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ENTROPY SOLUTIONS FOR NONLINEAR ELLIPTIC ANISOTROPIC PROBLEMS WITH HOMOGENEOUS NEUMANN BOUNDARY CONDITION

Idrissa Ibrango[1] and Stanislas Ouaro[2,†]

Abstract This study is about a nonlinear anisotropic problem with homogeneous Neumann boundary condition. We first prove, by using the technic of monotone operators in Banach spaces, the existence of weak solution, and by approximation methods, we achieve a result of existence and uniqueness of entropy solution.

Keywords Anisotropic Sobolev spaces, variable exponent, monotone operators, entropy solutions.

MSC(2000) 35J20, 35J25, 35D30, 35B38, 35J60.

1. Introduction

We consider in this paper the following nonlinear anisotropic elliptic Neumann boundary value problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u}{\partial x_i}\right) + b(u) = f & \text{in} \quad \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i\left(x, \frac{\partial u}{\partial x_i}\right)\eta_i = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded domain of $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary and $meas(\Omega) > 0$, b is a real function, surjective, continuous, non-decreasing defined on $\mathbb{R}$, in which $b(0) = 0$, $f \in L^1(\Omega)$ and $\eta = (\eta_1, \ldots, \eta_N)$ is the unit outward normal on $\partial\Omega$.

Anisotropic problems arise in many applications as reaction-diffusion systems, modeling of propagation of an epidemic disease. For example, Bendahmane et *al.* [2] considered a reaction-diffusion system with general anisotropic diffusivisities and transport effects. It is supplemented with either mixed boundary conditions or no-flux boundary conditions knowing that the model studied by Bendahmane is a modeling of Feline leukemia virus. The paper deal with also variable exponents. Indeed, the interest of the study of PDEs with variable exponents lies on the fact that

†the corresponding author. Email address: ouaro@yahoo.fr,souaro@univ-ouaga.bf(S. Ouaro)
1LAboratoire de Mathématiques et Informatique (LAMI), UFR. Sciences et Techniques, Université Polytechnique de Bobo-Dioulasso, 01 BP 1091 Bobo 01, Bobo-Dioulasso, Burkina Faso
2LAboratoire de Mathématiques et Informatique (LAMI), UFR. Sciences Exactes et Appliquées, Université de Ouagadougou, 03 BP 7021 Ouaga 03, Ouagadougou, Burkina Faso

most materials can be modelled with sufficient accuracy using classical Lebesgue and Sobolev spaces L^p and $W^{1,p}$, where p is a fixed constant. However, for some materials with inhomogeneities (blood for example), for instance, electrorheological fluids, this is not adequate, but rather the exponent should be able to vary.

All papers tackling the issues about (1.1) considered particular cases of function b. Indeed, in [5] , Bonzi et *al.* studied the following problems.

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u}{\partial x_i}\right) + |u|^{p_M(x)-2}u \ = f & \text{in} \quad \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i\left(x, \frac{\partial u}{\partial x_i}\right)\eta_i = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where $f \in L^1(\Omega)$.

In [5] , the authors use minimization technics used in [11] or [6] (see also [9, 12]) to show the existence and uniqueness of entropy solution. In this paper, as the function b is more general, it is not possible to use minimization tehnics to get the existence of solution. Therefore, we use the technic of monotone operators in Banach spaces (see [13]) to get the existence of entropy solutions of (1.1). For the uniqueness, since b is not necessarily invertible, then, we proved the uniqueness of the entropy solution in terms of $b(u)$ which is clearly equivalent to the uniqueness of u if and only if b is invertible.

Benboubker et *al* [3] studied an anisotropic problem with variable exponent where the boundary condition is the homogeneous Dirichlet boundary condition. Therefore, the good space where to choose the solution is the space $\mathcal{T}_0^{1,\vec{p}(.)}(\Omega)$ as the set of the mesurable functions $u : \Omega \longrightarrow \mathbb{R}$ such that $T_k(u) \in W_0^{1,\vec{p}(.)}(\Omega)$. In this space, it is possible to use the Poincaré inequality to get some useful inequality for the existence of entropy solutions. We do not have the uniqueness of entropy solutions (see [3]).

In our paper, we consider anisotropic elliptic problems with homogeneous Neumann boundary condition. Therefore, we have to choose the entropy solution in a new and more general space as in [3], which is the space $\mathcal{T}_{\mathcal{H}}^{1,\vec{p}(.)}(\Omega)$ to be defined later.

The remaining part of the paper is the following: in section 2, we introduce some preliminary results and in section 3, we study the existence and uniqueness of entropy solution.

2. Mathematical preliminaries

In order to bring evidence our main result, we first have to describe the data involved in our problem.

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary domain $\partial\Omega$ and $\vec{p}(.) = \big(p_1(.), \dots, p_N(.)\big)$ such that for any $i = 1, \dots, N$, $p_i(.) : \overline{\Omega} \longrightarrow \mathbb{R}$ is a continuous function with

$$1 < p_i^- := ess \inf_{x\in\Omega} p_i(x) \leq ess \sup_{x\in\Omega} p_i(x) := p_i^+ < \infty. \tag{2.1}$$

For any $i = 1, \dots, N$, let $a_i : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be a Carathéodory function satisfying:

• there exists a positive constant C_1 such that

$$|a_i(x,\xi)| \le C_1\big(j_i(x) + |\xi|^{p_i(x)-1}\big) \tag{2.2}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$, where j_i is a non-negative function in $L^{p'_i(.)}(\Omega)$, with $\frac{1}{p_i(x)} + \frac{1}{p'_i(x)} = 1$;
• for $\xi,\ \eta \in \mathbb{R}$ with $\xi \neq \eta$ and for almost every $x \in \Omega$, there exists a positive constant C_2 such that

$$\big(a_i(x,\xi) - a_i(x,\eta)\big)(\xi - \eta) \ge \begin{cases} C_2|\xi-\eta|^{p_i(x)} & \text{if } \ |\xi-\eta| \ge 1, \\ C_2|\xi-\eta|^{p_i^-} & \text{if } \ |\xi-\eta| < 1, \end{cases} \tag{2.3}$$

and
• there exists a positive constant C_3 such that

$$a_i(x,\xi).\xi \ge C_3|\xi|^{p_i(x)}, \tag{2.4}$$

for every $\xi \in \mathbb{R}$ and for almost every $x \in \Omega$.

The hypotheses on a_i are classical in the study of nonlinear problems (see [5,6]).

Throughout this paper, we assume that

$$\frac{\bar{p}(N-1)}{N(\bar{p}-1)} < p_i^- < \frac{\bar{p}(N-1)}{N-\bar{p}}, \qquad \frac{p_i^+ - p_i^- - 1}{p_i^-} < \frac{\bar{p}-N}{\bar{p}(N-1)}, \tag{2.5}$$

and

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1, \tag{2.6}$$

where $\dfrac{N}{\bar{p}} = \displaystyle\sum_{i=1}^{N} \frac{1}{p_i^-}$.

A prototype example based on our assumptions is the following anisotropic $\vec{p}(.)$-harmonic system

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-2} \frac{\partial u}{\partial x_i}\right) = f, \tag{2.7}$$

which, in the particular case when $p_i = p$ for any $i = 1, \dots, N$, is the p -Laplace equation.

We also recall in this section some definitions and basic properties of anisotropic Lebesgue and Sobolev spaces. Set

$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} p(x) > 1 \ \text{ for any } \ x \in \overline{\Omega} \right\}$$

and denotes by

$$p_M(x) := \max\big(p_1(x), \dots, p_N(x)\big) \ \text{ and } \ \ p_m(x) := \min\big(p_1(x), \dots, p_N(x)\big).$$

For any $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space is defined by

$$L^{p(.)}(\Omega) := \big\{ u : u \text{ is a mesurable real valued function such that } \int_\Omega |u|^{p(x)}dx \ < \ \infty \big\},$$

endowed with the so-called Luxemburg norm

$$|u|_{p(.)} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

The $p(.)$-*modular* of the $L^{p(\cdot)}(\Omega)$ space is the mapping $\rho_{p(.)} : L^{p(\cdot)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)} dx.$$

For any $u \in L^{p(\cdot)}(\Omega)$, the following inequality (see [7,8]) will be used later:

$$\min\left\{|u|_{p(.)}^{p^-};\ |u|_{p(.)}^{p^+}\right\} \leq \rho_{p(.)}(u) \leq \max\left\{|u|_{p(.)}^{p^-};\ |u|_{p(.)}^{p^+}\right\}. \tag{2.8}$$

For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, with $\dfrac{1}{p(x)} + \dfrac{1}{q(x)} = 1$ in Ω, we have the Hölder type inequality:

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(.)}|v|_{q(.)}. \tag{2.9}$$

Remark 2.1. An important condition on the exponent in the study of variable exponent spaces is the log-Hölder continuity condition which is the following.

$\alpha : \Omega \to \mathbb{R}$ is locally log-Hölder continuous on Ω if there exists $c_1 > 0$ such that

$$\Big|\alpha(x) - \alpha(y)\Big| \leq \frac{c_1}{log(e + 1/\|x - y\|)},$$

for all $x, y \in \Omega$.

The condition α is used to get more results such as the boundedness of maximal operators, some imbedding results or the concept of Lebesgue points, but in this paper, this condition is not needed.

If Ω is bounded and $p, q \in C_+(\overline{\Omega})$ such that $p(x) \leq q(x)$ for any $x \in \Omega$, then the embedding $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous (see [10, Theorem 2.8]).

Herein we need the anisotropic Sobolev space

$$W^{1,\vec{p}(.)}(\Omega) := \left\{u \in L^{p_M(\cdot)}(\Omega)\ :\ \frac{\partial u}{\partial x_i} \in L^{p_i(\cdot)}(\Omega),\quad i = 1, \ldots, N\right\}.$$

This is a separable and reflexive Banach space (see [11]) under the norm

$$||u||_{\vec{p}(.)} = |u|_{p_M(.)} + \sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)}.$$

We introduce the numbers

$$q = \frac{N(\bar{p} - 1)}{N - 1}, \qquad q^* = \frac{N(\bar{p} - 1)}{N - \bar{p}} = \frac{Nq}{N - q},$$

and define $P_-^\star,\ P_-^+,\ P_{-,\infty}\ \in \mathbb{R}^+$ by

$$P_-^\star = \frac{N}{\displaystyle\sum_{i=1}^{N}\frac{1}{p_i^-} - 1}, \quad P_-^+ = \max\{p_1^-, \ldots, p_N^-\} \quad \text{and} \quad P_{-,\infty} = \max\{p_-^+,\ P_-^\star\}.$$

We have the following embedding results (see [7, Corollary 2.1]).

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded open set and for all $i = 1, \ldots, N$, $p_i \in L^\infty(\Omega)$, $p_i(x) \geq 1$ a.e. in Ω. Then, for any $q \in L^\infty(\Omega)$ with $q(x) \geq 1$ a.e. in Ω such that*

$$ess \inf_{x\in\Omega} \big(p_M(x) - q(x)\big) > 0,$$

we have the compact embedding

$$W^{1,\vec{p}(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega). \tag{2.10}$$

The following result is due to Troisi (see [14]).

Theorem 2.2. *Let $p_1, \ldots, p_N \in [1, +\infty)$; $g \in W^{1,(p_1,\ldots,p_N)}(\Omega)$ and let*

$$\begin{cases} q = (\bar{p})^* & if \quad (\bar{p})^* < N, \\ q \in [1, +\infty) & if \quad (\bar{p})^* \geq N. \end{cases}$$

Then, there exists a constant $C_4 > 0$ depending on $N, p_1, \ldots, p_N$ if $\bar{p} < N$ and also on q and $meas(\Omega)$ if $\bar{p} \geq N$ such that

$$||g||_{L^q(\Omega)} \leq C_4 \prod_{i=1}^{N} \left[||g||_{L^{p_M}(\Omega)} + \left|\left| \frac{\partial g}{\partial x_i} \right|\right|_{L^{p_i}(\Omega)} \right]^{1/N}. \tag{2.11}$$

In this paper, we will use the Marcinkiewicz space $\mathcal{M}^q(\Omega)$ $(1 < q < +\infty)$ as the set of mesurable functions $g : \Omega \longrightarrow \mathbb{R}$ for which the distribution function

$$\lambda_g(k) = meas\big(\{x \in \Omega : \ |g(x)| > k\}\big), \quad k \geq 0, \tag{2.12}$$

satisfies an estimate of the form

$$\lambda_g(k) \leq Ck^{-q}, \text{ for some finite constant } \ C > 0. \tag{2.13}$$

We will use the following pseudo norm in $\mathcal{M}^q(\Omega)$:

$$||g||_{\mathcal{M}^q(\Omega)} := \inf\{C > 0 : \lambda_g(k) \leq Ck^{-q}, \ \ \forall\ k > 0\}. \tag{2.14}$$

Finally, we use throughout the paper, the truncation function T_k, $(k > 0)$, by

$$T_k(s) = \max\{-k; \min\{k; s\}\}. \tag{2.15}$$

It is clear that $\lim\limits_{k\to\infty} T_k(s) = s$ and $|T_k(s)| = \min\{|s|; k\}$.

We need the following lemma proved in [5] :

Lemma 2.1. *Let g be a nonnegative function in $W^{1,\vec{p}(.)}(\Omega)$. Assume $\bar{p} < N$ and there exists a constant $C > 0$ such that*

$$\int_\Omega |T_k(g)|^{p_M^-} dx + \sum_{i=1}^{N} \int_{\{|g|\leq k\}} \left| \frac{\partial g}{\partial x_i} \right|^{p_i^-} dx \leq C(1 + k), \quad \forall\ k > 0. \tag{2.16}$$

Then, there exists a constant D, depending on C, such that

$$||g||_{\mathcal{M}^{q^*}(\Omega)} \leq D, \tag{2.17}$$

where $q^* = N(\bar{p} - 1)/(N - \bar{p})$.

Set $\mathcal{T}^{1,\vec{p}(.)}(\Omega)$ as the set of the mesurable functions $u : \Omega \longrightarrow \mathbb{R}$ such that $T_k(u) \in W^{1,\vec{p}(.)}(\Omega)$. We define the space $\mathcal{T}_{\mathcal{H}}^{1,\vec{p}(.)}(\Omega)$ as the set of function $u \in \mathcal{T}^{1,\vec{p}(.)}(\Omega)$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,\vec{p}(.)}(\Omega)$ satisfying

$$u_n \longrightarrow u \quad \text{a.e.} \quad \text{in} \quad \Omega \tag{2.18}$$

and

$$\frac{\partial T_k(u_n)}{\partial x_i} \longrightarrow \frac{\partial T_k(u)}{\partial x_i} \quad \text{in} \quad L^1(\Omega) \ \ \forall \ \ k > 0. \tag{2.19}$$

In the sequel we denote $W^{1,\vec{p}(.)}(\Omega) = E$ to simplify.

3. Existence and uniqueness result

Definition 3.1. A mesurable function $u \in \mathcal{T}_{\mathcal{H}}^{1,\vec{p}(.)}(\Omega)$ is an entropy solution of (1.1) if $b(u) \in L^1(\Omega)$ and for every $k > 0$,

$$\int_\Omega \sum_{i=1}^N a_i\left(x, \frac{\partial u}{\partial x_i}\right) \frac{\partial}{\partial x_i} T_k(u-\varphi)dx + \int_\Omega b(u)T_k(u-\varphi)dx \leq \int_\Omega f(x)T_k(u-\varphi)dx, \tag{3.1}$$

for all $\varphi \in E \cap L^\infty(\Omega)$.

The existence result is the following theorem:

Theorem 3.1. *Assume (2.1)-(2.6). Then, there exists at least one entropy solution of the problem (1.1).*

Proof. The proof is done in three steps.
Step 1. The approximate problem.
For any $n \in \mathbb{N}^*$, we consider the approximate problem

$$(P_n)\begin{cases} -\displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u_n}{\partial x_i}\right) + T_n\big(b(u_n)\big) + \frac{1}{n}|u_n|^{p_M(x)-2}u_n = f_n & \text{in} \quad \Omega, \\ \displaystyle\sum_{i=1}^N a_i\left(x, \frac{\partial u_n}{\partial x_i}\right)\eta_i = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{3.2}$$

where $f_n = T_n(f) \in L^\infty(\Omega)$.
Note that

$$\begin{aligned} &f_n \underset{n\to+\infty}{\longrightarrow} f \quad \text{in} \quad L^1(\Omega) \quad \text{and} \quad a.e. \text{ in } \Omega, \quad \|f_n\|_\infty \leq \frac{\|f\|_1}{meas(\Omega)}, \\ &\text{and} \quad \|f_n\|_1 = \int_\Omega |f_n|dx \leq \int_\Omega |f|dx = ||f||_1. \end{aligned} \tag{3.3}$$

Definition 3.2. A mesurable function $u_n \in E$ is a weak solution for the problem (P_n) if

$$\sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u_n}{\partial x_i}\right)\frac{\partial v}{\partial x_i}dx + \int_\Omega T_n\big(b(u_n)\big)vdx + \frac{1}{n}\int_\Omega |u_n|^{p_M(x)-2}u_n v dx = \int_\Omega f_n v dx, \tag{3.4}$$

for every $v \in E$.

Let us prove the following lemma.

Lemma 3.1. *There exists at least one weak solution u_n for the problem (P_n).*

Proof. We define the operator A_n as follows:

$$\langle A_n(u),v\rangle = \langle A(u),v\rangle + \int_\Omega T_n\big(b(u)\big)v dx + \frac{1}{n}\int_\Omega |u|^{p_M(x)-2}uv dx, \quad \forall\, u,v\in E, \tag{3.5}$$

where

$$\langle A(u),v\rangle = \int_\Omega \sum_{i=1}^N a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial v}{\partial x_i}dx. \tag{3.6}$$

Assertion 1. *The operator A_n is of type M.*
• *The operator A is monotone.* Indeed, for $u,v\in E$, we have

$$\begin{aligned}
&\langle A(u)-A(v),u-v\rangle\\
&= \langle A(u),u-v\rangle + \langle A(v),v-u\rangle\\
&= \int_\Omega \sum_{i=1}^N a_i\Big(x,\frac{\partial u}{\partial x_i}\Big)\frac{\partial (u-v)}{\partial x_i}dx + \int_\Omega \sum_{i=1}^N a_i\Big(x,\frac{\partial v}{\partial x_i}\Big)\frac{\partial (v-u)}{\partial x_i}dx\\
&= \int_\Omega \sum_{i=1}^N \Big[a_i\Big(x,\frac{\partial u}{\partial x_i}\Big) - a_i\Big(x,\frac{\partial v}{\partial x_i}\Big)\Big]\Big(\frac{\partial u}{\partial x_i}-\frac{\partial v}{\partial x_i}\Big)dx,
\end{aligned}$$

then

$$\langle A(u)-A(v),u-v\rangle \geq 0, \tag{3.7}$$

since for $i=1,\dots,N,$ for almost every $x\in\Omega,$ $a_i(x,.)$ is monotone.

• *The operator A is hemicontinuous.* Indeed, let $\varphi: t\in\mathbb{R}\longmapsto \varphi(t)=\langle A(u+tv),v\rangle$ and let $t,\ t_0\in\mathbb{R}$ such that $t\longrightarrow t_0$. Put $w=u+tv\in E$ and $w_0=u+t_0v\in E$.

Therefore $||w-w_0||_{\vec{p}(.)} = ||(t-t_0)v||_{\vec{p}(.)} = |t-t_0|.||v||_{\vec{p}(.)}\longrightarrow 0,$ as $w\longrightarrow w_0$ in E.

We get

$$\begin{aligned}
&|\varphi(t)-\varphi(t_0)| = |\langle A(u+tv),v\rangle - \langle A(u+t_0v),v\rangle|\\
&\leq \sum_{i=1}^N \int_\Omega \Big|a_i\Big(x,\frac{\partial w}{\partial x_i}\Big) - a_i\Big(x,\frac{\partial w_0}{\partial x_i}\Big)\Big|\Big|\frac{\partial v}{\partial x_i}\Big|dx\\
&\leq N\max_{1\leq i\leq N}\Big[\Big(\frac{1}{p_i^-}+\frac{1}{(p_i')^-}\Big)\Big|a_i\Big(x,\frac{\partial w}{\partial x_i}\Big) - a_i\Big(x,\frac{\partial w_0}{\partial x_i}\Big)\Big|_{p_i'(.)}\Big|\frac{\partial v}{\partial x_i}\Big|_{p_i(.)}\Big].
\end{aligned}$$

Denote by $\psi_i(x,w)=a_i\big(x,\frac{\partial w}{\partial x_i}\big)$. Using assumption (2.2) and ([10], Theorems 4.1 and 4.2) we have $\psi_i(x,w)\longrightarrow \psi(x,w_0)$ in $L^{p_i'(.)}(\Omega)$. Then, we deduce that φ is continuous, namely the operator A is hemicontinuous.

Since the operator A is monotone and hemicontinuous, then according to the Lemma 2.1 in [13], A is of type M. Therefore, according to [1] the operator A_n is also of type M.

Assertion 2. *The operator A_n is coercive.*
We have to show that $\dfrac{\langle A_n(u),u\rangle}{||u||_{\vec{p}(.)}}\longrightarrow +\infty$ as $||u||_{\vec{p}(.)}\longrightarrow +\infty$. Indeed, let $u\in E$.

We have $T_n(b(u))u \geq 0$ for all $u \in E$.
Then

$$\langle A_n(u), u\rangle \geq \langle A(u), u\rangle + \frac{1}{n}\int_\Omega |u|^{p_M(x)}dx. \tag{3.8}$$

According to (2.4) we have

$$\langle A(u), u\rangle \geq C_3 \int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx.$$

Denote

$$\mathcal{I} = \left\{ i \in \{1,\ldots,N\} : \left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)} \leq 1 \right\} \quad \text{and} \quad \mathcal{J} = \left\{ i \in \{1,\ldots,N\} : \left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)} > 1 \right\}.$$

Then

$$\begin{aligned}
\frac{1}{C_3}\langle A(u), u\rangle &\geq \sum_{i\in\mathcal{I}} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx + \sum_{i\in\mathcal{J}} \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx \\
&\geq \sum_{i\in\mathcal{I}} \left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)}^{p_i^+} + \sum_{i\in\mathcal{J}} \left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)}^{p_i^-} \geq \sum_{i\in\mathcal{J}} \left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)}^{p_i^-} \\
&\geq \sum_{i\in\mathcal{J}} \left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)}^{p_m^-} \geq \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)}^{p_m^-} - \sum_{i\in\mathcal{I}} \left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)}^{p_m^-} \\
&\geq \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)}^{p_m^-} - N.
\end{aligned}$$

Using the convexity of the application $t \in \mathbb{R}^+ \longmapsto t^{p_m^-}$, $p_m^- > 1$, we obtain

$$\langle A(u), u\rangle \geq \frac{C_3}{N^{p_m^- - 1}} \left(\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)} \right)^{p_m^-} - C_3 N. \tag{3.9}$$

- Assume $|u|_{p_M(.)} \leq 1$. Then, combining (3.8) and (3.9) we get

$$\begin{aligned}
\langle A_n(u), u\rangle &\geq C\left[\left(\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)} \right)^{p_m^-} + |u|_{p_M(.)}^{p_m^-} \right] - 1 - C_3 N + \frac{1}{n}\int_\Omega |u|^{p_M(x)}dx \\
&\geq \frac{C}{2^{p_m^- - 1}} ||u||_{\vec{p}(.)}^{p_m^-} - 1 - C_3 N, \text{ where } C = \min\left\{ \frac{C_3}{N^{p_m^- - 1}}; 1 \right\}.
\end{aligned}$$

- Assume $|u|_{p_M(.)} > 1$. Then, (2.8) give $\int_\Omega |u|^{p_M(x)}dx \geq |u|_{p_M(.)}^{p_m^-}$.
  So, combining (3.8) and (3.9) we get

$$\begin{aligned}
\langle A_n(u), u\rangle &\geq C\left[\left(\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|_{p_i(.)} \right)^{p_m^-} + |u|_{p_M(.)}^{p_m^-} \right] - C_3 N \\
&\geq \frac{C}{2^{p_m^- - 1}} ||u||_{\vec{p}(.)}^{p_m^-} - C_3 N, \text{ where } C = \min\left\{ \frac{C_3}{N^{p_m^- - 1}}; \frac{1}{n} \right\}.
\end{aligned}$$

Consequently, since $p_m^- > 1$, the operator A_n is coercive.

Assertion 3. *The operator A_n is bounded.*
Indeed, let $u \in F \subset E$, where F is a bounded space and $v \in E$. According to (2.2) and (2.9) and as b is onto, we have

$$\begin{aligned}
|\langle A_n(u), v\rangle| &\leq \sum_{i=1}^{N}\int_\Omega \Big|a_i\Big(x, \frac{\partial u}{\partial x_i}\Big)\Big|\Big|\frac{\partial v}{\partial x_i}\Big|dx + \int_\Omega |b(u)||v|dx + \frac{1}{n}\int_\Omega |u|^{p_M(x)-2}|uv|dx \\
&\leq C_1\sum_{i=1}^{N}\Big(\int_\Omega j_i(x)\Big|\frac{\partial v}{\partial x_i}\Big|dx + \int_\Omega \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)-1}\Big|\frac{\partial v}{\partial x_i}\Big|dx\Big) \\
&\quad + \int_\Omega |b(u)||v|dx + \frac{1}{n}\int_\Omega |u|^{p_M(x)-1}|v|dx \\
&\leq C_1\sum_{i=1}^{N}\Big(\frac{1}{p_i^-} + \frac{1}{(p_i')^-}\Big)\Big|\frac{\partial v}{\partial x_i}\Big|_{p_i(.)}\Big(|j_i|_{p_i'(.)} + \Big||\frac{\partial u}{\partial x_i}|^{p_i(x)-1}\Big|_{p_i'(.)}\Big) \\
&\quad + \ C||v||_1 + \frac{1}{n}\Big(\frac{1}{p_M^-} + \frac{1}{(p_M')^-}\Big)||u|^{p_M(x)-1}|_{p_M'(.)}|v|_{p_M(.)},
\end{aligned}$$

where C is a positive constant.

Then the operator A_n is bounded.

The operator A_n is of type M, bounded and coercive on E to its dual E^*, then A_n is surjective (see [13], Corollary 2.2). Therefore, for $f_n \in E^*$, we can deduce the existence of a function $u_n \in E$ such that $\langle A_n(u_n), v\rangle = \langle f_n, v\rangle$ for all $v \in E$, namely

$$\sum_{i=1}^{N}\int_\Omega a_i\Big(x, \frac{\partial u_n}{\partial x_i}\Big)\frac{\partial v}{\partial x_i}dx + \int_\Omega T_n\big(b(u_n)\big)v dx + \frac{1}{n}\int_\Omega |u_n|^{p_M(x)-2}u_n v dx = \int_\Omega f_n v dx.$$

□

Our aim is to prove that these approximated solutions u_n tend, as n goes to infinity, to a mesurable function u which is an entropy solution of the problem (1.1). To start with, we establish some a priori estimates.

Step 2. A priori estimates

Assume that (2.1)-(2.6) holds and let u_n be a solution of problem (P_n). We have the following results:

Lemma 3.2. *There exists a positive constant C_5 which does not depend on n such that*

$$\sum_{i=1}^{N}\int_{\{|u_n|\leq k\}}\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p_i^-}dx \leq C_5(1+k) \tag{3.10}$$

for every $k > 0$.

Proof. Let us take $v = T_k(u_n)$ as test function in (3.4). Since

$$\int_\Omega T_n(b(u_n))T_k(u_n)dx + \frac{1}{n}\int_\Omega |u_n|^{p_M(x)-2}u_n T_k(u_n) \geq 0, \tag{3.11}$$

using relation (2.4), we obtain

$$C_3\sum_{i=1}^{N}\int_{\{|u_n|\leq k\}}\Big|\frac{\partial u_n}{\partial x_i}\Big|^{p_i(x)}dx \leq k||f||_1. \tag{3.12}$$

We have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-}dx\\
&=\sum_{i=1}^{N}\int_{\{|u_n|\le k;\ |\frac{\partial u_n}{\partial x_i}|>1\}}\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-}dx+\sum_{i=1}^{N}\int_{\{|u_n|\le k;\ |\frac{\partial u_n}{\partial x_i}|\le 1\}}\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-}dx\\
&\le\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i(x)}dx+N.meas(\Omega)\\
&\le\frac{1}{C_3}k||f||_1+N.meas(\Omega)\ \ \text{due to relation}\ \ (3.12)\\
&\le C_5(1+k)\ \ \text{with}\ \ C_5=\max\left\{\frac{1}{C_3}||f||_1;\ N.meas(\Omega)\right\}.
\end{aligned}$$

□

Lemma 3.3. *There exists a constant $C_6>0$ such that*

$$\int_\Omega |T_k(u_n)|^{p_M^-}dx+\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-}dx\le C_6(k+1). \tag{3.13}$$

Proof. We have

$$\begin{aligned}
\int_\Omega |T_k(u_n)|^{p_M^-}dx&=\int_{\{|T_k(u_n)|\le 1\}}|T_k(u_n)|^{p_M^-}dx+\int_{\{|T_k(u_n)|>1\}}|T_k(u_n)|^{p_M^-}dx\\
&\le meas(\Omega)+\int_{\{|T_k(u_n)|>1\}}k^{p_M^-}dx\\
&\le meas(\Omega)(1+k^{p_M^-}).
\end{aligned}$$

Then, using Lemma 3.2, we obtain

$$\begin{aligned}
&\int_\Omega |T_k(u_n)|^{p_M^-}dx+\sum_{i=1}^{N}\int_{\{|u_n|\le k\}}\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i^-}dx\\
&\le meas(\Omega)(1+N+k^{p_M^-})+k||f||_1\le C_6(1+k),
\end{aligned}$$

where $C_6=\max\left\{meas(\Omega)(1+N+k^{p_M^-});||f||_1\right\}$. □

Lemma 3.4. *For any $k>0$, there exists some constants $C_7,C_8>0$ such that*

(i) $||u_n||_{\mathcal{M}^{q^*}(\Omega)}\le C_7$;

(ii) $\left|\left|\frac{\partial u_n}{\partial x_i}\right|\right|_{\mathcal{M}^{p_i^- q/\bar{p}}(\Omega)}\le C_8;\qquad \forall\ i=1,\dots,N.$

Proof.

(i) is a consequence of lemmas 3.3 and 2.1.

(ii) • Let $\alpha \geq 1$. For any $k \geq 1$, we have

$$\begin{aligned}
&\lambda_{\frac{\partial u_n}{\partial x_i}}(\alpha) \\
&= meas\bigg(\bigg\{x \in \Omega : \bigg|\frac{\partial u_n}{\partial x_i}\bigg| > \alpha\bigg\}\bigg) \\
&= meas\bigg(\bigg\{\bigg|\frac{\partial u_n}{\partial x_i}\bigg| > \alpha; |u_n| \leq k\bigg\}\bigg) + meas\bigg(\bigg\{\bigg|\frac{\partial u_n}{\partial x_i}\bigg| > \alpha; |u_n| > k\bigg\}\bigg) \\
&\leq \int_{\left\{|\frac{\partial u_n}{\partial x_i}|>\alpha;\ |u_n|\leq k\right\}} dx + \lambda_{u_n}(k) \\
&\leq \int_{\{|u_n|\leq k\}} \bigg(\frac{1}{\alpha}\bigg|\frac{\partial u_n}{\partial x_i}\bigg|\bigg)^{p_i^-} dx + \lambda_{u_n}(k) \\
&\leq \alpha^{-p_i^-} C' k + C k^{-q^*}.
\end{aligned}$$

Then

$$\lambda_{\frac{\partial u_n}{\partial x_i}}(\alpha) \leq B\big(k\alpha^{-p_i^-} + k^{-q^*}\big), \tag{3.14}$$

with B a positive constant.
Let us consider the function

$$g : [1, +\infty[\longrightarrow \mathbb{R},\ x \longmapsto g(x) = \frac{x}{\alpha^{p_i^-}} + x^{-q^*}.$$

We have $g'(x) = 0$ for $x = \big(q^*\alpha^{p_i^-}\big)^{\frac{1}{q^*+1}}$. Thus, if we take $k = \big(q^*\alpha^{p_i^-}\big)^{\frac{1}{q^*+1}} \geq 1$ in (3.14) we get

$$\begin{aligned}
\lambda_{\frac{\partial u_n}{\partial x_i}}(\alpha) &\leq Bk\bigg(\frac{q^*+1}{q^*}\frac{1}{\alpha^{p_i^-}}\bigg) \\
&\leq M\alpha^{-\frac{q^*}{q^*+1}p_i^-} \\
&\leq M\alpha^{-p_i^- q/\bar{p}}, \quad \forall\ \alpha \geq 1 \ \text{ with } M \text{ a positive constant.}
\end{aligned}$$

• If $0 \leq \alpha < 1$, we have

$$\lambda_{\frac{\partial u_n}{\partial x_i}}(\alpha) = meas(\bigg\{\bigg|\frac{\partial u_n}{\partial x_i}\bigg| > \alpha\bigg\}) \leq meas(\Omega) \leq meas(\Omega)\alpha^{-p_i^- q/\bar{p}}.$$

Then

$$\lambda_{\frac{\partial u_n}{\partial x_i}}(\alpha) \leq (M + meas(\Omega))\alpha^{-p_i^- q/\bar{p}}, \quad \forall\ \alpha \geq 0.$$

Therefore, we deduce that there exists a positive constant C_8 such that

$$\bigg|\bigg|\frac{\partial u_n}{\partial x_i}\bigg|\bigg|_{\mathcal{M}^{p_i^- q/\bar{p}}(\Omega)} \leq C_8, \qquad \forall\ i = 1, \ldots, N.$$

□

Step 3. Existence of entropy solution
Using Lemma 3.4, we have the following useful lemma (see [5]).

Lemma 3.5. *For $i = 1, \ldots, N$, as $n \longrightarrow +\infty$, we have*

$$a_i\bigg(x, \frac{\partial u_n}{\partial x_i}\bigg) \longrightarrow a_i\bigg(x, \frac{\partial u}{\partial x_i}\bigg) \quad in \quad L^1(\Omega) \quad a.e \quad x \in \Omega. \tag{3.15}$$

In order to pass to the limit in relation (3.4), we also need the following convergence results which can be proved as in [4]:

Proposition 3.1. *Assume (2.1)-(2.6). If $u_n \in E$ is a weak solution of (P_n) then the sequence $(u_n)_{n\in\mathbb{N}^*}$ is Cauchy in measure. In particular, there exists a measurable function u and a sub-sequence still denoted by u_n such that $u_n \longrightarrow u$ in measure.*

Proposition 3.2. *Assume (2.1)-(2.6). If $u_n \in E$ is a weak solution of (P_n) then*

(i) *for all $i = 1, \ldots, N$, $\dfrac{\partial u_n}{\partial x_i}$ converges in measure to the weak partial gradient of u;*

(ii) *for all $i = 1, \ldots, N$ and $k > 0$, $a_i\left(x, \dfrac{\partial}{\partial x_i}T_k(u_n)\right)$ converges to $a_i\left(x, \dfrac{\partial}{\partial x_i}T_k(u)\right)$ in $L^1(\Omega)$ strongly and in $L^{p_i'(.)}(\Omega)$ weakly.*

We can now pass to the limit in relation (3.4).
Let $\varphi \in E \cap L^\infty(\Omega)$ and choosing $T_k(u_n - \varphi)$ as test function in (3.4), we get

$$\sum_{i=1}^{N}\int_\Omega a_i\left(x, \frac{\partial u_n}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u_n-\varphi)dx + \int_\Omega T_n(b(u_n))T_k(u_n-\varphi)dx$$
$$+\frac{1}{n}\int_\Omega |u_n|^{p_M(x)-2}u_n T_k(u_n-\varphi)dx = \int_\Omega f_n T_k(u_n-\varphi)dx. \tag{3.16}$$

For the right-hand side of (3.16) we have

$$\int_\Omega f_n(x)T_k(u_n-\varphi)dx \longrightarrow \int_\Omega f(x)T_k(u-\varphi)dx, \tag{3.17}$$

since f_n converges strongly to f in $L^1(\Omega)$ and $T_k(u_n - \varphi)$ converges weakly-* to $T_k(u - \varphi)$ in $L^\infty(\Omega)$ and a.e in Ω.

For the first term of (3.16) we have (see [5]):

$$\liminf_n \sum_{i=1}^{N}\int_\Omega a_i\left(x, \frac{\partial u_n}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u_n-\varphi)dx \geq \sum_{i=1}^{N}\int_\Omega a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u-\varphi)dx. \tag{3.18}$$

We focus our attention on the second term of (3.16). We have

$$T_n(b(u_n))T_k(u_n-\varphi) \longrightarrow b(u)T_k(u-\varphi) \ \text{ a.e. } \ x\in\Omega \tag{3.19}$$

and

$$|T_n(b(u_n))T_k(u_n-\varphi)| \leq k|b(u_n)|. \tag{3.20}$$

Now we show that $|b(u_n)| \leq \frac{\|f\|_1}{meas(\Omega)}$. Indeed, let us denote by

$$H_\epsilon(s) = \min\left(\frac{s^+}{\epsilon};1\right) \quad \text{and} \quad sign_0^+(s) = \begin{cases} 1 & \text{if } \ s>0, \\ 0 & \text{if } \ s\leq 0. \end{cases}$$

If γ is a maximal monotone operator defined on $\mathbb{R}$, we denote by γ_0 the main section of γ; i.e.,

$$\gamma_0(s) = \begin{cases} \text{minimal absolute value of } \gamma(s) & \text{if } \ \gamma(s)\neq\emptyset, \\ +\infty & \text{if } \ [s,+\infty)\cap D(\gamma)=\emptyset, \\ -\infty & \text{if } \ (-\infty,s]\cap D(\gamma)=\emptyset. \end{cases}$$

Remark that as ϵ goes 0, $H_\epsilon(s) = sign_0^+(s)$.

We take $\varphi = H_\epsilon(u_n - M)$ as test function in (3.4), for the weak solution u_n and $M > 0$ (a constant to be chosen later), to get

$$\sum_{i=1}^{N}\int_\Omega a_i\Big(x, \frac{\partial u_n}{\partial x_i}\Big)\frac{\partial}{\partial x_i}H_\epsilon(u_n - M)dx + \frac{1}{n}\int_\Omega |u_n|^{p_M(x)-2}u_n H_\epsilon(u_n - M)dx$$
$$+\int_\Omega T_n\big(b(u_n)\big)H_\epsilon(u_n - M)dx = \int_\Omega f_n H_\epsilon(u_n - M)dx. \qquad (3.21)$$

We have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i\Big(x, \frac{\partial u_n}{\partial x_i}\Big)\frac{\partial}{\partial x_i}H_\epsilon(u_n - M)dx\\
&= \frac{1}{\epsilon}\sum_{i=1}^{N}\int_{\left\{\frac{(u_n-M)^+}{\epsilon}<1\right\}} a_i\Big(x, \frac{\partial u_n}{\partial x_i}\Big)\frac{\partial}{\partial x_i}(u_n - M)^+dx\\
&= \frac{1}{\epsilon}\sum_{i=1}^{N}\int_{\left\{0<u_n-M<\epsilon\right\}} a_i\Big(x, \frac{\partial u_n}{\partial x_i}\Big)\frac{\partial}{\partial x_i}u_n dx\\
&\geq 0 \ \text{ according to } \ (2.4),
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega |u_n|^{p_M(x)-2}u_n H_\epsilon(u_n - M)dx\\
&= \int_{\left\{\frac{(u_n-M)^+}{\epsilon}<1\right\}} |u_n|^{p_M(x)-2}u_n\frac{(u_n - M)^+}{\epsilon}dx\\
&\quad + \int_{\left\{\frac{(u_n-M)^+}{\epsilon}\geq 1\right\}} |u_n|^{p_M(x)-2}u_n dx\\
&\geq \frac{1}{\epsilon}\int_{\left\{M<u_n<M+\epsilon\right\}} |u_n|^{p_M(x)-2}u_n(u_n - M)dx\\
&\geq 0.
\end{aligned}$$

Then, (3.21) give

$$\int_\Omega T_n\big(b(u_n)\big)H_\epsilon(u_n - M)dx \leq \int_\Omega f_n H_\epsilon(u_n - M)dx,$$

which is equivalent to

$$\int_\Omega \Big(T_n\big(b(u_n)\big) - T_n\big(b(M)\big)\Big)H_\epsilon(u_n - M)dx \leq \int_\Omega \Big(f_n - T_n\big(b(M)\big)\Big)H_\epsilon(u_n - M)dx.$$

Now we let ϵ goes to 0 in the above inequality to obtain

$$\int_\Omega \Big(T_n\big(b(u_n)\big) - T_n\big(b(M)\big)\Big)^+dx \leq \int_\Omega \Big(f_n - T_n\big(b(M)\big)\Big)sign_0^+(u_n - M)dx. \qquad (3.22)$$

Choosing $M = b_0^{-1}\big(||f_n||_\infty\big)$ in the above inequality (since b is surjective). We obtain

$$\int_\Omega \Big(T_n\big(b(u_n)\big) - T_n\big(||f_n||_\infty\big)\Big)^+ dx \leq \int_\Omega \Big(f_n - T_n\big(||f_n||_\infty\big)\Big) sign_0^+\big(u_n - b_0^{-1}(||f_n||_\infty)\big)dx. \tag{3.23}$$

For any $n > \frac{\|f\|_1}{meas(\Omega)}$, we have

$$\begin{aligned}&\int_\Omega \Big(f_n - T_n\big(||f_n||_\infty\big)\Big) sign_0^+\big(u_n - b_0^{-1}(||f_n||_\infty)\big)dx \\ &= \int_\Omega \Big(f_n - ||f_n||_\infty\Big) sign_0^+\big(u_n - b_0^{-1}(||f_n||_\infty)\big)dx \leq 0.\end{aligned}$$

Then, (3.23) gives

$$\int_\Omega \Big(T_n\big(b(u_n)\big) - ||f_n||_\infty\Big)^+ dx \leq 0 \quad \text{for all} \quad n > \frac{\|f\|_1}{meas(\Omega)}.$$

Hence, for all $n > \frac{\|f\|_1}{meas(\Omega)}$, we have $\Big(T_n\big(b(u_n)\big) - ||f_n||_\infty\Big)^+ = 0$ a.e. in Ω, which implies that

$$T_n\big(b(u_n)\big) \leq ||f_n||_\infty \quad \text{for all} \quad n > \frac{\|f\|_1}{meas(\Omega)}. \tag{3.24}$$

Let us remark that as u_n is a weak solution of (3.2), then $(-u_n)$ is a weak solution to the following problem

$$(\tilde{P}_n)\begin{cases} -\displaystyle\sum_{i=1}^N \frac{\partial}{\partial x_i}\tilde{a}_i\Big(x, \frac{\partial u_n}{\partial x_i}\Big) + T_n\big(\tilde{b}(u_n)\big) + \frac{1}{n}|u_n|^{p_M(x)-2}u_n = \tilde{f}_n & \text{in} \quad \Omega, \\ \displaystyle\sum_{i=1}^N \tilde{a}_i\Big(x, \frac{\partial u_n}{\partial x_i}\Big)\eta_i = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.25}$$

where $\tilde{a}_i(x,\xi) = -a_i(x,-\xi)$, $\tilde{b}(s) = -b(-s)$ and $\tilde{f}_n = -f_n$.

According to (3.24) we deduce that

$$T_n\big(-b(u_n)\big) \leq ||f_n||_\infty \quad \text{for all} \quad n > \frac{\|f\|_1}{meas(\Omega)}.$$

Therefore,

$$T_n\big(b(u_n)\big) \geq -||f_n||_\infty \quad \text{for all} \quad n > \frac{\|f\|_1}{meas(\Omega)}. \tag{3.26}$$

It follows from (3.24) and (3.26) that for all $n > \frac{\|f\|_1}{meas(\Omega)}$, $\big|T_n\big(b(u_n)\big)\big| \leq ||f_n||_\infty$ which implies

$$|b(u_n)| \leq ||f_n||_\infty \leq \frac{\|f\|_1}{meas(\Omega)} \quad \text{a.e. in} \quad \Omega.$$

We can now use the Lebesgue dominated convergence theorem to get

$$\lim_{n\to+\infty}\int_\Omega T_n(b(u_n))T_k(u_n-\varphi)dx = \int_\Omega b(u)T_k(u-\varphi)dx. \tag{3.27}$$

For the third term of (3.16), let us prove that

$$\liminf_n \frac{1}{n}\int_\Omega |u_n|^{p_M(x)-2}u_n T_k(u_n-\varphi)dx \geq 0.$$

We have

$$\begin{aligned}&\int_\Omega |u_n|^{p_M(x)-2}u_n T_k(u_n-\varphi)dx\\ =&\int_\Omega \Big(|u_n|^{p_M(x)-2}u_n-|\varphi|^{p_M(x)-2}\varphi\Big)T_k(u_n-\varphi)dx\\ &+\int_\Omega |\varphi|^{p_M(x)-2}\varphi T_k(u_n-\varphi)dx.\end{aligned}$$

Since the quantity $\Big(|u_n|^{p_M(x)-2}u_n-|\varphi|^{p_M(x)-2}\varphi\Big)T_k(u_n-\varphi)$ is non negative and since for all $x\in\Omega$, $\xi\longmapsto|\xi|^{p_M(x)-2}\xi$ is continuous, we get

$$\Big(|u_n|^{p_M(x)-2}u_n-|\varphi|^{p_M(x)-2}\varphi\Big)T_k(u_n-\varphi)\to\Big(|u|^{p_M(x)-2}u-|\varphi|^{p_M(x)-2}\varphi\Big)T_k(u-\varphi)$$

a.e. in Ω, and by Fatou's lemma, it follows that

$$\begin{aligned}&\liminf_n\int_\Omega \Big(|u_n|^{p_M(x)-2}u_n-|\varphi|^{p_M(x)-2}\varphi\Big)T_k(u_n-\varphi)dx\\ \geq&\int_\Omega \Big(|u|^{p_M(x)-2}u-|\varphi|^{p_M(x)-2}\varphi\Big)T_k(u-\varphi)dx.\end{aligned}\tag{3.28}$$

We have

$$\begin{aligned}&\int_\Omega \big||\varphi|^{p_M(x)-2}\varphi\big|dx=\int_\Omega |\varphi|^{p_M(x)-1}dx\\ \leq&\int_\Omega \Big(||\varphi||_\infty\Big)^{p_M(x)-1}dx\\ \leq&\int_{\{||\varphi||_\infty\leq1\}}\Big(||\varphi||_\infty\Big)^{p_M(x)-1}dx+\int_{\{||\varphi||_\infty>1\}}\Big(||\varphi||_\infty\Big)^{p_M(x)-1}dx\\ \leq& \,meas(\Omega)+\big(||\varphi||_\infty\big)^{p_M^+-1}meas(\Omega)<+\infty.\end{aligned}$$

Hence, $|\varphi|^{p_M(x)-2}\varphi\in L^1(\Omega)$.
Since $T_k(u_n-\varphi)$ converge weakly-* to $T_k(u-\varphi)$ in $L^\infty(\Omega)$ and $|\varphi|^{p_M(x)-2}\varphi\in L^1(\Omega)$, it follows that

$$\lim_n\int_\Omega |\varphi|^{p_M(x)-2}\varphi T_k(u_n-\varphi)dx=\int_\Omega |\varphi|^{p_M(x)-2}\varphi T_k(u-\varphi)dx.\tag{3.29}$$

By adding (3.28) and (3.29), we get

$$\liminf_n\int_\Omega |u_n|^{p_M(x)-2}u_nT_k(u_n-\varphi)dx\geq\int_\Omega |u|^{p_M(x)-2}uT_k(u-\varphi)dx.\tag{3.30}$$

Since

$$\int_\Omega |u|^{p_M(x)-2}uT_k(u-\varphi)dx\leq k\int_\Omega |u|^{p_M(x)-1}dx<+\infty,$$

we get finally

$$\liminf_n \frac{1}{n}\int_\Omega |u_n|^{p_M(x)-2}u_n T_k(u_n-\varphi)dx \geq 0. \tag{3.31}$$

Combining (3.17), (3.18), (3.27) and (3.31) we obtain

$$\sum_{i=1}^N \int_\Omega a_i\left(x, \frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u-\varphi)dx + \int_\Omega b(u)T_k(u-\varphi)dx \leq \int_\Omega f(x)T_k(u-\varphi)dx. \tag{3.32}$$

Then u is an entropy solution of (1.1). □

Theorem 3.2. *Assume that (2.1)-(2.6) hold true and let u be an entropy solution of (1.1). Then, u is unique.*

Proof. The proof is done in two steps.
Step 1. A priori estimates

Lemma 3.6. *Assume (2.1)-(2.6) holds and $f \in L^1(\Omega)$. Let u be an entropy solution of (1.1). Then*

$$\sum_{i=1}^N \int_{\{|u|\leq k\}} \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)} dx \leq \frac{k}{C_3}||f||_1 \tag{3.33}$$

and there exists a positive constant C_9 such that

$$||b(u)||_1 \leq C_9.meas(\Omega) + ||f||_1. \tag{3.34}$$

Proof. Let us take $\varphi = 0$ in the entropy inequality (3.1).

- By the fact that $\int_\Omega b(u)T_k(u)dx \geq 0$ and using the relation (2.4), we get (3.33).
- Using the fact that $\sum_{i=1}^N \int_\Omega a_i\big(x, \frac{\partial u}{\partial x_i}\big)\frac{\partial}{\partial x_i}T_k(u)dx \geq 0$, relation (3.1) gives

$$\int_\Omega b(u)T_k(u)dx \leq \int_\Omega f(x)T_k(u)dx. \tag{3.35}$$

By (3.35), we deduce that

$$\int_{\{|u|\leq k\}} b(u)T_k(u)dx + \int_{\{|u|>k\}} b(u)T_k(u)dx \leq k||f||_1,$$

which imply that

$$\int_{\{|u|>k\}} b(u)T_k(u)dx \leq k||f||_1$$

or

$$\int_{\{u>k\}} b(u)dx + \int_{\{u<-k\}} -b(u)dx \leq ||f||_1.$$

Therefore

$$\int_{\{|u|>k\}} |b(u)|dx \leq ||f||_1.$$

So, we obtain

$$\int_\Omega |b(u)|dx = \int_{\{|u|\leq k\}} |b(u)|dx + \int_{\{|u|>k\}} |b(u)|dx$$

$$\leq \int_{\{|u|\leq k\}} |b(u)|dx + ||f||_1.$$

Since the function b is non-decreasing, then

$$\int_{\{|u|\leq k\}} |b(u)|dx \leq \max\{b(k); |b(-k)|\}.meas(\Omega).$$

Consequently, there exists a constant $C_9 = \max\{b(k); |b(-k)|\}$ such that

$$||b(u)||_1 \leq C_9.meas(\Omega) + ||f||_1.$$

□

Lemma 3.7. *Assume (2.1)-(2.6) holds and let $f \in L^1(\Omega)$. If u is an entropy solution of (1.1), then there exists a constant D which depends on f and Ω such that*

$$meas\{|u| > k\} \leq \frac{D}{\min(b(k), |b(-k)|)}, \quad \forall\ k > 0 \tag{3.36}$$

and a constant $D' > 0$ which depends on f and Ω such that

$$meas\left\{\left|\frac{\partial u}{\partial x_i}\right| > k\right\} \leq \frac{D'}{k^{\frac{1}{(p_M^-)'}}}, \quad \forall\ k \geq 1. \tag{3.37}$$

Proof. • For any $k > 0$, the relation (3.34) gives

$$\int_{\{|u|>k\}} \min(b(k), |b(-k)|)dx \leq \int_{\{|u|>k\}} |b(u)|dx \leq C_9.meas(\Omega) + ||f||_1.$$

Therefore,

$$\min(b(k), |b(-k)|).mes\{|u| > k\} \leq C_9.meas(\Omega) + ||f||_1 = D;$$

that is

$$meas\{|u| > k\} \leq \frac{D}{\min(b(k), |b(-k)|)}.$$

• See [4] for the poof of (3.37). □

Lemma 3.8. *Assume (2.1)-(2.6) holds and let $f \in L^1(\Omega)$. If u is an entropy solution of (1.1), then*

$$\lim_{h\longrightarrow+\infty} \int_\Omega |f|\chi_{\{|u|>h-t\}}dx = 0, \tag{3.38}$$

where $h > 0$ and $t > 0$.

Proof. Since the function b is surjective, according to (3.36), we have $\lim_{h\longrightarrow+\infty} meas\{|u| > h - t\} = 0$ and as $f \in L^1(\Omega)$, it follows by using the Lebesgue dominated convergence theorem that $\lim_{h\longrightarrow+\infty} \int_\Omega |f|\chi_{\{|u|>h-t\}}dx = 0$. □

Lemma 3.9. *Assume (2.1)-(2.6) holds and let $f \in L^1(\Omega)$. If u is an entropy solution of (1.1), then there exists a positive constant K such that*

$$\rho_{p_i'(.)}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-1}\chi_F\right) \leq K, \quad \forall\ i = 1,\ldots,N, \tag{3.39}$$

where $F = \{h < |u| \leq h + k\}$, $h > 0$, $k > 0$.

Proof. Let $\varphi = T_h(u)$ as test function in the entropy inequality (3.1). We get

$$\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u-T_h(u))dx+\int_\Omega b(u)T_k(u-T_h(u))dx$$
$$\leq \int_\Omega f(x)T_k(u-T_h(u))dx.$$

Thus,

$$\sum_{i=1}^{N}\int_{\{h<|u|\leq h+k\}} a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial u}{\partial x_i}dx \leq k||f||_1$$

and using (2.4), we have

$$\int_F \left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)}dx \leq \frac{C_3}{k}||f||_1, \quad \forall\ i=1\ldots,N.$$

Consequently,

$$\rho_{p_i'(.)}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-1}\chi_F\right)\leq K, \quad \forall\ i=1,\ldots,N.$$

□

Step 2. **Uniqueness of entropy solution**.

Let $h>0$ and u,v be two entropy solutions of (1.1). We write the entropy inequality corresponding to the solution u, with $T_h(v)$ as test function, and to the solution v, with $T_h(u)$ as test function:

$$\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u-T_h(v))dx+\int_\Omega b(u)T_k(u-T_h(v))dx$$
$$\leq \int_\Omega f(x)T_k(u-T_h(v))dx \tag{3.40}$$

and

$$\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial v}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(v-T_h(u))dx+\int_\Omega b(v)T_k(v-T_h(u))dx$$
$$\leq \int_\Omega f(x)T_k(v-T_h(u))dx. \tag{3.41}$$

Upon addition, we get

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u-T_h(v))dx\\&+\sum_{i=1}^{N}\int_\Omega a_i\left(x,\frac{\partial v}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(v-T_h(u))dx\\&+\int_\Omega b(u)T_k(u-T_h(v))dx+\int_\Omega b(v)T_k(v-T_h(u))dx\\ \leq &\int_\Omega f(x)\big[T_k(u-T_h(v))+T_k(v-T_h(u))\big]dx.\end{aligned} \tag{3.42}$$

Define

$$E_1 = \{|u-v| \le k;\ |v| \le h\};\quad E_2 = E_1 \cap \{|u| \le h\} \ \text{ and } \ E_3 = E_1 \cap \{|u| > h\}.$$

We start with the first integral in (3.42). We have

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\{|u-T_h(v)|\le k\}} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u - T_h(v))dx \\
&= \sum_{i=1}^{N} \int_{\{|u-T_h(v)|\le k\}\cap\{|v|\le h\}} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u - T_h(v))dx \\
&\quad + \sum_{i=1}^{N} \int_{\{|u-T_h(v)|\le k\}\cap\{|v|> h\}} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u - T_h(v))dx \\
&= \sum_{i=1}^{N} \int_{\{|u-v|\le k\}\cap\{|v|\le h\}} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial (u-v)}{\partial x_i} dx \\
&\quad + \sum_{i=1}^{N} \int_{\{|u-hsign(v)|\le k\}\cap\{|v|> h\}} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial u}{\partial x_i} dx \\
&\ge \sum_{i=1}^{N} \int_{E_1} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} (u-v)dx \\
&= \sum_{i=1}^{N} \int_{E_2} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} (u-v)dx + \sum_{i=1}^{N} \int_{E_3} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} (u-v)dx.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\{|u-T_h(v)|\le k\}} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} T_k(u - T_h(v))dx \\
&\ge \sum_{i=1}^{N} \int_{E_2} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial}{\partial x_i} (u-v)dx - \sum_{i=1}^{N} \int_{E_3} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial v}{\partial x_i} dx. \qquad (3.43)
\end{aligned}$$

According to (2.2) and the Hölder type inequality we have

$$\begin{aligned}
&\Big|\sum_{i=1}^{N} \int_{E_3} a_i\Big(x, \frac{\partial u}{\partial x_i}\Big) \frac{\partial v}{\partial x_i} dx\Big| \\
&\le C_1 \sum_{i=1}^{N} \int_{E_3} \Big(j_i(x) + \Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)-1}\Big)\Big|\frac{\partial v}{\partial x_i}\Big| dx \\
&\le C_1 \sum_{i=1}^{N} \Big(|j_i|_{p_i'(.)} + \Big|\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-1}\Big|_{p_i'(.),\{h<|u|\le h+k\}}\Big)\big|\frac{\partial v}{\partial x_i}\big|_{p_i(.),\{h-k<|v|\le h\}},
\end{aligned}$$

where

$$\Big|\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-1}\Big|_{p_i'(.),\{h<|u|\le h+k\}} = \Big\|\big|\frac{\partial u}{\partial x_i}\big|^{p_i(x)-1}\Big\|_{L^{p_i'(\cdot)}(\{h<|u|\le h+k\})}.$$

For $i=1,\ldots,N$, the quantity $\left(|j_i|_{p_i'(.)}+\left|\left|\frac{\partial u}{\partial x_i}\right|^{p_i(x)-1}\right|_{p_i'(.),\{h<|u|\leq h+k\}}\right)$ is finite according to relation (2.8) and Lemma 3.9.

According to Lemma 3.8, the quantity $\left|\frac{\partial v}{\partial x_i}\right|_{p_i(.),\{h-k<|v|\leq h\}}$ converges to zero as h goes to infinity. Consequently the last integral of (3.43) converges to zero as h goes to infinity. Then

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{|u-T_h(v)|\leq k\}}a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}T_k(u-T_h(v))dx\\ &\geq I_h+\sum_{i=1}^{N}\int_{E_2}a_i\left(x,\frac{\partial u}{\partial x_i}\right)\frac{\partial}{\partial x_i}(u-v)dx \qquad (3.44)\end{aligned}$$

with $\lim_{h\longrightarrow+\infty}I_h=0$.

We may adopt the same procedure to treat the second term in (3.42) to obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{|v-T_h(u)|\leq k\}}a_i\big(x,\frac{\partial v}{\partial x_i}\big)\frac{\partial}{\partial x_i}T_k(v-T_h(u))dx\\ &\geq J_h-\sum_{i=1}^{N}\int_{E_2}a_i\big(x,\frac{\partial v}{\partial x_i}\big)\frac{\partial}{\partial x_i}(u-v)dx, \qquad (3.45)\end{aligned}$$

with $\lim_{h\longrightarrow+\infty}J_h=0$.

For the two other terms in the left-hand side of (3.42), we denote

$$K_h=\int_\Omega b(u)T_k(u-T_h(v))dx+\int_\Omega b(v)T_k(v-T_h(u))dx.$$

We have

$$b(u)T_k(u-T_h(v))\longrightarrow b(u)T_k(u-v) \quad \text{a.e} \quad \text{since} \quad h\longrightarrow+\infty$$

and

$$\big|b(u)T_k(u-T_h(v))\big|\leq k|b(u)|\in L^1(\Omega).$$

Then, by the Lebesgue dominated convergence theorem, we obtain

$$\lim_{h\longrightarrow+\infty}\int_\Omega b(u)T_k(u-T_h(v))dx=\int_\Omega b(u)T_k(u-v)dx.$$

In the same way, we get

$$\lim_{h\longrightarrow+\infty}\int_\Omega b(v)T_k(v-T_h(u))dx=\int_\Omega b(v)T_k(v-u)dx.$$

Then

$$\lim_{h\longrightarrow+\infty}K_h=\int_\Omega\big(b(u)-b(v)\big)T_k(u-v)dx. \qquad (3.46)$$

Now, considering the right-hand side of inequality (3.42), we have

$$\lim_{h\longrightarrow+\infty}f(x)\Big(T_k(u-T_h(v))+T_k(v-T_h(u))\Big)=0 \quad \text{a.e}$$

and

$$\big|f(x)\big(T_k(u-T_h(v))+T_k(v-T_h(u))\big)\big| \le 2k|f| \in L^1(\Omega).$$

By the Lebesgue dominated convergence theorem, we obtain

$$\lim_{h\longrightarrow+\infty}\int_\Omega f(x)\Big(T_k\big(u-T_h(v)\big)+T_k\big(v-T_h(u)\big)\Big)dx=0. \tag{3.47}$$

After passing to the limit as h goes to $+\infty$ in (3.42) we get

$$\begin{aligned}&\sum_{i=1}^N\int_{\{|u-v|\le k\}}\Big(a_i\big(x,\frac{\partial u}{\partial x_i}\big)-a_i\big(x,\frac{\partial v}{\partial x_i}\big)\Big)\frac{\partial}{\partial x_i}(u-v)dx\\&+\int_\Omega\big(b(u)-b(v)\big)T_k(u-v)dx\le 0.\end{aligned} \tag{3.48}$$

Since b and $a_i(x,.)$ are monotone then

$$\int_\Omega\big(b(u)-b(v)\big)T_k(u-v)dx=0 \tag{3.49}$$

and

$$\int_{\{|u-v|\le k\}}\sum_{i=1}^N\Big(a_i\big(x,\frac{\partial u}{\partial x_i}\big)-a_i\big(x,\frac{\partial v}{\partial x_i}\big)\Big)\frac{\partial}{\partial x_i}(u-v)dx=0. \tag{3.50}$$

We deduce from (3.49) that

$$\lim_{k\longrightarrow 0}\int_\Omega\big(b(u)-b(v)\big)\frac{1}{k}T_k(u-v)dx=\int_\Omega|b(u)-b(v)|dx=0. \tag{3.51}$$

According to (2.3), we deduce from (3.50) that, for $i=1,...,N$,

$$\left|\frac{\partial}{\partial x_i}(u-v)\right|=0 \ \text{ a.e } x\in\Omega \ \text{ that is to say } \ u-v=C \ \text{ a.e } \ x\ \in\Omega,$$

where C is a positive constant. Therefore

$$u-v=C \ \text{ a.e } x\in\Omega$$

and

$$b(u)=b(v).$$

□

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