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IMPROVED BI-ACCELERATOR DERIVATIVE FREE WITH MEMORY FAMILY FOR SOLVING NONLINEAR EQUATIONS

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Abstract The object of the present paper is to accelerate the R-order convergence of with memory derivative free family given by Lotfi et al. (2014) without adding any new evaluations. To achieve this goal one more iterative parameter is introduced, which is calculated with the help of Newton's interpolatory polynomial. It is shown that the R-order convergence of the proposed scheme is increased from 12 to 14 without any extra evaluation. Smooth as well as non-smooth examples are presented to confirm theoretical result and significance of the new scheme.

Keywords Nonlinear equation, Newton's interpolatory polynomial, with and without memory method, R-order convergence, computational order of convergence.

MSC(2010) 65H05, 65B99.

1. Introduction

Finding the root of a nonlinear equations or systems is an interesting task in numerical analysis and applied scientific branches, which has attracted so much attention recently. In the last years, iterative techniques have been applied in many diverse fields as economics, engineering, physics, dynamical models, and so on. Some nice applications of iterative methods has been presented in preliminary orbit determination for deducing the orbit of the minor planet Ceres [3, 4], global positioning system [1], integral equations [13], nonlinear partial differential equation [15] and denoising [2] etc. Newton's method is the most well-known method for solving nonlinear equations. However, the existence of first derivative is compulsory for the convergence of Newton's method, which bounds its applications in practice. To overcome on this difficulty, Steffensen replaced the first derivative of the function in the Newton's iterate by forward finite difference approximation. Both the methods possess the quadratic convergence and the same efficiency but second one is derivative free. Multipoint iterative methods for solving nonlinear equations are of great practical importance since they overcome theoretical limits of one-point methods concerning the convergence order and computational efficiency. In the case of multipoint without memory methods, this requirement is closely connected with results of Kung and Traub [6], who conjectured that the order of convergence of any multipoint method without memory, consuming $n + 1$ function evaluations per iteration, cannot exceed the bound 2^n (called optimal order). Multipoint methods

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with this property are usually called optimal methods. The vast literature is available on optimal methods for computing the solution of nonlinear equations. In the recent past, researchers have focused to optimize the existing non-optimal iterative methods without any additional evaluation of functions and derivatives.

The convergence of the multipoint optimal without memory methods can be accelerated without additional computations using information from the points at which old data are reused. Let m_k represent the $r+1$ quantities $x_k, t_1(x_k), t_2(x_k), ...,$ $t_r(x_k)$ and define an iterative process by

$$x_{k+1} = F(m_k; m_{k-1}, m_{k-2}, ..., m_{k-r}).$$

Following Traub's terminology [17], F is called a multipoint iterative function with memory. To compare iterative methods theoretically, Owtrowski [12] introduced the idea of efficiency index and given by $d^{1/n}$, where d is the order of convergence and n number of function evaluations per iteration. In other words, we can say that an iterative method with higher efficiency index is more efficient. Probably Traub initiated the idea of with memory method in his book [17]. To accelerate the convergence order of Steffensen method without using additional evaluation γ is recursively calculated by self-accelerating method. Let γ_0 is given initial parameter and consider

$$\begin{aligned} \phi_k &= \frac{f(x_k + \gamma_k f(x_k)) - f(x_k)}{\gamma_k f(x_k)}, \ k = 0, 1, 2, ..., \\ x_{k+1} &= x_k - \frac{f(x_k)}{\phi_k}. \end{aligned} \tag{1.1}$$

where

$$\gamma_k = -\frac{1}{\phi_{k-1}}, \ k = 1, 2, ...$$

Traub derived that order of convergence of this method is 2.414. And thus the order of convergence of (1.1) with memory is more than that of Steffensen method, which also needs two function evaluations per iteration. Motivated by this idea some researchers have developed with memory methods to increase the efficiency of the existing optimal order without memory methods by using iterative parameter, such as ([9, 10, 14]).

With memory methods of higher efficiencies with low computational load can be established introducing multi accelerators [8, 16]. To get more efficient method, we first modify slightly existing optimal without memory methods in such a way that their given corresponding error equations have the most appropriate forms for achieving as high as possible efficiency index when they are extended to methods with memory. Then, based on interpolation, we consider some accelerators so that R-order of convergence raised to higher and higher level and therefore efficiency index will be certainly improved. In the just recent paper [7], a family of three-point derivative free without memory method of optimal eighth-order has been established and then one self-accelerating parameter is introduced to get modified with memory family of increased order twelve without any extra evaluation. This paper is devoted to the further improvement of with memory family by inserting one more iterative parameter.

In this paper we present an improvement of the existing with memory family, constructed by introducing one more iterative parameter which is calculated with help of Newton's interpolatory polynomial of degree five. In section 2, a family of three-point methods with memory with improved order of convergence from 12 to 14 without adding more evaluations is presented. The comparisons of absolute errors and computational efficiencies are given in section 3 to illustrate convergence behavior. Finally, we give the concluding remark.

2. Improved Family and Convergence Analysis

In the convergence analysis of the new method, we employ the notation used in Traub's book [17]: if m_k and n_k are null sequences and $m_k/n_k \to C$, where C is a non-zero constant, we shall write $m_k = O(n_k)$ or $m_k \sim Cn_k$. We also use the concept of R-order of convergence introduced by Ortega and Rheinboldt [11]. Let x_k be a sequence of approximations generated by an iterative method (IM). If this sequence converges to a zero ξ of function f with the R-order $O_R((IM), \xi) \geq r$, we will write

$$e_{k+1} \sim D_{k,r} e_k^r,$$

where $D_{k,r}$ tends to the asymptotic error constant D_r of the iterative method (IM) when $k \to \infty$. Very recently, Lofti et al. established the following iterative method without memory [7]:

$$\begin{aligned}
w_k &= x_k + \beta f(x_k), \quad k = 0, 1, 2, ... \\
y_k &= x_k - \frac{f(x_k)}{f[x_k, w_k]}, \\
z_k &= y_k - H(u_k, v_k) \frac{f(y_k)}{f[y_k, w_k]}, \\
x_{k+1} &= z_k - W(s_k) \frac{f(z_k)}{f[z_k, y_k] + f[w_k, z_k, y_k](z_k - y_k)},
\end{aligned} \tag{2.1}$$

where $\beta \in R$, $u_k = f(y_k)/f(x_k)$, $v_k = f(y_k)/f(w_k)$, $s_k = f(z_k)/f(x_k)$. The authors claim that in their paper [7] that this method achieve eighth-order convergence when the weight functions satisfy the following conditions

$$\begin{aligned}
&H(0, 0) = H_u(0, 0) = H_{uu}(0, 0) = 1, \\
&H_v(0, 0) = H_{vv}(0, 0) = 0, \\
&H_{uv}(0, 0) = 2, \\
&W(0) = W'(0) = 1.
\end{aligned} \tag{2.2}$$

And it's error expression is

$$e_{k+1} = c_2^2 c_3 (1 + \beta c_1)^4 ((9 + \beta c_1 (7 + \beta c_1)) c_2^3 + 2c_2 c_3 - c_4) e_k^8 + O(e_k^9), \tag{2.3}$$

where $c_i = \frac{f^{(i)}(\xi)}{i!}$. But for the above conditions on the weight functions, it is of seventh-order convergence with error expression given by

$$e_{k+1} = -\frac{(c_2^4 (1 + \beta c_1)^5 ((1 + \beta c_1) c_2^2 - 2c_1 c_3))}{4c_1^6} e_k^7 + O(e_k^8). \tag{2.4}$$

In fact it of the eighth-order convergence when the weight functions satisfy the conditions

$$\begin{aligned} &H(0,0) = H_u(0,0) = 1,\\ &H_v(0,0) = H_{vv}(0,0) = 0,\\ &H_{uu}(0,0) = H_{uv}(0,0) = 2,\\ &W(0) = W'(0) = 1. \end{aligned} \tag{2.5}$$

Any way this may be typing mistake. To improve convergence order as well as efficiency index without adding any new function evaluations, the authors in the same paper replaced real parameter β by iterative parameter β_k and assumed $(1 + \beta_k c_1) = 0$. They have done this using $\beta_k = \frac{-1}{c_1}$. But ξ is unknown here. Fortunately more accurate approximations of ξ are obtained by sequence x_k , w_k etc. and therefore one can get better approximation of c_1. In their work they used $c_1 \approx \widetilde{c}_1$ and $\widetilde{c}_1 = N_4'(x_k)$, where $N_4(t) = N_4(t; x_k, z_{k-1}, y_{k-1}, w_{k-1}, x_{k-1})$. And thus one parameter family with memory is given by
For given x_0, β_0, consider

$$\begin{aligned} w_k &= x_k + \beta_k f(x_k), \quad k = 0,1,2,...\\ y_k &= x_k - \frac{f(x_k)}{f[x_k, w_k]},\\ z_k &= y_k - H(u_k, v_k)\frac{f(y_k)}{f[y_k, w_k]},\\ x_{k+1} &= z_k - W(s_k)\frac{f(z_k)}{f[z_k, y_k] + f[w_k, z_k, y_k](z_k - y_k)},\\ \beta_{k+1} &= \frac{-1}{N_4'(x_{k+1})}, \end{aligned} \tag{2.6}$$

where u_k, v_k, s_k are defined as previous. They showed that under the same conditions on the weight functions the order of this scheme has been increased to twelve. The result of which is that the efficiency index is improved form $8^{1/4} = 1.6817$ to $12^{1/4} = 1.8612$. Motivated by this paper one natural question raised in our mind. *Is it possible to find more efficient method using the same number of evaluations?* We have found the answer of this question in positive. To justify our answer, we consider the following two-parametric with memory scheme
For given x_0, β_0, α_0 consider

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f[x_k, w_k] + \alpha_k f(w_k)}, \quad n = 0,1,2,...\\ z_k &= y_k - H(u_k, v_k)\frac{f(y_k)}{f[y_k, w_k] + \alpha_k f(w_k)},\\ x_{k+1} &= z_k - W(s_k)\frac{f(z_k)}{f[z_k, y_k] + f[w_k, z_k, y_k](z_k - y_k) + \alpha_k f(z_k)}, \end{aligned} \tag{2.7}$$

where $w_k = x_k + \beta_k f(x_k)$, u_k, v_k, s_k are defined as above. Under the same conditions on the weight functions the error expression of this family is given by

$$e_{k+1} = M_3(1 + \beta_k c_1)^4(\alpha_k c_1 + c_2)^3 e_k^7 + O(e_k^8), \tag{2.8}$$

where $M_3 = -\frac{2\alpha_k(2\alpha_k^2 c_1 + 2\alpha_k c_2 + c_3)}{c_1^4}$. Later we will show that by imposing one more iterative parameter in the existing with memory family R-order convergence is increased by two. To show this, we consider

$$\begin{aligned} \beta_k &= -\tfrac{1}{c_1} \approx \tfrac{-1}{N_4'(x_k)}, \\ \alpha_k &= -\tfrac{c_2}{c_1} \approx \tfrac{-N_5''(w_k)}{N_5'(w_k)}, \end{aligned} \tag{2.9}$$

where
$N_4(t) = N_4(t; x_k, z_{k-1}, y_{k-1}, x_{k-1}, w_{k-1})$,
$N_5(t) = N_5(t; w_k, x_k, z_{k-1}, y_{k-1}, w_{k-1}, x_{k-1})$.

And hence finally the proposed modified family is given by
For given x_0, β_0, α_0 consider

$$\begin{aligned} w_k &= x_k + \beta_k f(x_k), \quad k = 0, 1, 2, ... \\ y_k &= x_k - \frac{f(x_k)}{f[x_k, w_k] + \alpha_k f(w_k)}, \\ z_k &= y_k - H(u_k, v_k)\frac{f(y_k)}{f[y_k, w_k] + \alpha_k f(w_k)}, \\ x_{k+1} &= z_k - W(s_k)\frac{f(z_k)}{f[z_k, y_k] + f[w_k, z_k, y_k](z_k - y_k) + \alpha_k f(z_k)}, \\ \beta_{k+1} &= \frac{-1}{N_4'(x_{k+1})}, \\ \alpha_{k+1} &= \frac{-N_5''(w_{k+1})}{N_5'(w_{k+1})}, \end{aligned} \tag{2.10}$$

where u_k, v_k, s_k, $N_4(t)$ and $N_5(t)$ are defined as above. Now we denote

$$e_k = x_k - \xi,\ e_{k,z} = z_k - \xi,\ e_{k,y} = y_k - \xi,\ e_{k,w} = w_k - \xi,$$

where ξ is the exact root. Before going to main result, we first prove the following two lemmas:

Lemma 2.1. *The estimate* $1 + \beta_k c_1 \sim -\frac{c_5}{c_1}\ e_{k-1}\ e_{k-1,z}\ e_{k-1,y}\ e_{k-1,w}$.

Proof. Suppose that there are s nodes $t_0, t_1\ .\ .\ .\ , t_s$ from the interval $D = [a, b]$, where a is the minimum and b is the maximum of these nodes, respectively. Then for some $\zeta \in D$, the error of Newton's interpolation polynomial $N_s(t)$ of degree s is given by

$$f(t) - N_s(t) = \frac{f^{(s+1)}(\zeta)}{s+1!}\prod_{j=0}^{s}(t - t_j). \tag{2.11}$$

For $s = 4$ the above equation assumes the form (keeping in the mind $t_0 = x_k$, $t_1 = z_{k-1}$, $t_2 = y_{k-1}$, $t_3 = x_{k-1}$, $t_4 = w_{k-1}$)

$$f(t) - N_4(t) = \frac{f^5(\zeta_4)}{5!}\left\{(t - x_k)(t - z_{k-1})(t - y_{k-1})(t - x_{k-1})(t - w_{k-1})\right\}. \tag{2.12}$$

Differentiating equation (2.12) with respect to t and putting $t = x_k$, we get

$$f'(x_k) - N_4'(x_k) = \frac{f^5(\zeta_4)}{5!}\left\{(x_k - z_{k-1})(x_k - y_{k-1})(x_k - x_{k-1})(x_k - w_{k-1})\right\} \tag{2.13}$$

Now

$$x_k - z_{k-1} = (x_k - \xi) - (z_{k-1} - \xi) = e_k - e_{k-1,z}.$$

Similarly

$$\begin{aligned} x_k - y_{k-1} &= e_k - e_{k-1,y}, \\ x_k - x_{k-1} &= e_k - e_{k-1}, \\ x_k - w_{k-1} &= e_k - e_{k-1,w}. \end{aligned}$$

Using these relations in the equation (2.13) and simplifying we get

$$N_4'(x_k) \sim c_1 + 2c_2 e_k - c_5 \; e_{k-1} \; e_{k-1,z} \; e_{k-1,y} \; e_{k-1,w}. \tag{2.14}$$

And thus

$$\begin{aligned} &1 + \beta_k c_1 = 1 - \frac{c_1}{N_4'(x_k)} \sim 1 - \frac{c_1}{c_1 + 2c_2 e_k - c_5 \; e_{k-1} \; e_{k-1,z} \; e_{k-1,y} \; e_{k-1,w}}, \\ &\text{or} \\ &1 + \beta_k c_1 \sim -\frac{c_5}{c_1} \; e_{k-1} \; e_{k-1,z} \; e_{k-1,y} \; e_{k-1,w}. \end{aligned} \tag{2.15}$$

Hence we have the result.

□

Lemma 2.2. *The estimate* $\alpha_k c_1 + c_2 \sim c_6 \; e_{k-1} \; e_{k-1,z} \; e_{k-1,y} \; e_{k-1,w}$.

Proof. For $s = 5$ and $t_0 = w_k$, $t_1 = x_k$, $t_2 = z_{k-1}$, $t_3 = y_{k-1}$, $t_4 = w_{k-1}$, $t_5 = x_{k-1}$, the equation (2.11) becomes

$$\begin{aligned} &f(t) - N_5(t) \\ =&\frac{f^6(\zeta_5)}{6!}\left\{(t - w_k)(t - x_k)(t - z_{k-1})(t - y_{k-1})(t - w_{k-1})(t - x_{k-1})\right\}. \end{aligned} \tag{2.16}$$

Differentiating equation (2.16) with respect to t and putting $t = w_k$, we have

$$\begin{aligned} &f'(w_k) - N_5'(w_k) \\ =&\frac{f^6(\zeta_5)}{6!}\left\{(w_k - x_k)(w_k - z_{k-1})(w_k - y_{k-1})(w_k - w_{k-1})(w_k - x_{k-1})\right\}. \end{aligned} \tag{2.17}$$

The above equation can be rewritten as

$$N_5'(w_k) \sim c_1 + 2c_2 e_{k,w} + c_6 \; e_k \; e_{k-1,z} \; e_{k-1,y} \; e_{k-1,w} \; e_{k-1}. \tag{2.18}$$

Now twice differentiating (2.16) with respect to t and putting $t = w_k$ and proceeding in the same we get

$$N_5''(w_k) \sim 2c_2\left[1 + \frac{3c_3}{c_2} e_{k,w} - \frac{c_6}{c_2} \; e_{k-1,z} \; e_{k-1,y} \; e_{k-1,w} \; e_{k-1}\right]. \tag{2.19}$$

Hence

$$\begin{aligned} \alpha_k c_1 + c_2 &= c_2 - \frac{N_5''(w_k)}{N_5'(w_k)} c_1 \\ &\sim c_2 - \frac{2c_2\left[1 + \frac{3c_3}{c_2} e_{k,w} - \frac{c_6}{c_2} \; e_{k-1,z} \; e_{k-1,y} \; e_{k-1,w} \; e_{k-1}\right]}{2\left[c_1 + 2c_2 e_{k,w} + c_6 \; e_k \; e_{k-1,z} \; e_{k-1,y} \; e_{k-1,w} \; e_{k-1}\right]} c_1 \end{aligned}$$

or

$$\alpha_k c_1 + c_2 \sim c_6\ e_{k-1}\ e_{k-1,z}\ e_{k-1,y}\ e_{k-1,w}. \tag{2.20}$$

Thus we proved the lemma. $\square$

By using the above lemmas now we are going to prove the main result.

Theorem 2.1. *If an initial approximation x_0 is sufficiently close to a simple zero ξ of f, then the R-order of convergence of three-point method* (2.10) *with memory is at least fourteen.*

Proof. First we assume that the R-order of convergence of sequence x_k, w_k, y_k, z_k is at least r, r_1, r_2 and r_3, respectively. Hence

$$e_{k+1} \sim D_{k,r} e_k^r \sim D_{k,r}(D_{k-1,r} e_{k-1}^r)^r \sim D_{k,r} D_{k-1,r}^r e_{k-1}^{r^2}, \tag{2.21}$$

and

$$e_{k,w} \sim D_{k,r_1} e_k^{r_1} \sim D_{k,r_1}(D_{k-1,r} e_{k-1}^r)^{r_1} \sim D_{k,r_1} D_{k-1,r}^{r_1} e_{k-1}^{rr_1}. \tag{2.22}$$

Similarly

$$e_{k,y} \sim D_{k,r_2} D_{k-1,r}^{r_2} e_{k-1}^{rr_2}, \tag{2.23}$$

$$e_{k,z} \sim D_{k,r_3} D_{k-1,r}^{r_3} e_{k-1}^{rr_3}. \tag{2.24}$$

By virtue of the above equation, lemma (2.1) and (2.2) implies that

$$1 + \beta_k c_1 \sim -\frac{c_5}{c_1}(D_{k-1,r_3})(D_{k-1,r_2})(D_{k-1,r_1}) e_{k-1}^{r_3+r_2+r_1+1}, \tag{2.25}$$

and

$$\alpha_k c_1 + c_2 \sim c_6 (D_{k-1,r_3})(D_{k-1,r_2})(D_{k-1,r_1}) e_{k-1}^{r_3+r_2+r_1+1}, \tag{2.26}$$

respectively. For the scheme (2.10), it can be derived that

$$e_{k,w} \sim (1 + \beta_k c_1) e_k, \tag{2.27}$$

$$e_{k,y} \sim M_1 (1 + \beta_k c_1)(\alpha_k c_1 + c_2) e_k^2, \tag{2.28}$$

where $M_1 = 1/c_1$,

$$e_{k,z} \sim M_2 (1 + \beta_k c_1)^2 (\alpha_k c_1 + c_2) e_k^4, \tag{2.29}$$

where $M_2 = -\frac{2\alpha_k^2 c_1 + 2\alpha_k c_2 + c_3}{c_1^2}$ and

$$e_{k+1} \sim M_3 (1 + \beta_k c_1)^4 (\alpha_k c_1 + c_2)^3 e_k^7, \tag{2.30}$$

where M_3 is given just below to (2.8). Combining the eqns. (2.27) and (2.25), we have

$$e_{k,w} \sim -\frac{c_5}{c_1}(D_{k-1,r_3})(D_{k-1,r_2})(D_{k-1,r_1}) e_{k-1}^{r_3+r_2+r_1+1} (D_{k-1,r} e_{k-1}^r),$$

or

$$e_{k,w} \sim -\frac{c_5}{c_1}(D_{k-1,r_3})(D_{k-1,r_2})(D_{k-1,r_1})(D_{k-1,r}) e_{k-1}^{r_3+r_2+r_1+r+1}. \tag{2.31}$$

Using (2.25) and (2.26) in eqn. (2.28) we get

$$\begin{aligned}
e_{k,y} &\sim -M_1\left(\frac{c_5}{c_1}\right)(D_{k-1,r_3})(D_{k-1,r_2})(D_{k-1,r_1})e_{k-1}^{r_3+r_2+r_1+1}\\
&c_6(D_{k-1,r_3})(D_{k-1,r_2})(D_{k-1,r_1})e_{k-1}^{r_3+r_2+r_1+1}(D_{k-1,r}e_{k-1}^{r})^2,
\end{aligned}$$

or

$$e_{k,y} \sim N_1(D^2_{k-1,r_3})(D^2_{k-1,r_2})(D^2_{k-1,r_1})(D^2_{k-1,r})e_{k-1}^{2(r_3+r_2+r_1+1)+2r}, \tag{2.32}$$

where $N_1 = -M_1\left(\frac{c_5c_6}{c_1}\right)$. Similarly

$$e_{k,z} \sim N_2(D^3_{k-1,r_3})(D^3_{k-1,r_2})(D^3_{k-1,r_1})(D^4_{k-1,r})e_{k-1}^{3(r_3+r_2+r_1+1)+4r}, \tag{2.33}$$

where $N_2 = M_2\left(\frac{c_5}{c_1}\right)^2.c_6$ and

$$e_{k+1} \sim N_3(D^7_{k-1,r_3})(D^7_{k-1,r_2})(D^7_{k-1,r_1})(D^7_{k-1,r})e_{k-1}^{7(r_3+r_2+r_1+1)+7r}, \tag{2.34}$$

where $N_3 = M_3\left(\frac{c_5}{c_1}\right)^4.c_6^3$. Now comparing the equal powers of e_{k-1} in eqns. (2.22)-(2.31), (2.23)-(2.32), (2.24)-(2.33) and (2.21)-(2.34), we find the following system of nonlinear equations:

$$\begin{aligned}
&rr_1 - r - (r_3 + r_2 + r_1 + 1) = 0,\\
&rr_2 - 2r - 2(r_3 + r_2 + r_1 + 1) = 0,\\
&rr_3 - 4r - 3(r_3 + r_2 + r_1 + 1) = 0,\\
&r^2 - 7r - 7(r_3 + r_2 + r_1 + 1) = 0.
\end{aligned}$$

Solving these equations, we get $r_1 = 2, r_2 = 4, r_3 = 7, r = 14$. And hence we proved the main result. □

Note 1: The efficiency index of the proposed scheme (2.10) is $14^{1/4} = 1.9343$ which is more than $12^{1/4} = 1.8612$ of with memory family given by Lotfi et al. [7].

3. Numerical Examples and Conclusion

In this section we compare the proposed with memory family to existing scheme by taking some particular choices of weight functions $H(u_k, v_k)$ and $W(s_k)$. Lotfi et al. [7] have considered the following weight functions for numerical testing in the scheme (2.6):

$$\begin{aligned}
&H_1(u_k, v_k) = 1 + u_k + 2u_kv_k + u_k^2,\\
&H_2(u_k, v_k) = \frac{1}{1 - u_k - 2u_kv_k},\\
&W_1(s_k) = \cos(s_k) + \sin(s_k), W_2(s_k) = \frac{1}{1 - s_k},\\
&W_3(s_k) = 1 + s_k, W_4(s_k) = e^{s_k}.
\end{aligned}$$

To show the efficiency of the proposed method here we choose the same weight functions for the family (2.10). Since the scheme is derivative free, so we apply the proposed family to solve a smooth as well as a non-smooth nonlinear equations and compared with the existing with memory methods. Numerical testing have been carried out using variable precision arithmetic in *MATHEMATICA 8* with 500 significant digits. The computational order of convergence (COC) is defined by

$$COC = \frac{\ln(|f(x_k)/f(x_{k-1})|)}{\ln(|f(x_{k-1})/f(x_{k-2})|)}.$$

To test the performance of new method consider the following two nonlinear functions (which are taken from [5] and [7]):

$$1. f_1(x) = \sin(\pi x)e^{x^2+x\cos(x)-1} + x\log(x\sin(x)+1),$$

$$2. f_2(x) = \begin{cases} 10(x^4+x), & x<0 \\ -10(x^3+x), & x \geq 0. \end{cases}$$

The absolute error for the first three iterations are given in Tables 1-2. In the table $ae \pm b$ stand for $a \times 10^{\pm b}$. Note that a large number of three-step derivative free (with and without memory) methods are available in the literature. But the methods which have been tested for non-smooth functions are very very rare and this clearly reveals the significance of this article. From the the theoretical result, we can conclude that the order of convergence of the without memory family can be made more higher than the existing with memory family by imposing more self-accelerating parameters without any additional calculations and the computational efficiency of the presented with memory method becomes high. The R-order of convergence is increased from 12 to 14 in accordance with the quality of the applied accelerating method proposed in this paper. We can see that the self-accelerating parameters play a key role in increasing the order of convergence of the iterative method.

Table 1. Comparison of the absolute error in first, second and third iterations for $f_1(x)$ with $x_0 = 0.6$, $\xi = 0$, $\beta_0 = 0.1$, $\alpha_0 = 0.01$.

Method	$\lvert x_1-\xi\rvert$	$\lvert x_2-\xi\rvert$	$\lvert x_3-\xi\rvert$	COC
Lotfi et al. Method [7] with H_1, W_1	0.16408e-1	0.34379e-20	0.25814e-245	12.057
Lotfi et al. Method [7] with H_1, W_2	0.11711e-1	0.58655e-21	0.15632e-254	12.105
Lotfi et al. Method [7] with H_1, W_3	0.14751e-1	0.20948e-20	0.67504e-248	12.074
Lotfi et al. Method [7] with H_1, W_4	0.13088e-1	0.11136e-20	0.34311e-251	12.091
Lotfi et al. Method [7] with H_2, W_1	0.28676e-1	0.24880e-18	0.53628e-223	12.004
Lotfi et al. Method [7] with H_2, W_2	0.18955e-1	0.53482e-19	0.51066e-231	12.087
Lotfi et al. Method [7] with H_2, W_3	0.25205e-1	0.18570e-18	0.15906e-224	12.035
Lotfi et al. Method [7] with H_2, W_4	0.21713e-1	0.10425e-18	0.15467e-227	12.065
Proposed Method with H_1, W_1	0.16158e-1	0.12243e-25	0.58421e-365	14.072
Proposed Method with H_1, W_2	0.11234e-1	0.27610e-29	0.32316e-418	14.089
Proposed Method with H_1, W_3	0.14420e-1	0.26632e-26	0.99116e-375	14.091
Proposed Method with H_1, W_4	0.12675e-1	0.20826e-27	0.18137e-391	14.122
Proposed Method with H_2, W_1	0.27043e-1	0.71237e-23	0.17535e-325	14.031
Proposed Method with H_2, W_2	0.17473e-1	0.10211e-24	0.32654e-352	14.101
Proposed Method with H_2, W_3	0.23627e-1	0.26998e-23	0.14620e-331	14.056
Proposed Method with H_2, W_4	0.20192e-1	0.59893e-24	0.51508e-341	14.080

Table 2. Comparison of the absolute error in first, second and third iterations for $f_2(x)$ with $x_0 = -0.8$, $\xi = -1$, $\beta_0 = 1$, $\alpha_0 = 0.01$.

Method	$\lvert x_1 - \xi\rvert$	$\lvert x_2 - \xi\rvert$	$\lvert x_3 - \xi\rvert$	COC
Lotfi et al. Method [7] with H_1, W_1	0.18654e+0	0.17935e-6	0.27411e-77	11.478
Lotfi et al. Method [7] with H_1, W_2	0.75427e+0	0.16246e+0	0.61778e-1	1.1238
Lotfi et al. Method [7] with H_1, W_3	0.34083e+0	0.22151e-4	0.34568e-52	10.733
Lotfi et al. Method [7] with H_1, W_4	0.51834e+0	0.14925e-3	0.30384e-42	9.8489
Lotfi et al. Method [7] with H_2, W_1	0.12278e+0	0.13520e-8	0.92299e-103	11.682
Lotfi et al. Method [7] with H_2, W_2	0.95479e-1	0.57537e-10	0.32577e-119	11.747
Lotfi et al. Method [7] with H_2, W_3	0.11261e+0	0.48192e-9	0.38838e-108	11.709
Lotfi et al. Method [7] with H_2, W_4	0.99856e-1	0.10515e-9	0.45221e-116	11.738
Proposed Method with H_1, W_1	0.18185e+0	0.64162e-9	0.58263e-126	13.607
Proposed Method with H_1, W_2	0.71281e+1	0.46083e+1	0.37102e+1	0.4717
Proposed Method with H_1, W_3	0.33349e+0	0.24624e-6	0.87589e-90	13.052
Proposed Method with H_1, W_4	0.50787e+0	0.17376e-5	0.66509e-78	12.386
Proposed Method with H_2, W_1	0.12565e+0	0.91112e-11	0.78990e-152	13.770
Proposed Method with H_2, W_2	0.97215e-1	0.39968e-12	0.77183e-171	13.840
Proposed Method with H_2, W_3	0.11510e+0	0.31706e-11	0.30163e-158	13.797
Proposed Method with H_2, W_4	0.10173e+0	0.70077e-12	0.20027e-167	13.829

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