f_2)[x_1, x_2] \leq M_1 + d^+ M_2 - d^- m_2.$$

Since $k_1 \leq f_1[x_1, x_2, x_3] \leq K_1$ and $0 \leq f_2(x) \leq 1, \forall x \in I$,

$$k_1 f_2(x_3) \leq f_2(x_3) f_1[x_1, x_2, x_3] \leq K_1 f_2(x_3). \quad (2.4)$$

$k_1 \leq 0 \leq K_1$ implies

$$k_1 \leq k_1 f_2(x_3) \leq 0, 0 \leq K_1 f_2(x_3) \leq K_1.$$

Hence, by (2.4),

$$k_1 \leq f_2(x_3) f_1[x_1, x_2, x_3] \leq K_1. \quad (2.5)$$

Similarly,

$$c^+ k_2 - c^- K_2 \leq f_1(x_2) f_2[x_1, x_2, x_3] \leq d^+ K_2 - d^- k_2 \quad (2.6)$$

because

$$c \leq f_1(x) \leq d, k_2 \leq f_2[x_1, x_2, x_3] \leq K_2, 0 \leq k_2 \leq K_2.$$

Since

$$m_1 \leq f_1[x_1, x_2] \leq M_1, m_2 \leq f_2[x_1, x_2] \leq M_2$$

and $0 \leq m_2 \leq M_2$, we have

$$m_1 f_2[x_1, x_2] \leq f_1[x_1, x_2] f_2[x_1, x_2] \leq M_1 f_2[x_1, x_2].$$

Hence

$$m_1 M_2 \leq f_1[x_1, x_2] f_2[x_1, x_2] \leq M_1 M_2 \quad (2.7)$$

because $m_1 \leq 0 \leq M_1$. By Lemma 2.1 and summarizing (2.5), (2.6) and (2.7), we get

$$k_1 + c^+ k_2 - c^- K_2 + m_1 M_2 \leq (f_1 f_2)[x_1, x_2, x_3] \leq K_1 - d^- k_2 + d^+ K_2 + M_1 M_2.$$

Consequently,

$$\begin{aligned} f_1 f_2 \in C(I, J^*; m_1 + c^+ m_2 - c^- M_2, M_1 + d^+ M_2 - d^- m_2, \\ k_1 + c^+ k_2 - c^- K_2 + m_1 M_2, K_1 - d^- k_2 + d^+ K_2 + M_1 M_2). \end{aligned}$$

The proof is completed. \square

Lemma 2.3. ([20, Lemma 2.2]). *Let I and J be compact intervals in \mathbb{R} such that $J \subset I$. Both $f_j : I \rightarrow J (j = 1, 2)$ are homeomorphisms such that $|f_j[x, y]| \leq \varsigma$ for all distinct $x, y \in I$, where $\varsigma > 0$ is a constant. Then (i) $\|f_1^i - f_2^i\| \leq \sum_{j=0}^{i-1} \varsigma^j \|f_1 - f_2\|, \forall i = 1, 2, \dots$, (ii) $\|f_1 - f_2\| \leq \varsigma \|f_1^{-1} - f_2^{-1}\|$.*

Lemma 2.4. ([20, Lemma 2.4]). *Suppose that $f_j \in C(I, I; m_j, M_j, k_j, K_j), j = 1, 2$. Then*

- (i) $af_1 + bf_2 \in C(I, I; a^+m_1 - a^-M_1 + b^+m_2 - b^-M_2, -a^-m_1 + a^+M_1 - b^-m_2 + b^+M_2, a^+k_1 - a^-K_1 + b^+k_2 - b^-K_2, -a^-k_1 + a^+K_1 - b^-k_2 + b^+K_2)$ for $a, b \in \mathbb{R}$;
- (ii) $f_2 \circ f_1 \in C(I, I; m_2m_1, M_2M_1, M_2k_1 + k_2M_1^2, M_2K_1 + K_2M_1^2)$ if $0 \leq m_1 \leq M_1, 0 \leq m_2 \leq M_2, k_1 \leq 0 \leq K_1$ and $k_2 \leq 0 \leq K_2$;
- (iii) $f_2 \circ f_1 \in C(I, I; m_2m_1, M_2M_1, m_2k_1 + k_2M_1^2, M_2K_1 + K_2M_1^2)$ if $0 \leq m_1 \leq M_1, 0 \leq m_2 \leq M_2, 0 \leq k_1 \leq K_1$ and $k_2 \leq 0 \leq K_2$;
- (iv) $f_2 \circ f_1 \in C(I, I; m_2m_1, M_2M_1, M_2k_1 + k_2M_1^2, m_2K_1 + K_2M_1^2)$ if $0 \leq m_1 \leq M_1, 0 \leq m_2 \leq M_2, k_1 \leq K_1 \leq 0$ and $k_2 \leq 0 \leq K_2$;
- (v) $f_2 \circ f_1 \in C(I, I; M_2M_1, m_2m_1, m_2K_1 + k_2m_1^2, m_2k_1 + K_2m_1^2)$ if $m_1 \leq M_1 \leq 0, m_2 \leq M_2 \leq 0, k_1 \leq 0 \leq K_1$ and $k_2 \leq 0 \leq K_2$;
- (vi) $f_2 \circ f_1 \in C(I, I; M_2m_1, m_2M_1, M_2k_1 + k_2m_1^2, M_2K_1 + K_2m_1^2)$ if $m_1 \leq M_1 \leq 0, 0 \leq m_2 \leq M_2, k_1 \leq 0 \leq K_1$ and $k_2 \leq 0 \leq K_2$;
- (vii) $f_2 \circ f_1 \in C(I, I; M_2m_1, m_2M_1, m_2k_1 + k_2m_1^2, M_2K_1 + K_2m_1^2)$ if $m_1 \leq M_1 \leq 0, 0 \leq m_2 \leq M_2, 0 \leq k_1 \leq K_1$ and $k_2 \leq 0 \leq K_2$;
- (viii) $f_2 \circ f_1 \in C(I, I; M_2m_1, m_2M_1, M_2k_1 + k_2m_1^2, m_2K_1 + K_2m_1^2)$ if $m_1 \leq M_1 \leq 0, 0 \leq m_2 \leq M_2, k_1 \leq K_1 \leq 0$ and $k_2 \leq 0 \leq K_2$.

Lemma 2.5. ([20, Lemma 2.5]). Let $f \in C(I, I; m, M, k, K)$ and $J := f(I)$. Then

- (i) $f^{-1} \in C(J, I; 1/M, 1/m, -K/m^3, -k/m^3)$ if $0 < m \leq M$ and $k \leq 0 \leq K$;
- (ii) $f^i \in C(I, I; m^i, M^i, k \sum_{j=i-1}^{2(i-1)} M^j, K \sum_{j=i-1}^{2(i-1)} M^j)$ if $0 \leq m \leq M$ and $k \leq 0 \leq K$ for $i = 1, 2, \dots$

Lemma 2.6. ([20, Lemma 2.6]). Let $f \in C(I, I; m, M, k, K)$, where $m \leq M \leq 0, k \leq 0 \leq K$. Then

$$\begin{aligned} f^{2i} &\in C(I, I; M^{2i}, m^{2i}, (Km + km^2)S_{i-1}(m), (km + Km^2)S_{i-1}(m)), \\ f^{2i+1} &\in C(I, I; m^{2i+1}, M^{2i+1}, Km^3S_{i-1}(m) + kS_i(m), km^3S_{i-1}(m) + KS_i(m)) \end{aligned}$$

for all $i = 1, 2, \dots$, where $S_l = \sum_{j=0}^l m^{2(j+l)}$ for $l = 0, 1, 2, \dots$

3. Increasing and decreasing solutions

Before discussing convexity, we prove the existence of increasing and decreasing solutions of Eq.(1.2). We need the following hypothesis:

(H1) $\lambda_i \in C(I, J_i, \alpha_i, \beta_i)$, where $J_i := [c_i, d_i]$ and $\sum_{i=1}^n \lambda_i(1) = 1, \alpha_i \leq 0 \leq \beta_i$.

Theorem 3.1. Suppose that (H1) holds and $F \in C_+(I, I; 0, M_1)$, where $M_1 \in (0, +\infty)$ is a constant. If

$$m_I M \geq M_1 \tag{3.1}$$

for a constant $M \in (0, +\infty)$, where

$$m_I := \alpha_1 + c_1^+ - c_1^- + \sum_{i=1}^{n-1} (\alpha_{i+1} - c_{i+1}^- M^i).$$

Then Eq.(1.2) has a increasing solution $f \in C_+(I, I; 0, M)$. Additionally, if

$$\sum_{i=1}^{n-1} \gamma_{i+1} \sum_{j=0}^{i-1} M^j < m_I, \quad (3.2)$$

where $\gamma_i := \max\{|c_i|, |d_i|\}$, $i = 2, 3, \dots, n$, then the solution f is unique in $C_+(I, I; 0, M)$ and depends continuously on F .

Proof. Define $L : C_+(I, I; 0, M) \rightarrow C(I)$ by

$$Lf(x) = \lambda_1(x)x + \lambda_2(x)f(x) + \dots + \lambda_n(x)f^{n-1}(x), \quad f \in C_+(I, I; 0, M).$$

By Lemma 2.5 (ii),

$$0 \leq f^i[x_1, x_2] \leq M^i, \quad i = 1, 2, \dots, n-1.$$

Hence, by Lemma 2.2 (i),

$$\alpha_{i+1} - c_{i+1}^- M^i \leq (\lambda_{i+1} f^i)[x_1, x_2] \leq \beta_{i+1} + d_{i+1}^+ M^i, \quad i = 1, 2, \dots, n-1.$$

Let $h(x) = \lambda_1(x)x$, $x \in I$. By Lemma 2.2 (i),

$$\alpha_1 + c_1^+ - c_1^- \leq h[x_1, x_2] \leq \beta_1 + d_1^+ - d_1^-. \quad (3.3)$$

By (H1) and (3.1), $Lf \in C(I, I, m_I, M_I)$, where

$$M_I := \beta_1 + d_1^+ - d_1^- + \sum_{i=1}^{n-1} (\beta_{i+1} + d_{i+1}^+ M^i),$$

and Lf is an orientation-preserving homeomorphism from I onto I . By Lemma 2.5 (i),

$$(Lf)^{-1} \in C(I, I; \frac{1}{M_I}, \frac{1}{m_I}). \quad (3.4)$$

Define a mapping $\mathcal{T} : C_+(I, I; 0, M) \rightarrow C(I)$ by

$$\mathcal{T}f(x) = (Lf)^{-1} \circ F(x), \quad f \in C_+(I, I; 0, M). \quad (3.5)$$

Note that $Lf(0) = 0, Lf(1) = 1$ by (H1). Hence, by $F(0) = 0, F(1) = 1$, we get $\mathcal{T}f(0) = 0, \mathcal{T}f(1) = 1$ and $\mathcal{T}f(I) \subset I$ because $\mathcal{T}f$ is increasing. By Lemma 2.4 (ii) and (3.4) we have

$$\mathcal{T}f \in C_+(I, I; 0, \frac{M_1}{m_I}),$$

which implies that \mathcal{T} is self-mapping on $C_+(I, I; 0, M)$ by (3.1). For $f_1, f_2 \in C_+(I, I; 0, M)$,

$$\begin{aligned} \|\mathcal{T}f_2 - \mathcal{T}f_1\| &= \|(Lf_2)^{-1} \circ F - (Lf_1)^{-1} \circ F\| \\ &= \|(Lf_2)^{-1} - (Lf_1)^{-1}\| \\ &\leq \frac{1}{m_I} \|Lf_2 - Lf_1\| \\ &\leq \frac{1}{m_I} \sum_{i=1}^{n-1} \gamma_{i+1} \|f_2^i - f_1^i\| \\ &\leq \frac{1}{m_I} \sum_{i=1}^{n-1} \gamma_{i+1} \sum_{j=0}^{i-1} M^j \|f_2 - f_1\| \end{aligned}$$

by Lemma 2.3. Hence \mathcal{T} is a continuous mapping. $C_+(I, I; 0, M)$ is a compact convex subset of $C(I)$. Schauder's fixed point theorem guarantees that \mathcal{T} has a fixed point $f \in C_+(I, I; 0, M)$ which is a solution of equation (1.2). By (3.2), similar to the proof of Theorem 3.1 in [20], uniqueness and continuous dependence of the solution can be proved. This completes the proof. \square

The following is devoted to decreasing solutions. We need the following hypotheses:

- (H2) $\lambda_i \in C(I, J_i, \alpha_i, \beta_i)$, where $J_i := [c_i, d_i]$, $\lambda_i(0) = \lambda_i(1) = \lambda_i$ and $\alpha_i \leq 0 \leq \beta_i$.
- (H3) $0 \leq \sum_{\text{even } i} \lambda_i \leq \sum_{\text{odd } i} \lambda_i \leq 1$.
- (H4) $F \in C(I, I; -M_1, 0)$ satisfies $F(0) = \sum_{\text{odd } i} \lambda_i$ and $F(1) = \sum_{\text{even } i} \lambda_i$, where $M_1 > 0$ is a constant.

Theorem 3.2. *Suppose that (H2), (H3) and (H4) hold. If*

$$m_D M \geq M_1 \tag{3.6}$$

for a constant $M \in (0, +\infty)$, where

$$m_D := c_1^+ - c_1^- + \sum_{i=1}^n \alpha_i - \sum_{\text{odd } i, \neq 1} c_i^- M^{i-1} - \sum_{\text{even } i} d_i^+ M^{i-1}.$$

Then Eq.(1.2) has a decreasing solution $f \in C_-(I, I; -M, 0)$. Additionally, if

$$\sum_{i=1}^{n-1} \gamma_{i+1} \sum_{j=0}^{i-1} M^j < m_D, \tag{3.7}$$

then the solution f is unique in $C_-(I, I; -M, 0)$ and depends continuously on F .

Proof. Define a mapping $L : C_-(I, I; -M, 0) \rightarrow C(I)$ as in Theorem 3.1. By Lemma 2.6,

$$0 \leq f^{2i}[x_1, x_2] \leq M^{2i}, -M^{2i+1} \leq f^{2i+1}[x_1, x_2] \leq 0, i = 1, 2, \dots$$

Similar to Theorem 3.1, by Lemma 2.2 (i),

$$\alpha_{2i+1} - c_{2i+1}^- M^{2i} \leq (\lambda_{2i+1} f^{2i})[x_1, x_2] \leq \beta_{2i+1} + d_{2i+1}^+ M^{2i}, i = 1, 2, \dots,$$

and by Lemma 2.2 (iv),

$$\alpha_{2i+2} - d_{2i+2}^+ M^{2i+1} \leq (\lambda_{2i+2} f^{2i+1})[x_1, x_2] \leq \beta_{2i+2} + c_{2i+2}^- M^{2i+1}, i = 0, 2, \dots$$

By (3.3), (H2), (H3), (H4) and (3.6) we have $Lf \in C(I, I_1, m_D, M_D)$, where

$$M_D := d_1^+ - d_1^- + \sum_{i=1}^n \beta_i + \sum_{\text{odd } i, \neq 1} d_i^+ M^{i-1} + \sum_{\text{even } i} c_i^- M^{i-1},$$

and Lf is an orientation-preserving homeomorphism from I onto $I_1 := [F(1), F(0)]$. Clearly $I_1 \subset I$ by (H3). By Lemma 2.5 (i),

$$(Lf)^{-1} \in C(I_1, I; \frac{1}{M_D}, \frac{1}{m_D}). \quad (3.8)$$

Define a mapping $\mathcal{T} : C_-(I, I; -M, 0) \rightarrow C(I)$ as in (3.5). Clearly $\mathcal{T}f(0) = 1, \mathcal{T}f(1) = 0$. By Lemma 2.4 (vi), (3.8) and $F \in C(I, I; -M_1, 0)$, we get

$$\mathcal{T}f \in C_-(I, I; -\frac{M_1}{m_D}, 0),$$

which implies that \mathcal{T} is self-mapping on $C_-(I, I; -M, 0)$ by (3.6). The proof of continuity of \mathcal{T} is similar to Theorem 3.1. $C_-(I, I; -M, 0)$ is a compact convex subset of $C(I)$. Schauder's fixed point theorem guarantees that \mathcal{T} has a fixed point $f \in C_-(I, I; -M, 0)$ which is a solution of equation (1.2). The remaining part of the proof is the same as the proof of Theorem 3.1. This completes the proof. \square

4. Convexity of solutions

In this section, we will discuss convexity of increasing solutions and convexity of decreasing solutions.

4.1. Convexity of increasing solutions

First we give convexity of increasing solutions. We need the following hypothesis:

(H5) $\lambda_i \in C(I, J_i; \alpha_i, \beta_i, \mu_i, \nu_i)$, where $J_i := [c_i, d_i]$ and $\sum_{i=1}^n \lambda_i(1) = 1, \alpha_i \leq 0 \leq \beta_i, \mu_i \leq 0 \leq \nu_i$.

Theorem 4.1. *Suppose that (H5) holds and $F \in C_+(I, I; 0, M_1, k_1, K_1)$, where $M_1 \in (0, +\infty)$ is a constant and $0 \leq k_1 \leq K_1$. If*

$$m_I M \geq M_1, \frac{k_1}{M_I} - \frac{K_{Icv} M_1^2}{m_I^3} \geq 0, \frac{K_1}{m_I} - \frac{k_{Icv} M_1^2}{m_I^3} \leq K \quad (4.1)$$

for constants $M, K \in (0, +\infty)$, where

$$k_{Icv} := \mu_1 + \alpha_1 + \sum_{i=1}^{n-1} (\mu_{i+1} - c_{i+1}^- K \sum_{j=i-1}^{2(i-1)} M^j + \alpha_{i+1} M^i),$$

$$K_{Icv} := \nu_1 + \beta_1 + \sum_{i=1}^{n-1} (\nu_{i+1} + d_{i+1}^+ K \sum_{j=i-1}^{2(i-1)} M^j + \beta_{i+1} M^i).$$

Then Eq.(1.2) has a convex solution $f \in C_+(I, I; 0, M, 0, K)$. Additionally, if (3.2) is satisfied then Eq.(1.2) has a unique increasing convex solution $f \in C_+(I, I; 0, M, 0, K)$, which continuously depends on F .

Proof. Define $L : C_+(I, I; 0, M, 0, K) \rightarrow C(I)$ as in Theorem 3.1. By Lemma 2.5 (ii),

$$f^i \in C(I, I; 0, M^i, 0, K \sum_{j=i-1}^{2(i-1)} M^j).$$

Hence, by Lemma 2.2 (i),

$$\alpha_{i+1} - c_{i+1}^- M^i \leq (\lambda_{i+1} f^i)[x_1, x_2] \leq \beta_{i+1} + d_{i+1}^+ M^i, \tag{4.2}$$

and

$$\begin{aligned} & \mu_{i+1} - c_{i+1}^- K \sum_{j=i-1}^{2(i-1)} M^j + \alpha_{i+1} M^i \\ & \leq (\lambda_{i+1} f^i)[x_1, x_2, x_3] \\ & \leq \nu_{i+1} + d_{i+1}^+ K \sum_{j=i-1}^{2(i-1)} M^j + \beta_{i+1} M^i, \end{aligned} \tag{4.3}$$

where $i = 1, 2, \dots, n - 1$. Let $h(x) = \lambda_1(x)x, x \in I$. By Lemma 2.2 (i),

$$\mu_1 + \alpha_1 \leq h[x_1, x_2, x_3] \leq \nu_1 + \beta_1. \tag{4.4}$$

By (H5) and the first inequality of (4.1), summarizing (4.2), (4.3), (3.3) and (4.4) we get $Lf \in C(I, I, m_I, M_I, k_{Icv}, K_{Icv})$ and Lf is an orientation-preserving homeomorphism from I onto I . By Lemma 2.5 (i),

$$(Lf)^{-1} \in C\left(I, I; \frac{1}{M_I}, \frac{1}{m_I}, -\frac{K_{Icv}}{(m_I)^3}, -\frac{k_{Icv}}{(m_I)^3}\right). \tag{4.5}$$

Define a mapping $\mathcal{T} : C_+(I, I; 0, M, 0, K) \rightarrow C(I)$ as in (3.5). Clearly $\mathcal{T}f(0) = 0, \mathcal{T}f(1) = 1$. By Lemma 2.4 (iii) and (4.5) we have

$$\mathcal{T}f \in C_+\left(I, I; 0, \frac{M_1}{m_I}, \frac{k_1}{M_I} - \frac{K_{Icv}M_1^2}{m_I^3}, \frac{K_1}{m_I} - \frac{k_{Icv}M_1^2}{m_I^3}\right),$$

which implies that \mathcal{T} is self-mapping on $C_+(I, I; 0, M, 0, K)$ by (4.1). The remaining part of the proof is the same as the proofs of Theorem 3.1. We complete the proof. \square

Concaveness of increasing solutions can be discussed similarly.

Theorem 4.2. Suppose that (H5) holds and $F \in C_+(I, I; 0, M_1, k_1, K_1)$, where $M_1 \in (0, +\infty)$ is a constant and $k_1 \leq K_1 \leq 0$. If

$$m_I M \geq M_1, \frac{k_1}{m_I} - \frac{K_{Icc}M_1^2}{m_I^3} + K \geq 0, \frac{K_1}{M_I} - \frac{k_{Icc}M_1^2}{m_I^3} \leq 0 \tag{4.6}$$

for constants $M, K \in (0, +\infty)$, where

$$k_{Icc} := \mu_1 + \alpha_1 + \sum_{i=1}^{n-1} (\mu_{i+1} - d_{i+1}^+ K \sum_{j=i-1}^{2(i-1)} M^j + \alpha_{i+1} M^i),$$

$$K_{Icc} := \nu_1 + \beta_1 + \sum_{i=1}^{n-1} (\nu_{i+1} + c_{i+1}^- K \sum_{j=i-1}^{2(i-1)} M^j + \beta_{i+1} M^i).$$

Then Eq.(1.2) has a concave solution $f \in C_+(I, I; 0, M, -K, 0)$. Additionally, if (3.2) is satisfied then Eq.(1.2) has a unique increasing concave solution $f \in C_+(I, I; 0, M, -K, 0)$, which continuously depends on F .

Example 4.1. Consider the equation

$$\left(-\frac{1}{40} \left(x - \frac{1}{2}\right)^2 + 1\right) f(x) + \frac{1}{40} \left(x - \frac{1}{2}\right)^2 f^2(x) = x^2, \quad x \in I := [0, 1], \quad (4.7)$$

where

$$\lambda_1(x) = -\frac{1}{40} \left(x - \frac{1}{2}\right)^2 + 1, \quad \lambda_2(x) = \frac{1}{40} \left(x - \frac{1}{2}\right)^2, \quad F(x) = x^2.$$

Obviously, equation (4.7) is the form of Eq.(1.2) and

$$\lambda_1 \in C \left([0, 1], \left[\frac{159}{160}, 1 \right], -\frac{1}{40}, \frac{1}{40}, -\frac{1}{40}, 0 \right),$$

$$\lambda_2 \in C \left([0, 1], \left[0, \frac{1}{160} \right], -\frac{1}{40}, \frac{1}{40}, 0, \frac{1}{40} \right),$$

$$F \in C_+(I, I, 0, 2, 1, 1).$$

$\sum_{i=1}^n \lambda_i(1) := 159/160 + 1/160 = 1, \alpha_1 = -1/40 \leq 0 \leq \beta_1 = 1/40, \alpha_2 = -1/40 \leq 0 \leq \beta_2 = 1/40, \mu_1 = -1/40 \leq \nu_1 = 0, \mu_2 = 0 \leq \nu_2 = 1/40$ imply that H5 is satisfied. Since

$$m_I := \alpha_1 + c_1 + \sum_{i=1}^{n-1} (\alpha_{i+1} - c_{i+1}^- M^i) = \frac{151}{160},$$

$$M_I := \beta_1 + d_1 + \sum_{i=1}^{n-1} (\beta_{i+1} + d_{i+1}^+ M^i) = \frac{21}{20} + \frac{1}{160} M,$$

$$k_{Icv} := \mu_1 + \alpha_1 + \sum_{i=1}^{n-1} (\mu_{i+1} - c_{i+1}^- K \sum_{j=i-1}^{2(i-1)} M^j + \alpha_{i+1} M^i) = -\frac{1}{20} - \frac{1}{40} M,$$

$$K_{Icv} := \nu_1 + \beta_1 + \sum_{i=1}^{n-1} (\nu_{i+1} + d_{i+1}^+ K \sum_{j=i-1}^{2(i-1)} M^j + \beta_{i+1} M^i) = \frac{8 + K + 4M}{160},$$

by calculation inequalities (4.1) hold when $M = 32/15, K = 2$, by Theorem 4.1 we see that Eq. (4.7) has a increasing convex solution $f \in C_+(I, I, 0, 32/15, 0, 2)$.

4.2. Convexity of decreasing solutions

Suppose that

(H6) $\lambda_i \in C(I, J_i; \alpha_i, \beta_i, \mu_i, \nu_i)$, where $J_i := [c_i, d_i]$ and $\lambda_i(0) = \lambda_i(1) = \lambda_i$, $c_i d_i \geq 0$, $\alpha_i \leq 0 \leq \beta_i$, $\mu_i \leq 0 \leq \nu_i$, $i = 1, 2, \dots, n$.

(H7) $F \in C(I, I; -M_1, 0, k_1, K_1)$ satisfies $F(0) = \sum_{\text{odd } i} \lambda_i$ and $F(1) = \sum_{\text{even } i} \lambda_i$, where $M_1 > 0$ is a constant and $0 \leq k_1 \leq K_1$.

Theorem 4.3. *Suppose that (H3), (H6) and (H7) hold and if*

$$m_D M \geq M_1, \frac{k_1}{M_D} - \frac{K_{Dcv} M_1^2}{m_D^3} \geq 0, \frac{K_1}{m_D} - \frac{k_{Dcv} M_1^2}{m_D^3} \leq K \quad (4.8)$$

for constants $M, K \in (0, +\infty)$, where

$$\begin{aligned} k_{Dcv} &:= \alpha_1 - c_2^- K - M\beta_2 + \sum_{i=1}^n \mu_i - \sum_{\text{odd } i, \neq 1} (d_i^+ K M S_{\frac{i-3}{2}}(M) + c_i^- K M^2 S_{\frac{i-3}{2}}(M) \\ &\quad - \alpha_i M^{i-1}) - \sum_{\text{even } i, \neq 2} (d_i^+ K M^3 S_{\frac{i-4}{2}}(M) + c_i^- K S_{\frac{i-2}{2}}(M) + \beta_i M^{i-1}), \\ K_{Dcv} &:= \beta_1 + d_2^+ K - M\alpha_2 + \sum_{i=1}^n \nu_i + \sum_{\text{odd } i, \neq 1} (c_i^- K M S_{\frac{i-3}{2}}(M) + d_i^+ K M^2 S_{\frac{i-3}{2}}(M) \\ &\quad + \beta_i M^{i-1}) + \sum_{\text{even } i, \neq 2} (c_i^- K M^3 S_{\frac{i-4}{2}}(M) + d_i^+ K S_{\frac{i-2}{2}}(M) - \alpha_i M^{i-1}). \end{aligned}$$

Then Eq.(1.2) has a convex solution $f \in C_-(I, I; -M, 0, 0, K)$. Additionally, if (3.7) is satisfied then Eq.(1.2) has a unique decreasing convex solution $f \in C_-(I, I; -M, 0, 0, K)$, which continuously depends on F .

Proof. Define $L : C_-(I, I; -M, 0, 0, K) \rightarrow C(I)$ as in Theorem 3.1. By Lemma 2.6, for $f \in C_-(I, I; -M, 0, 0, K)$ and positive integer i we have

$$f^{2i} \in C(I, I; 0, M^{2i}, -K M S_{i-1}(M), K M^2 S_{i-1}(M)),$$

and

$$f^{2i+1} \in C(I, I; -M^{2i+1}, 0, -K M^3 S_{i-1}(M), K S_i(M)).$$

Hence, by Lemma 2.2 (iii) and (iv), we have

$$\begin{aligned} \alpha_{2i+1} - c_{2i+1}^- M^{2i} &\leq (\lambda_{2i+1} f^{2i})[x_1, x_2] \leq \beta_{2i+1} + d_{2i+1}^+ M^{2i}, \\ \alpha_{2i+2} - d_{2i+2}^+ M^{2i+1} &\leq (\lambda_{2i+2} f^{2i+1})[x_1, x_2] \leq \beta_{2i+2} + c_{2i+2}^- M^{2i+1}, \\ \alpha_2 - d_2^+ M &\leq (\lambda_2 f)[x_1, x_2] \leq \beta_2 + c_2^- M, \end{aligned} \quad (4.9)$$

and

$$\begin{aligned}
& \mu_{2i+1} - d_{2i+1}^+ KMS_{i-1}(M) - c_{2i+1}^- KM^2S_{i-1}(M) + \alpha_{2i+1}M^{2i} \\
\leq & (\lambda_{2i+1}f^{2i})[x_1, x_2, x_3] \leq \nu_{2i+1} + c_{2i+1}^- KMS_{i-1}(M) + d_{2i+1}^+ KM^2S_{i-1}(M) \\
& + \beta_{2i+1}M^{2i}, \\
& \mu_{2i+2} - d_{2i+2}^+ KM^3S_{i-1}(M) - c_{2i+2}^- KS_i(M) - \beta_{2i+2}M^{2i+1} \\
\leq & (\lambda_{2i+2}f^{2i+1})[x_1, x_2, x_3] \leq \nu_{2i+2} + c_{2i+2}^- KM^3S_{i-1}(M) + d_{2i+2}^+ KS_i(M) \\
& - \alpha_{2i+2}M^{2i+1}, \\
& \mu_2 - c_2^- K - M\beta_2 \leq (\lambda_2f)[x_1, x_2, x_3] \leq \nu_2 + d_2^+ K - \alpha_2M,
\end{aligned} \tag{4.10}$$

where $i = 1, 2, \dots$. By (H3), (H6), (H7) and the first inequalities of (4.8), summarizing (4.9), (4.10), (3.3) and (4.4) we get $Lf \in C(I, I_1, m_D, M_D, k_{Dcv}, K_{Dcv})$ and Lf is an orientation-preserving homeomorphism from I onto $I_1 := [F(1), F(0)]$. By Lemma 2.5 (i),

$$(Lf)^{-1} \in C\left(I_1, I; \frac{1}{M_D}, \frac{1}{m_D}, -\frac{K_{Dcv}}{(m_D)^3}, -\frac{k_{Dcv}}{(m_D)^3}\right). \tag{4.11}$$

Define a mapping $\mathcal{T} : C_-(I, I; -M, 0, 0, K) \rightarrow C(I)$ as in (3.5). Clearly $\mathcal{T}f(0) = 1, \mathcal{T}f(1) = 0$. By Lemma 2.4 (vii) and (4.11) we have

$$\mathcal{T}f \in C_+\left(I, I; -\frac{M_1}{m_D}, 0, \frac{k_1}{M_D} - \frac{K_{Dcv}M_1^2}{m_D^3}, \frac{K_1}{m_D} - \frac{k_{Dcv}M_1^2}{m_D^3}\right),$$

which implies that \mathcal{T} is self-mapping on $C_-(I, I; -M, 0, 0, K)$ by (4.8). The remaining part of the proof is the same as the proofs of Theorem 3.1. This completes the proof. \square

We similarly give concaveness of decreasing solutions with the hypothesis:

(H8) $F \in C(I, I; -M_1, 0, k_1, K_1)$ satisfies $F(0) = \sum_{\text{oddi}} \lambda_i$ and $F(1) = \sum_{\text{eveni}} \lambda_i$, where $M_1 > 0$ is a constant and $k_1 \leq K_1 \leq 0$.

Theorem 4.4. *Suppose that (H3), (H6) and (H8) hold and if*

$$m_D M \geq M_1, \frac{k_1}{m_D} - \frac{K_{Dcc}M_1^2}{m_D^3} + K \geq 0, \frac{K_1}{M_D} - \frac{k_{Dcc}M_1^2}{m_D^3} \leq 0 \tag{4.12}$$

for constants $M, K \in (0, +\infty)$, where

$$\begin{aligned}
k_{Dcc} & := \alpha_1 - d_2^+ K - M\beta_2 + \sum_{i=1}^n \mu_i - \sum_{\text{oddi}, \neq 1} (d_i^+ KM^2S_{\frac{i-3}{2}}(M) + c_i^- KMS_{\frac{i-3}{2}}(M) \\
& \quad - \alpha_i M^{i-1}) - \sum_{\text{eveni}, \neq 2} (d_i^+ KS_{\frac{i-2}{2}}(M) + c_i^- KM^3S_{\frac{i-4}{2}}(M) + \beta_i M^{i-1}), \\
K_{Dcc} & := \beta_1 + c_2^- K - M\alpha_2 + \sum_{i=1}^n \nu_i + \sum_{\text{oddi}, \neq 1} (c_i^- KM^2S_{\frac{i-3}{2}}(M) + d_i^+ KMS_{\frac{i-3}{2}}(M) \\
& \quad + \beta_i M^{i-1}) + \sum_{\text{eveni}, \neq 2} (c_i^- KS_{\frac{i-2}{2}} + d_i^+ KM^3S_{\frac{i-4}{2}} - \alpha_i M^{i-1}).
\end{aligned}$$

Then Eq.(1.2) has a concave solution $f \in C_-(I, I; -M, 0, -K, 0)$. Additionally, if (3.7) is satisfied then Eq.(1.2) has a unique decreasing concave solution $f \in C_-(I, I; -M, 0, -K, 0)$, which continuously depends on F .

Example 4.2. Consider the equation

$$\lambda_1(x)f(x) + \lambda_2(x)f^2(x) = F(x), \quad x \in I := [0, 1], \tag{4.13}$$

where

$$\begin{aligned} \lambda_1(x) &= -\frac{1}{40} \left(x - \frac{1}{2}\right)^2 + \frac{161}{160}, \\ \lambda_2(x) &= \frac{1}{40} \left(x - \frac{1}{2}\right)^2 - \frac{1}{160}, \\ F(x) &= -x^2 + 1. \end{aligned}$$

Obviously, equation (4.13) is the form of Eq.(1.2) and

$$\begin{aligned} \lambda_1 &\in C \left([0, 1], \left[1, \frac{161}{160}\right], -\frac{1}{40}, \frac{1}{40}, -\frac{1}{40}, 0 \right), \\ \lambda_2 &\in C \left([0, 1], \left[-\frac{1}{160}, 0\right], -\frac{1}{40}, \frac{1}{40}, 0, \frac{1}{40} \right), \\ F &\in C_+(I, I, -2, 0, -1, -1). \end{aligned}$$

$0 \leq \sum_{\text{even } i} \lambda_i := \lambda_2 = 0 \leq \sum_{\text{odd } i} \lambda_i := \lambda_1 = 1 \leq 1$ implies that (H3) is satisfied. $c_1 d_1 = 161/160 > 0, c_2 d_2 = 0 \geq 0, \alpha_1 := -1/40 \leq 0 \leq \beta_1 := 1/40, \alpha_2 := -1/40 \leq 0 \leq \beta_2 := 1/40, \mu_1 = -1/40 \leq \nu_1 = 0, \mu_2 = 0 \leq \nu_2 = 1/40$ imply that (H6) is satisfied. $F(0) = 1 = \sum_{\text{odd } i} \lambda_i = \lambda_1$ and $F(1) = 0 = \sum_{\text{even } i} \lambda_i = \lambda_2$ imply that (H8) is satisfied. Since

$$\begin{aligned} m_D &:= c_1 + \sum_{i=1}^n \alpha_i - \sum_{\text{odd } i, \neq 1} c_i^- M^{i-1} - \sum_{\text{even } i} d_i^+ M^{i-1} = \frac{19}{20}, \\ M_D &:= d_1 + \sum_{i=1}^n \beta_i + \sum_{\text{odd } i, \neq 1} d_i^+ M^{i-1} + \sum_{\text{even } i} c_i^- M^{i-1} = \frac{167 + M}{160}, \\ k_{Dcc} &:= \alpha_1 - d_2^+ K - M\beta_2 + \sum_{i=1}^n \mu_i - \sum_{\text{odd } i, \neq 1} (d_i^+ K M^2 S_{\frac{i-3}{2}}(M) + c_i^- K M S_{\frac{i-3}{2}}(M) \\ &\quad - \alpha_i M^{i-1}) - \sum_{\text{even } i, \neq 2} (d_i^+ K S_{\frac{i-2}{2}}(M) + c_i^- K M^3 S_{\frac{i-4}{2}}(M) + \beta_i M^{i-1}) \\ &= -\frac{1}{20} - \frac{M}{40}, \\ K_{Dcc} &:= \beta_1 + c_2^- K - M\alpha_2 + \sum_{i=1}^n \nu_i + \sum_{\text{odd } i, \neq 1} (c_i^- K M^2 S_{\frac{i-3}{2}}(M) + d_i^+ K M S_{\frac{i-3}{2}}(M) \\ &\quad + \beta_i M^{i-1}) + \sum_{\text{even } i, \neq 2} (c_i^- K S_{\frac{i-2}{2}} + d_i^+ K M^3 S_{\frac{i-4}{2}} - \alpha_i M^{i-1}) \\ &= \frac{1}{20} + \frac{K}{160} + \frac{M}{40}, \end{aligned}$$

by calculation inequalities (4.12) hold when $M = 40/19, K = 2$, by Theorem 4.4 we see that Eq. (4.13) has a decreasing concave solution $f \in C_-(I, I, -M, 0, -K, 0)$.

We end the paper with remarks that in the special case where all $\lambda_i(x)$ are constants, our theorems imply the results (at the cases $a = 0, b = 1$) in [20] and it is difficult to simplify (4.1), (4.6), (4.8) and (4.12) similar to [20].

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