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CONTROLLABILITY OF THE KORTEWEG-DE VRIES-BURGERS EQUATION*

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Abstract In this paper, we investigate the controllability of the Korteweg-de Vries-Burgers equation on a periodic domain $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. With the aid of the classical duality approach and a fixed-point argument, the local exact controllability is established.

Keywords Korteweg-de Vries-Burgers equation, controllability, periodic domain.

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1. Introduction

The Korteweg-de Vries-Burgers (KdV-B) equation

$$\begin{cases} u_t + u_{xxx} - u_{xx} + uu_x = 0, & x \in \mathbb{T},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{T} \end{cases} \tag{1.1}$$

has been derived as a model for the propagation of weakly nonlinear dispersive long waves in some physical contexts when dissipative effects occur (see [10]). The well-posedness of (1.1) has been studied in [7–9]. In these works, the existence of the solution is obtained by performing a fixed point argument on the corresponding integral equation.

As far as we know, the discussion of the KdV-B equation is mainly about the well-posedness. In this paper, we will study the KdV-B equation from a control point of view with a forcing term $h = h(x,t)$ added to the equation as a control input:

$$u_t + u_{xxx} - u_{xx} + uu_x = h, \quad x \in \mathbb{T},\ t \in \mathbb{R}^+. \tag{1.2}$$

It is natural to propose the following problem:

Problem. For any time $T > 0$, any two states u_0 and u_1 in a certain space, can one find an appropriate control h that drives the solution of (1.2) from u_0 at $t = 0$ to u_1 at $t = T$?

The problems were first investigated by Russel and Zhang in [13,14] (also in [4]) for the Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} + uu_x = h, \quad x \in \mathbb{T},\ t \in \mathbb{R}^+. \tag{1.3}$$

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They obtained that there exists a control h which is supported in a given open set $\omega \subset \mathbb{T}$ such that (1.3) is globally exactly controllable. The exact controllability of nonlinear third order dispersion equation with infinite distributed delay is obtained in [5].

However, since the linear KdV-B equation possesses a regularizing effect, the exact controllability may not hold with a localized control. Therefore, we consider a control acts on the entire domain $\mathbb{T}$.

Throughout the paper, for any $s \in \mathbb{R}$,

$$H_0^s(\mathbb{T}) = \{u \in H^s(\mathbb{T});\ [u] := \frac{1}{2\pi}\int_{\mathbb{T}} u(x)dx = 0\}.$$

Let$(u,v)_0 = \int_{\mathbb{T}} u(x)v(x)dx$ denote the usual scalar product in $L^2(\mathbb{T})$ and $(u,v)_s = ((1-\partial_x^2)^{\frac{s}{2}}u, (1-\partial_x^2)^{\frac{s}{2}}v)_0$ denote the scalar product in $H^s(\mathbb{T})$ with corresponding norm $\|u\|_s = (u,u)_s^{\frac{1}{2}}$.

The main results in this paper are stated as follows:

Theorem 1.1. *Let $s \geq 0, T > 0$ be given. There exists a $\delta > 0$ such that for any $u_0, u_1 \in H_0^s(\mathbb{T})$ satisfying*

$$\|u_0\|_s \leq \delta, \quad \|u_1\|_s \leq \delta,$$

one may find a control $h \in L^2(0,T;H_0^{s-1}(\mathbb{T}))$ such that (1.2) admits a unique solution $u \in C([0,T], H_0^s(\mathbb{T})) \cap L^2(0,T,H_0^{s+1}(\mathbb{T})))$ for which $u(0) = u_0$ and $u(T) = u_1$.

The rest of this paper is organized as follows. In Section 2 we get the well-posedness of system (1.1). Section 3 is devoted to the exact controllability.

2. Well-posedness

In this section, attention will be given to the well-posedness of (1.1). The well-posedness of the KdV-B equation was investigated in many articles ([7–9]). In these works, the existence of the solution is obtained by performing a fixed point argument on the corresponding integral equation. One of the main points is to find a "good" function space in which the fixed point argument will be performed. For the KdV equation, J. Bourgain [2] introduced new function spaces, adapted to the linear operator $\partial_t + \partial_x^3$, for which there are good "bilinear" estimates for the nonlinear term uu_x. Using these spaces, Bourgain was able to establish the well-posedness of KdV equation in the spatially periodic setting. Then this method was applied to the KdV-B equation, and obtained the well-posedness of (1.1) in $H_0^s(\mathbb{T})(s > -1)$. Since we only investigate the well-posedness of (1.1) in $H_0^s(\mathbb{T})(s \geq 0)$, there is no need to introduce corresponding Bourgain spaces.

2.1. Linear system

We first consider the inhomogeneous linear system

$$\begin{cases} u_t + u_{xxx} - u_{xx} = f, & x \in \mathbb{T},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{T}. \end{cases} \tag{2.1}$$

Let $s \in \mathbb{R}$ and let A be the linear operator defined by

$$Au = -u_{xxx} + u_{xx}$$

with the domain $\mathcal{D}(A) = H_0^{s+3}(\mathbb{T}) \subset H_0^s(\mathbb{T})$. Clearly, A is densely defined closed operator in $H_0^s(\mathbb{T})$.

For any $u \in \mathcal{D}(A)$, it is easy to deduce that

$$\begin{aligned}(Au, u)_s &= (-u_{xxx} + u_{xx}, u)_s \\ &= -\left((1-\partial_x^2)^{\frac{s}{2}}\partial_x^3 u, (1-\partial_x^2)^{\frac{s}{2}} u\right)_0 + \left((1-\partial_x^2)^{\frac{s}{2}}\partial_x^2 u, (1-\partial_x^2)^{\frac{s}{2}} u\right)_0 \\ &= -\|u_x\|_s^2 \\ &\le 0.\end{aligned}$$

Similarly, for any $v \in \mathcal{D}(A^*)$, $(A^*v, v)_s \le 0$, where $A^*v = v_{xxx} + v_{xx}$ and $\mathcal{D}(A^*) = H_0^{s+3}(\mathbb{T})$. This implies that both A and its adjoint A^* are dissipative. Thus A generates a semigroup $\{S(t)\}_{t\geq 0}$ in $H_0^s(\mathbb{T})$ by [11].

For $s \ge 0$ and $T > 0$. Let $Y_{s,I} = C(\overline{I}; H_0^s(\mathbb{T})) \cap L^2(I; H_0^{s+1}(\mathbb{T}))$ be endowed with the norm

$$\|v\|_{Y_{s,I}} = \|v\|_{L^\infty(I;H^s(\mathbb{T}))} + \|v\|_{L^2(I;H^{s+1}(\mathbb{T}))}.$$

For simplicity, we denote $Y_{s,I}$ by $Y_{s,T}$ if $I = (0,T)$.

Lemma 2.1. *For any $T > 0$, $u_0 \in H_0^s(\mathbb{T})$ and $f \in L^2(0,T; H_0^{s-1}(\mathbb{T}))$, the solution of (2.1) satisfies*

$$\|u\|_{Y_{s,T}} \le C\left(\|u_0\|_s + \|f\|_{L^2(0,T;H_0^{s-1}(\mathbb{T}))}\right), \tag{2.2}$$

where C is independent of T.

Proof. To have enough regularity in the following computations, we assume that $u_0 \in H_0^{s+3}(\mathbb{T})$ and that $f \in L^2(0,T; H_0^{s+3}(\mathbb{T}))$, so that the solution u of (2.1) satisfies $u \in C([0,T]; H_0^{s+3}(\mathbb{T})) \cap C^1([0,T]; H_0^s(\mathbb{T}))$.

Taking the scalar product of each term by u in $H_0^s(\mathbb{T})$ results in

$$\begin{aligned}\frac{1}{2}\|u(\cdot,t)\|_s^2 + \int_0^t \|u_x\|_s^2 d\tau &= \frac{1}{2}\|u_0\|_s^2 + \int_0^t (f,u)_s d\tau \\ &\le \frac{1}{2}\|u_0\|_s^2 + \int_0^t \|f\|_{s-1}\|u\|_{s+1} d\tau \\ &\le \frac{1}{2}\|u_0\|_s^2 + C\int_0^t \|f\|_{s-1}\|u_x\|_s d\tau \\ &\le \frac{1}{2}\|u_0\|_s^2 + C\int_0^t \|f\|_{s-1}^2 d\tau + \frac{1}{2}\int_0^t \|u_x\|_s^2 d\tau.\end{aligned}$$

Consequently,

$$\|u(\cdot,t)\|_s^2 + \int_0^t \|u_x\|_s^2 d\tau \le C\left(\|u_0\|_s^2 + \int_0^t \|f\|_{s-1}^2 d\tau\right).$$

Taking supremum in $t \in (0,T)$, it follows that

$$\|u\|_{L^\infty(0,T;H^s(\mathbb{T}))}^2 + \|u\|_{L^2(0,T;H^{s+1}(\mathbb{T}))}^2 \le C\left(\|u_0\|_s^2 + \|f\|_{L^2(0,T;H^{s-1}(\mathbb{T}))}^2\right).$$

This is also true for $u_0 \in H_0^s(\mathbb{T})$ and $f \in L^2(0,T; H_0^{s-1}(\mathbb{T}))$. □

2.2. Nonlinear system

We now present our first well-posedness result for (1.1).

Proposition 2.1. *For any $T > 0$ and any $u_0 \in H_0^0(\mathbb{T})$, (1.1) admits a unique solution $u \in Y_{0,T}$ which also satisfies*

$$\|u\|_{Y_{0,T}} \leq C\|u_0\|_0, \tag{2.3}$$

where C is independent of T. Moreover, the corresponding solution map is locally Lipschitz continuous: for any $u_0, v_0 \in H_0^0(\mathbb{T})$, the corresponding solutions u and v of (1.1) satisfy

$$\|u - v\|_{Y_{0,T}} \leq \alpha_{0,T}(\|u_0\|_0 + \|v_0\|_0)\|u_0 - v_0\|_0, \tag{2.4}$$

where $\alpha_{0,T} : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function.

Proof. We borrow some ideas from [12]. For given u_0, define the map Γ on the closed ball $B_{\theta,R} = \{u \in Y_{0,\theta} : \|u\|_{Y_{0,\theta}} \leq R\}$ of $Y_{0,\theta}$ by

$$\Gamma(u) = S(t)u_0 - \int_0^t S(t-\tau)(uu_x)(\tau)d\tau.$$

Notice Lemma 2.1 and the fact that

$$\begin{aligned}\int_0^\theta \|u\|_{L^\infty(\mathbb{T})}^2 d\tau &\leq C\int_0^\theta \|u\|_1\|u\|_0 d\tau \\ &\leq C\theta^{\frac{1}{2}}\|u\|_{L^\infty(0,\theta;L^2(\mathbb{T}))}\|u\|_{L^2(0,\theta;H^1(\mathbb{T}))}.\end{aligned}$$

There exist constants C_1, C_2, such that

$$\begin{aligned}\|\Gamma(u)\|_{Y_{0,\theta}} &\leq C_1\|u_0\|_0 + C_2\theta^{\frac{1}{4}}\|u\|_{Y_{0,\theta}}^2, \\ \|\Gamma(u) - \Gamma(v)\|_{Y_{0,\theta}} &\leq C_2\theta^{\frac{1}{4}}\|u + v\|_{Y_{0,\theta}}\|u - v\|_{Y_{0,\theta}}\end{aligned}$$

for any $u, v \in B_{\theta,R}$.

Choosing $R = 2C_1\|u_0\|_0$ and $\theta > 0$ so that $2C_2\theta^{\frac{1}{4}}R \leq \frac{1}{2}$, then

$$\|\Gamma(u)\|_{Y_{0,\theta}} \leq R \quad \text{ and } \quad \|\Gamma(u) - \Gamma(v)\|_{Y_{0,\theta}} \leq \frac{1}{2}\|u - v\|_{Y_{0,\theta}}$$

for any $u, v \in B_{\theta,R}$. Thus Γ is a contractive mapping on $B_{\theta,R}$. Its fixed point $u = \Gamma(u)$ is the unique solution of (1.1).

Multiply both sides of the first equation in (1.1) by u and integrate with respect x over the interval $\mathbb{T}$. An integration by parts leads to

$$\frac{1}{2}\frac{d}{dt}\|u(\cdot,t)\|_0^2 + \|u_x(\cdot,t)\|_0^2 = 0. \tag{2.5}$$

This implies

$$\sup_{0\leq t\leq\theta} \|u(\cdot,t)\|_0 \leq \|u_0\|_0.$$

By the standard extension argument, one may extend θ to T and obtain a nondecreasing continuous function $\alpha_{0,T} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|u\|_{Y_{0,T}} \leq \alpha_{0,T}(\|u_0\|_0)\|u_0\|_0. \tag{2.6}$$

Similarly, we can obtain (2.4).

On the other hand, integrate (2.5) with respect time over the interval $(0,t)$, then take supremum in $t \in (0,T)$, we can obtain (2.3) which is better than (2.6). □

Next, we show that (1.1) is well-posed in $Y_{3,T}$.

Proposition 2.2. *For any $u_0 \in H_0^3(\mathbb{T})$, (1.1) admits a unique solution $u \in Y_{3,T}$. Moreover, there exists a nondecreasing continuous function $\alpha_{3,T} : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\|u\|_{Y_{3,T}} \leq \alpha_{3,T}(\|u_0\|_0)\|u_0\|_3.$$

Proof. By Proposition 2.1, (1.1) admits a unique solution $u \in Y_{0,T}$, we just need to show further that $u \in Y_{3,T}$. For this purpose, let $v = u_t$, then

$$\begin{cases} v_t + v_{xxx} - v_{xx} + (uv)_x = 0, & x \in \mathbb{T},\ t > 0, \\ v(x,0) = v_0(x), & x \in \mathbb{T}, \end{cases} \tag{2.7}$$

where $v_0 = -u_0''' + u_0'' - u_0 u_0' \in H_0^0(\mathbb{T})$.

Proceeding as in the proof of Proposition 2.1, we see that (2.7) admits a unique solution $v \in Y_{0,T}$. Moreover

$$\|v\|_{Y_{0,T}} \leq \alpha_{0,T}(\|u_0\|_0)\|v_0\|_0. \tag{2.8}$$

Now we claim that (2.8) holds.

In fact, it is not difficult that

$$\begin{aligned} \|u\|_{L^\infty(0,T;H^1(\mathbb{T}))} &\leq C\|u\|_{H^1(0,T;H^1(\mathbb{T}))} \\ &\leq C\left(\|u\|_{L^2(0,T;H^1(\mathbb{T}))} + \|v\|_{L^2(0,T;H^1(\mathbb{T}))}\right), \end{aligned}$$

and

$$\|u_x\|_0^2 = (u_x, u_x)_0 = -(u, u_{xx})_0 \leq \|u\|_0 \|u_{xx}\|_0.$$

Then for any $\varepsilon > 0$,

$$\begin{aligned} \|uu_x\|_{L^\infty(0,T;L^2(\mathbb{T}))} &\leq \|u\|_{L^\infty(0,T;L^2(\mathbb{T}))}\|u_x\|_{L^\infty(0,T;L^\infty(\mathbb{T}))} \\ &\leq \|u\|_{L^\infty(0,T;L^2(\mathbb{T}))}\|u_x\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\mathbb{T}))}\|u_{xx}\|^{\frac{1}{2}}_{L^\infty(0,T;L^2(\mathbb{T}))} \\ &\leq \|u\|^{\frac{5}{4}}_{L^\infty(0,T;L^2(\mathbb{T}))}\|u_{xx}\|^{\frac{3}{4}}_{L^\infty(0,T;L^2(\mathbb{T}))} \\ &\leq \varepsilon\|u_{xx}\|_{L^\infty(0,T;L^2(\mathbb{T}))} + C(\varepsilon)\|u\|^5_{L^\infty(0,T;L^2(\mathbb{T}))}. \end{aligned}$$

Therefore from the equation $v = -u_{xxx} + u_{xx} - uu_x$, we have

$$\begin{aligned}\|u\|_{L^\infty(0,T;H^2(\mathbb{T}))} \leq & C\|u_x\|_{L^\infty(0,T;H^1(\mathbb{T}))} \\ \leq & C\Big(\|u_x - u\|_{L^\infty(0,T;H^1(\mathbb{T}))} + \|u\|_{L^\infty(0,T;H^1(\mathbb{T}))}\Big) \\ \leq & C\Big(\|u_{xxx} - u_{xx}\|_{L^\infty(0,T;L^2(\mathbb{T}))} + \|u\|_{L^\infty(0,T;H^1(\mathbb{T}))}\Big) \\ \leq & C\Big(\|v + uu_x\|_{L^\infty(0,T;L^2(\mathbb{T}))} + \|u\|_{L^\infty(0,T;H^1(\mathbb{T}))}\Big) \\ \leq & C\Big(\|v\|_{L^\infty(0,T;L^2(\mathbb{T}))} + \|u\|_{L^\infty(0,T;H^1(\mathbb{T}))} \\ & + \varepsilon\|u_{xx}\|_{L^\infty(0,T;L^2(\mathbb{T}))} + C(\varepsilon)\|u\|^5_{L^\infty(0,T;L^2(\mathbb{T}))}\Big). \end{aligned} \tag{2.9}$$

Choosing $\varepsilon = \frac{1}{2C}$, according to (2.3) and (2.8), there exists a nondecreasing continuous function $\widetilde{\alpha}_{3,T} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|u\|_{L^\infty(0,T;H^2(\mathbb{T}))} \leq \widetilde{\alpha}_{3,T}(\|u_0\|_0)\|u_0\|_3.$$

This implies

$$\begin{aligned}\|uu_x\|_{L^2(0,T;H^1(\mathbb{T}))} &\leq C\|u\|_{L^2(0,T;H^1(\mathbb{T}))}\|u\|_{L^\infty(0,T;H^2(\mathbb{T}))} \\ &\leq C\|u_0\|_0\widetilde{\alpha}_{3,T}(\|u_0\|_0)\|u_0\|_3.\end{aligned} \tag{2.10}$$

Computations similar (but more complicated) to those in (2.9) give

$$\|u\|_{L^\infty(0,T;H^3(\mathbb{T}))} \leq C\left(\|v\|_{L^\infty(0,T;L^2(\mathbb{T}))} + \|u\|^9_{L^\infty(0,T;H^2(\mathbb{T}))} + \|u\|_{L^\infty(0,T;H^2(\mathbb{T}))}\right).$$

Using the same argument as in (2.9), we get

$$\|u\|_{L^2(0,T;H^4(\mathbb{T}))} \leq C\left(\|v\|_{L^2(0,T;H^1(\mathbb{T}))} + \|uu_x\|_{L^2(0,T;H^1(\mathbb{T}))} + \|u\|_{L^\infty(0,T;H^3(\mathbb{T}))}\right).$$

Consequently, there exists a nondecreasing continuous function $\alpha_{3,T} : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|u\|_{L^\infty(0,T;H^3(\mathbb{T}))} + \|u\|_{L^2(0,T;H^4(\mathbb{T}))} \leq \alpha_{3,T}(\|u_0\|_0)\|u_0\|_3.$$

Finally, we can state the main result in this section. □

Theorem 2.1. *For any $u_0 \in H^s_0(\mathbb{T})$, (1.1) admits a unique solution $u \in Y_{s,T}$. Moreover, there exists a nondecreasing continuous function $\alpha_{s,T} : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\|u\|_{Y_{s,T}} \leq \alpha_{s,T}(\|u_0\|_0)\|u_0\|_s.$$

Proof. The cases $s = 0$ and $s = 3$ have been proved in Proposition 2.1 and Proposition 2.2. The cases of $0 < s < 3$ follows by (2.4) and the nonlinear interpolation theory [1]. The other cases of s can be proved similarly. □

3. Controllability

In this section, we consider the exact controllability of the KdV-B equation

$$\begin{cases} u_t + u_{xxx} - u_{xx} + uu_x = h, & x \in \mathbb{T},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{T}. \end{cases} \tag{3.1}$$

The exact controllability of the linear system can be obtained by the classical duality approach. Then by a fixed-point argument, we get the exact controllability of the nonlinear system.

First, let us consider the linear system

$$\begin{cases} u_t + u_{xxx} - u_{xx} = h, & x \in \mathbb{T},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{T}. \end{cases} \tag{3.2}$$

Proposition 3.1. *Let $s \geq 0$, $T > 0$ be given. For any $u_0, u_1 \in H_0^s(\mathbb{T})$, there exists a control input $h \in L^2(0,T;H_0^{s-1}(\mathbb{T}))$ such that (3.2) admits a solution $u \in Y_{s,T}$ satisfying*

$$u(x,0) = u_0(x),\ u(x,T) = u_1(x)$$

for any $x \in \mathbb{T}$.

Proof. According to Lemma 2.1, solution of (3.2) belongs to $Y_{s,T}$ for $u_0 \in H_0^s(\mathbb{T})$ and $h \in L^2(0,T;H_0^{s-1}(\mathbb{T}))$.

Motivated by [6], we consider the adjoint system

$$\begin{cases} -v_t - v_{xxx} - v_{xx} = 0, & x \in \mathbb{T},\ t > 0, \\ v(x,T) = v_T(x), & x \in \mathbb{T}. \end{cases} \tag{3.3}$$

First we claim that for any $T > 0$, the system (3.3) admits a unique solution $v \in Y_{-s,T}$.

In fact, let $\widetilde{v}(x,t) = v(x,T-t)$, then $\widetilde{v}$ solves

$$\begin{cases} \widetilde{v}_t - \widetilde{v}_{xxx} - \widetilde{v}_{xx} = 0, & x \in \mathbb{T},\ t > 0, \\ \widetilde{v}(x,0) = v_T(x), & x \in \mathbb{T}. \end{cases}$$

Similar as in Section 2, the linear operator $\widetilde{A}$ defined by $\widetilde{A}v = v_{xxx} + v_{xx}$ also generates a semigroup in $H_0^{-s}(\mathbb{T})$. Thus for any $v_T \in H_0^{-s}(\mathbb{T})$, $v \in C([0,T];H_0^{-s}(\mathbb{T}))$. Moreover, by a similar method as in Lemma 2.1, we have

$$\|v\|_{Y_{-s,T}} \leq C\|v_T\|_{-s}. \tag{3.4}$$

Taking the duality product of each term of (3.3) by v yields

$$\langle v, u\rangle_{-s,s} \,|_0^T = \int_0^T \langle v, h\rangle_{-s+1,s-1} dt,$$

where $\langle \cdot\,,\cdot\rangle_{-s,s}$ denotes the duality pairing $\langle \cdot\,,\cdot\rangle_{H_0^{-s}(\mathbb{T}),H_0^s(\mathbb{T})}$. Without loss of generality, we can assume that $u_0 = 0$. Following the classical duality approach, it is sufficient to prove the following observability inequality

$$\|v_T\|_{-s}^2 \leq C \int_0^T \|v\|_{-s+1}^2 dt. \tag{3.5}$$

Taking the scalar product of each term of by tv in $H_0^{-s}(\mathbb{T})$ results in

$$\frac{T}{2}\|v_T\|_{-s}^2 = \frac{1}{2}\int_0^T \|v\|_{-s}^2 dt + \int_0^T t\|v_x\|_{-s}^2 dt.$$

Combined with the Poincaré inequality, this gives (3.5). □

Now we can obtain the local exact controllability of the nonlinear system (3.1).

Proof of Theorem 1.1. From Proposition 3.1, by a classical functional analysis argument [3], one can construct a continuous operator $\Phi : H_0^s(\mathbb{T}) \to L^2(0,T;H_0^{s-1}(\mathbb{T}))$ such that for any $u_1 \in H_0^s(\mathbb{T})$, the solution u of (3.2) associated with $u_0 = 0$ and $h = \Phi(u_1)$ satisfies $u(T) = u_1$. Let us denote by $W(h)$ the corresponding trajectory. We know from Lemma 2.1 that W is continuous from $L^2(0,T;H_0^{s-1}(\mathbb{T}))$ into $Y_{s,T}$.

Pick any $u_0, u_1 \in H_0^s(\mathbb{T})$ satisfying $\|u_0\|_s \le \delta,\quad \|u_1\|_s \le \delta$ with δ to be determined. For any $u \in Y_{s,T}$, we set

$$\omega(u) = -\int_0^T S(T-\tau)(uu_x)(\tau)d\tau.$$

It is easy to deduce that

$$\begin{aligned}
\|\omega(u)\|_s \le& C\|\int_0^t S(t-\tau)(uu_x)(\tau)d\tau\|_{Y_{s,T}} \le C\|u\|^2_{Y_{s,T}},\\
\|\omega(u)-\omega(v)\|_s \le& C\|\int_0^t S(t-\tau)(uu_x - vv_x)(\tau)d\tau\|_{Y_{s,T}}\\
\le& C\|u+v\|_{Y_{s,T}}\|u-v\|_{Y_{s,T}}.
\end{aligned}$$

If we choose $h = \Phi(u_1 - S(T)u_0 - \omega(u))$, then

$$S(t)u_0 - \int_0^t S(t-\tau)(uu_x)(\tau)d\tau + W(h)(t) = \begin{cases} u_0, & \text{if } t = 0,\\ u_1, & \text{if } t = T.\end{cases}$$

We are led to consider the nonlinear map

$$\Gamma(u) = S(t)u_0 - \int_0^t S(t-\tau)(uu_x)(\tau)d\tau + W(h)(t).$$

Let

$$B_R = \{u \in Y_{s,T} :\ \|u\|_{Y_{s,T}} \le R\}.$$

For all $u, v \in B_R$,

$$\begin{aligned}
&\|\Gamma(u)\|_{Y_{s,T}}\\
\le& C\left(\|u_0\|_s + \|uu_x\|_{L^1(0,T;H^s(\mathbb{T}))} + \|\Phi(u_1 - S(T)u_0 - \omega(u))\|_{L^2(0,T;H^{s-1}(\mathbb{T}))}\right)\\
\le& C_1(\|u_0\|_s + \|u_1\|_s) + C_2\|u\|_{Y^2_{s,T}}\\
\le& 2C_1\delta + C_2R^2,\\
&\|\Gamma(u)-\Gamma(v)\|_{Y_{s,T}}\\
\le& C\|\Phi(\omega(u)-\omega(v))\|_{L^2(0,T;H^{s-1}(\mathbb{T}))} + \|uu_x - vv_x\|_{L^1(0,T;H^s(\mathbb{T}))}\\
\le& C_2\|u+v\|_{Y_{s,T}}\|u-v\|_{Y_{s,T}}\\
\le& 2C_2R\|u-v\|_{Y_{s,T}}.
\end{aligned}$$

Picking $R = 4C_1\delta$ and $\delta = (8C_1C_2)^{-1}$, we obtain that for u_0, u_1 satisfying $\|u_0\|_s \le \delta,\ \|u_1\|_s \le \delta$ and $u, v \in B_R$ that

$$\|\Gamma(u)-\Gamma(v)\|_{Y_{s,T}} \le R, \quad \|\Gamma(u)\|_{Y_{s,T}} \le \frac{1}{2}\|u-v\|_{Y_{s,T}}.$$

It follows from the contraction mapping theorem that Γ has a unique fixed point u in B_R. Then u satisfies (3.1) with $h = \Phi(u_1 - S(T)u_0 - \omega(u))$ and $u(T) = u_1$, as desired.

The proof of Theorem 1.1 is completed. $\square$

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