

SOME NEW INTEGRAL INEQUALITIES OF HERMITE–HADAMARD TYPE FOR $(\alpha, m; P)$ -CONVEX FUNCTIONS ON CO-ORDINATES*

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Abstract In the paper, the authors introduce the notion “ $(\alpha, m; P)$ -convex function on co-ordinates” and establish new integral inequalities of Hermite–Hadamard type for $(\alpha, m; P)$ -convex functions on co-ordinates in a rectangle from the plane $\mathbb{R}_0 \times \mathbb{R}$.

Keywords $(\alpha, m; P)$ -convex function on co-ordinates, integral inequality of Hermite–Hadamard type.

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1. Introduction

Let us recall some definitions of various convex functions.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, +\infty) \rightarrow \mathbb{R}$ is said to be convex on an interval I if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.2 ([5]). We say that a map $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and satisfies

$$f(\lambda x + (1 - \lambda)y) \leq f(x) + f(y)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.3 ([9]). For $f : [0, b] \rightarrow \mathbb{R}$ and $m \in (0, 1]$, if

$$f(\lambda x + m(1 - \lambda)y) \leq \lambda f(x) + m(1 - \lambda)f(y)$$

is valid for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$, then we say that f is an m -convex function on $[0, b]$.

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Definition 1.4 ([7]). For $f : [0, b] \rightarrow \mathbb{R}$ and $(\alpha, m) \in (0, 1] \times (0, 1]$, if

$$f(\lambda x + m(1 - \lambda)y) \leq \lambda^\alpha f(x) + m(1 - \lambda^\alpha)f(y)$$

is valid for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$, then we say that f is an (α, m) -convex function on $[0, b]$.

Definition 1.5 ([3, 4]). A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ is said to be convex on co-ordinates in Δ for $a < b$ and $c < d$ if the partial functions

$$f_y : [a, b] \rightarrow \mathbb{R}, \quad f_y(u) = f_y(u, y) \quad \text{and} \quad f_x : [c, d] \rightarrow \mathbb{R}, \quad f_x(v) = f_x(x, v)$$

are convex for all $x \in (a, b)$ and $y \in (c, d)$.

Definition 1.6 ([3, 4]). A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ is said to be convex on co-ordinates in Δ for $a < b$ and $c < d$ if the inequality

$$\begin{aligned} & f(tx + (1 - t)z, \lambda y + (1 - \lambda)w) \\ & \leq t\lambda f(x, y) + t(1 - \lambda)f(x, w) + (1 - t)\lambda f(z, y) + (1 - t)(1 - \lambda)f(z, w) \end{aligned}$$

holds for all $t, \lambda \in [0, 1]$ and $(x, y), (z, w) \in \Delta$.

Some integral inequalities of Hermite–Hadamard type for the above mentioned convex functions may be recited as follows.

Theorem 1.1 ([6]). Let $f : \mathbb{R}_0 = [0, \infty) \rightarrow \mathbb{R}$ be m -convex and $m \in (0, 1]$. If $f \in L([a, b])$ for $0 \leq a < b < \infty$, then

$$\frac{1}{b - a} \int_a^b f(x) \, dx \leq \min \left\{ \frac{f(a) + mf(b/m)}{2}, \frac{mf(a/m) + f(b)}{2} \right\}.$$

Theorem 1.2 ([5, Theorem 3.1]). Let $f \in P(I)$, $a, b \in I$ with $a < b$, and $f \in L([a, b])$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a} \int_a^b f(x) \, dx \leq 2(f(a) + f(b)).$$

Both inequalities are the best possible.

Theorem 1.3 ([3, 4, Theorem 2.2]). Let $f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R}$ be convex on co-ordinates in Δ for $a < b$ and $c < d$. Then

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \leq \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) \, dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) \, dy \right] \\ & \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dy \, dx \\ & \leq \frac{1}{4} \left[\frac{1}{b-a} \left(\int_a^b f(x, c) \, dx + \int_a^b f(x, d) \, dx \right) + \frac{1}{d-c} \left(\int_c^d f(a, y) \, dy \right. \right. \\ & \quad \left. \left. + \int_c^d f(b, y) \, dy \right) \right] \\ & \leq \frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{4}. \end{aligned} \tag{1.1}$$

For more information on new results in this topic, please refer to [1, 2, 8, 10] and plenty of references therein.

In this paper, we will introduce a new notion “ $(\alpha, m; P)$ -convex function on co-ordinates” and establish some new integral inequalities of Hermite–Hadamard type for $(\alpha, m; P)$ -convex functions on co-ordinates in a rectangle from the plane $\mathbb{R}_0 \times \mathbb{R}$.

2. Integral inequalities of Hermite–Hadamard type

Motivated by Definitions 1.2, 1.4, and 1.6, we now introduce a new notion “ $(\alpha, m; P)$ -convex function on co-ordinates” as follows.

Definition 2.1. For $m, \alpha \in (0, 1]$, a mapping $f : [0, b] \times [c, d] \rightarrow \mathbb{R}_0$ is called co-ordinated $(\alpha, m; P)$ -convex for $0 < b$ and $c < d$, if the inequality

$$\begin{aligned} & f(tx + m(1 - t)z, \lambda y + (1 - \lambda)w) \\ & \leq t^\alpha [f(x, y) + f(x, w)] + m(1 - t^\alpha) [f(z, y) + f(z, w)] \end{aligned}$$

holds for all $t, \lambda \in [0, 1]$ and $(x, y), (z, w) \in [0, b] \times [c, d]$.

Now we establish some new integral inequalities of Hermite–Hadamard type for $(\alpha, m; P)$ -convex functions on co-ordinates in a rectangle from the plane $\mathbb{R}_0 \times \mathbb{R}$.

Theorem 2.1. Let $f : [0, \frac{b}{m}] \times [c, d] \rightarrow \mathbb{R}_0$ be integrable for $0 \leq a < b, c < d$, and $m \in (0, 1]$ and let $f \in L_1([0, \frac{b}{m}] \times [c, d])$. If f is co-ordinated $(\alpha, m; P)$ -convex on $[0, \frac{b}{m}] \times [c, d]$ for $\alpha \in (0, 1]$, then

$$f\left(\frac{m(a + b)}{2}, \frac{c + d}{2}\right) \leq \frac{\int_c^d \int_a^b [f(mx, y) + m(2^\alpha - 1)f(x, y)] dx dy}{2^{\alpha-1}(b - a)(d - c)}$$

and

$$\begin{aligned} & \frac{1}{(b - a)(d - c)} \int_c^d \int_a^b f(x, y) dx dy \\ & \leq \frac{1}{\alpha + 1} \left[f(a, c) + f(a, d) + m\alpha f\left(\frac{b}{m}, c\right) + m\alpha f\left(\frac{b}{m}, d\right) \right]. \end{aligned}$$

Proof. The $(\alpha, m; P)$ -convexity of f gives

$$\begin{aligned} & f\left(\frac{m(a + b)}{2}, \frac{c + d}{2}\right) \\ & = f\left(\frac{m[ta + (1 - t)b + (1 - t)a + tb]}{2}, \frac{\lambda c + (1 - \lambda)d + (1 - \lambda)c + \lambda d}{2}\right) \\ & \leq \frac{1}{2^\alpha} \{ f(m[ta + (1 - t)b], \lambda c + (1 - \lambda)d) + f(m[ta + (1 - t)b], \\ & \quad (1 - \lambda)c + \lambda d) + m(2^\alpha - 1) [f((1 - t)a + tb, \lambda c + (1 - \lambda)d) \\ & \quad + f((1 - t)a + tb, (1 - \lambda)c + \lambda d)] \}. \end{aligned} \tag{2.1}$$

Putting $x = ta + (1 - t)b$ and $y = \lambda c + (1 - \lambda)d$ for $0 \leq t, \lambda \leq 1$ and integrating the

inequality (2.1) on $[0, 1] \times [0, 1]$ over (t, λ) lead to

$$\begin{aligned} & f\left(\frac{m(a+b)}{2}, \frac{c+d}{2}\right) \\ & \leq \frac{1}{2^\alpha} \int_0^1 \int_0^1 \left\{ f(m[ta + (1-t)b], \lambda c + (1-\lambda)d) \right. \\ & \quad + f(m[ta + (1-t)b], (1-\lambda)c + \lambda d) + m(2^\alpha - 1) [f((1-t)a + tb, \\ & \quad \left. \lambda c + (1-\lambda)d) + f((1-t)a + tb, (1-\lambda)c + \lambda d)] \right\} dt d\lambda \\ & = \frac{1}{2^{\alpha-1}(b-a)(d-c)} \int_c^d \int_a^b [f(mx, y) + m(2^\alpha - 1)f(x, y)] dx dy. \quad (2.2) \end{aligned}$$

Similarly,

$$\begin{aligned} & \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dx dy \\ & = \int_0^1 \int_0^1 f(ta + (1-t)b, \lambda c + (1-\lambda)d) dt d\lambda \\ & \leq \int_0^1 \int_0^1 \left\{ t^\alpha [f(a, c) + f(a, d)] + m(1-t^\alpha) \left[f\left(\frac{b}{m}, c\right) + f\left(\frac{b}{m}, d\right) \right] \right\} dt d\lambda \\ & = \frac{1}{\alpha+1} \left[f(a, c) + f(a, d) + m\alpha f\left(\frac{b}{m}, c\right) + m\alpha f\left(\frac{b}{m}, d\right) \right]. \end{aligned}$$

Theorem 2.1 is thus proved. \square

Corollary 2.1. *Under the assumptions of Theorem 2.1,*

1. *if $m = 1$, then*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) & \leq \frac{2}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dx dy \\ & \leq \frac{2}{\alpha+1} [f(a, c) + f(a, d) + \alpha f(b, c) + \alpha f(b, d)]; \end{aligned}$$

2. *if $\alpha = 1$, then*

$$f\left(\frac{m(a+b)}{2}, \frac{c+d}{2}\right) \leq \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b [f(mx, y) + mf(x, y)] dx dy$$

and

$$\begin{aligned} & \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dx dy \\ & \leq \frac{1}{2} \left[f(a, c) + f(a, d) + mf\left(\frac{b}{m}, c\right) + mf\left(\frac{b}{m}, d\right) \right]; \end{aligned}$$

3. *if $m = \alpha = 1$, then*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) & \leq \frac{2}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dx dy \\ & \leq f(a, c) + f(a, d) + f(b, c) + f(b, d). \end{aligned}$$

Theorem 2.2. Let $f : [0, \frac{b}{m^2}] \times [c, d] \rightarrow \mathbb{R}_0$ be integrable for $0 \leq a < b$, $c < d$, and $m \in (0, 1]$ and let $f \in L_1([0, \frac{b}{m^2}] \times [c, d])$. If f is co-ordinated $(\alpha, m; P)$ -convex on $[0, \frac{b}{m^2}] \times [c, d]$ for $\alpha \in (0, 1]$, then

$$\begin{aligned} & \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) \, dx \, dy \\ & \leq \frac{1}{2^{\alpha+1}(b-a)} \int_a^b Q(x, c, d) \, dx + \frac{1}{(\alpha+1)(d-c)} \int_c^d \left[f(a, y) + m\alpha f\left(\frac{b}{m}, y\right) \right] \, dy \\ & \leq \frac{1}{2^{\alpha-1}(\alpha+1)} \left[Q(a, c, d) + m\alpha Q\left(\frac{b}{m}, c, d\right) \right], \end{aligned}$$

where

$$Q(x, c, d) = f(x, c) + f(x, d) + m(2^\alpha - 1) \left[f\left(\frac{x}{m}, c\right) + f\left(\frac{x}{m}, d\right) \right]$$

for $x \in [a, \frac{b}{m}]$.

Proof. Letting $y = \lambda c + (1 - \lambda)d$ for $0 \leq \lambda \leq 1$ and employing the $(\alpha, m; P)$ -convexity of f reveal

$$\begin{aligned} & \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) \, dx \, dy = \frac{1}{b-a} \int_0^1 \int_a^b f\left(\frac{x+x}{2}, \lambda c + (1-\lambda)d\right) \, dx \, d\lambda \\ & \leq \frac{1}{2^\alpha(b-a)} \int_0^1 \int_a^b \left\{ f(x, c) + f(x, d) + m(2^\alpha - 1) \left[f\left(\frac{x}{m}, c\right) + f\left(\frac{x}{m}, d\right) \right] \right\} \, dx \, d\lambda \\ & = \frac{1}{2^\alpha(b-a)} \int_a^b Q(x, c, d) \, dx. \end{aligned}$$

Taking $x = ta + (1 - t)b$ for $0 \leq t \leq 1$ and utilizing the $(\alpha, m; P)$ -convexity of f figure out

$$\begin{aligned} & \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) \, dx \, dy \\ & \leq \frac{1}{2^\alpha(b-a)} \int_a^b \left\{ f(x, c) + f(x, d) + m(2^\alpha - 1) \left[f\left(\frac{x}{m}, c\right) + f\left(\frac{x}{m}, d\right) \right] \right\} \, dx \\ & \leq \frac{1}{2^{\alpha-1}} \int_0^1 \left[t^\alpha Q(a, c, d) + m(1 - t^\alpha) Q\left(\frac{b}{m}, c, d\right) \right] \, dt \\ & = \frac{1}{2^{\alpha-1}(\alpha+1)} \left[Q(a, c, d) + m\alpha Q\left(\frac{b}{m}, c, d\right) \right]. \end{aligned}$$

Similarly,

$$\begin{aligned} & \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) \, dx \, dy \\ & \leq \frac{2}{d-c} \int_c^d \int_0^1 \left[t^\alpha f(a, y) + m(1 - t^\alpha) f\left(\frac{b}{m}, y\right) \right] \, dt \, dy \\ & = \frac{2}{(\alpha+1)(d-c)} \int_c^d \left[f(a, y) + m\alpha f\left(\frac{b}{m}, y\right) \right] \, dy \\ & \leq \frac{1}{2^{\alpha-1}(\alpha+1)} \int_0^1 \left[Q(a, c, d) + m\alpha Q\left(\frac{b}{m}, c, d\right) \right] \, d\lambda \\ & = \frac{1}{2^{\alpha-1}(\alpha+1)} \left[Q(a, c, d) + m\alpha Q\left(\frac{b}{m}, c, d\right) \right]. \end{aligned}$$

The proof of Theorem 2.2 is completed. □

Corollary 2.2. *Under the conditions of Theorem 2.2,*

1. *if $m = 1$, then*

$$\begin{aligned} & \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) \, dx \, dy \\ & \leq \frac{1}{2(b-a)} \int_a^b [f(x, c) + f(x, d)] \, dx \\ & \quad + \frac{1}{(\alpha+1)(d-c)} \int_c^d [f(a, y) + \alpha f(b, y)] \, dy \\ & \leq \frac{2}{\alpha+1} \{f(a, c) + f(a, d) + \alpha[f(b, c) + f(b, d)]\}; \end{aligned}$$

2. *if $\alpha = 1$, then*

$$\begin{aligned} & \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) \, dx \, dy \\ & \leq \frac{1}{4(b-a)} \int_a^b Q(x, c, d) \, dx + \frac{1}{2(d-c)} \int_c^d \left[f(a, y) + mf\left(\frac{b}{m}, y\right) \right] \, dy \\ & \leq \frac{1}{2} \left\{ f(a, c) + f(a, d) + m \left[f\left(\frac{b}{m}, c\right) + f\left(\frac{b}{m}, d\right) \right] \right. \\ & \quad \left. + m \left[f\left(\frac{a}{m}, c\right) + f\left(\frac{a}{m}, d\right) + m \left(f\left(\frac{b}{m^2}, c\right) + f\left(\frac{b}{m^2}, d\right) \right) \right] \right\}. \end{aligned}$$

Theorem 2.3. *Let $f : [0, b] \times [c, d] \rightarrow \mathbb{R}_0$ be integrable for $0 \leq a < b$ and $c < d$ and let $f \in L_1([0, b] \times [c, d])$. If f is co-ordinated $(\alpha, m; P)$ -convex on $[0, b] \times [c, d]$ for $\alpha \in (0, 1]$ and $m \in (0, 1]$, then*

$$\begin{aligned} & f\left(\frac{m^2(a+b)}{2}, \frac{c+d}{2}\right) \\ & \leq \frac{1}{2^\alpha(b-a)} \int_a^b \left[f\left(m^2x, \frac{c+d}{2}\right) + m(2^\alpha - 1)f\left(mx, \frac{c+d}{2}\right) \right] \, dx \\ & \quad + \frac{1}{2^\alpha(d-c)} \int_c^d \left[f\left(\frac{m^2(a+b)}{2}, y\right) + m(2^\alpha - 1)f\left(\frac{m(a+b)}{2}, y\right) \right] \, dy \\ & \leq \frac{1}{2^{2\alpha-2}(b-a)(d-c)} \int_c^d \int_a^b [f(m^2x, y) \\ & \quad + 2m(2^\alpha - 1)f(mx, y) + m^2(2^\alpha - 1)^2f(x, y)] \, dx \, dy. \end{aligned}$$

Proof. Similar to the proof of (2.2), by the $(\alpha, m; P)$ -convexity of f , we see that

$$\begin{aligned} & f\left(\frac{m^2(a+b)}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2^{\alpha-1}} \int_0^1 \left[f\left(m^2[ta + (1-t)b], \frac{c+d}{2}\right) \right. \\ & \quad \left. + m(2^\alpha - 1)f\left(m[(1-t)a + tb], \frac{c+d}{2}\right) \right] \, dt \\ & = \frac{1}{2^{\alpha-1}(b-a)} \int_a^b \left[f\left(m^2x, \frac{c+d}{2}\right) + m(2^\alpha - 1)f\left(mx, \frac{c+d}{2}\right) \right] \, dx \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{2^{2\alpha-1}(b-a)} \int_0^1 \int_a^b \{f(m^2x, \lambda c + (1-\lambda)d) + f(m^2x, (1-\lambda)c + \lambda d) \\ &\quad + 2m(2^\alpha - 1)[f(mx, \lambda c + (1-\lambda)d) + f(mx, (1-\lambda)c + \lambda d)] \\ &\quad + m^2(2^\alpha - 1)^2[f(x, \lambda c + (1-\lambda)d) + f(x, (1-\lambda)c + \lambda d)]\} dx d\lambda \\ &= \frac{1}{2^{2\alpha-2}(b-a)(d-c)} \int_c^d \int_a^b [f(m^2x, y) + 2m(2^\alpha - 1)f(mx, y) \\ &\quad + m^2(2^\alpha - 1)^2f(x, y)] dx dy \end{aligned}$$

and

$$\begin{aligned} &f\left(\frac{m^2(a+b)}{2}, \frac{c+d}{2}\right) \\ &\leq \frac{1}{2^\alpha} \int_0^1 \left\{ f\left(\frac{m^2(a+b)}{2}, \lambda c + (1-\lambda)d\right) + f\left(\frac{m^2(a+b)}{2}, (1-\lambda)c + \lambda d\right) \right. \\ &\quad \left. + m(2^\alpha - 1) \left[f\left(\frac{m(a+b)}{2}, \lambda c + (1-\lambda)d\right) + f\left(\frac{m(a+b)}{2}, (1-\lambda)c + \lambda d\right) \right] \right\} d\lambda \\ &= \frac{1}{2^{\alpha-1}(d-c)} \int_c^d \left[f\left(\frac{m^2(a+b)}{2}, y\right) + m(2^\alpha - 1)f\left(\frac{m(a+b)}{2}, y\right) \right] dy \\ &\leq \frac{1}{2^{2\alpha-2}(d-c)} \int_c^d \int_0^1 \{f(m^2[ta + (1-t)b], y) + m(2^\alpha - 1)f(m[ta + (1-t)b], y) \\ &\quad + m(2^\alpha - 1)[f(m[ta + (1-t)b], y) + m(2^\alpha - 1)f(ta + (1-t)b, y)]\} dt dy \\ &= \frac{1}{2^{2\alpha-2}(b-a)(d-c)} \int_c^d \int_a^b [f(m^2x, y) + 2m(2^\alpha - 1)f(mx, y) \\ &\quad + m^2(2^\alpha - 1)^2f(x, y)] dx dy. \end{aligned}$$

Theorem 2.3 is thus proved. □

Corollary 2.3. *Let $f : [0, \frac{b}{m^2}] \times [c, d] \rightarrow \mathbb{R}_0$ be integrable for $0 \leq a < b, c < d$, and $m \in (0, 1]$ and $f \in L_1([0, \frac{b}{m^2}] \times [c, d])$. If f is co-ordinated $(\alpha, m; P)$ -convex on $[0, \frac{b}{m^2}] \times [c, d]$ for $\alpha \in (0, 1]$, then*

$$\begin{aligned} &f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ &\leq \frac{1}{2^\alpha(b-a)} \int_a^b \left[f\left(x, \frac{c+d}{2}\right) + m(2^\alpha - 1)f\left(\frac{x}{m}, \frac{c+d}{2}\right) \right] dx \\ &\quad + \frac{1}{2^\alpha(d-c)} \int_c^d \left[f\left(\frac{a+b}{2}, y\right) + m(2^\alpha - 1)f\left(\frac{a+b}{2m}, y\right) \right] dy \\ &\leq \frac{1}{2^{2\alpha-2}(b-a)(d-c)} \int_c^d \int_a^b \left[f(x, y) + 2m(2^\alpha - 1)f\left(\frac{x}{m}, y\right) \right. \\ &\quad \left. + m^2(2^\alpha - 1)^2f\left(\frac{x}{m^2}, y\right) \right] dx dy. \end{aligned}$$

Corollary 2.4. *Under the conditions of Theorems 2.2 and 2.3, if $m = 1$, then*

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy$$

$$\begin{aligned}
&\leq \frac{4}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) \, dx \, dy \\
&\leq \frac{2}{b-a} \int_a^b [f(x, c) + f(x, d)] \, dx + \frac{2}{d-c} \int_c^d [f(a, y) + f(b, y)] \, dy \\
&\leq 4[f(a, c) + f(a, d) + f(b, c) + f(b, d)].
\end{aligned}$$

Corollary 2.5. *Under the conditions of Theorem 2.3, if $\alpha = 1$, then*

$$\begin{aligned}
&f\left(\frac{m^2(a+b)}{2}, \frac{c+d}{2}\right) \\
&\leq \frac{1}{2(b-a)} \int_a^b \left[f\left(m^2x, \frac{c+d}{2}\right) + mf\left(mx, \frac{c+d}{2}\right) \right] dx \\
&\quad + \frac{1}{2(d-c)} \int_c^d \left[f\left(\frac{m^2(a+b)}{2}, y\right) + mf\left(\frac{m(a+b)}{2}, y\right) \right] dy \\
&\leq \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b [f(m^2x, y) + 2mf(mx, y) + m^2f(x, y)] \, dx \, dy.
\end{aligned}$$

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