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SOME NEW INTEGRAL INEQUALITIES OF HERMITE–HADAMARD TYPE FOR $(\alpha, m; P)$-CONVEX FUNCTIONS ON CO-ORDINATES*

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Abstract In the paper, the authors introduce the notion "$(\alpha, m; P)$-convex function on co-ordinates" and establish new integral inequalities of Hermite–Hadamard type for $(\alpha, m; P)$-convex functions on co-ordinates in a rectangle from the plane $\mathbb{R}_0 \times \mathbb{R}$.

Keywords $(\alpha, m; P)$-convex function on co-ordinates, integral inequality of Hermite–Hadamard type.

MSC(2010) 26A51, 26D15, 26D20, 26E60, 41A55.

1. Introduction

Let us recall some definitions of various convex functions.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, +\infty) \to \mathbb{R}$ is said to be convex on an interval I if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.2 ([5])**.** We say that a map $f : I \subseteq \mathbb{R} \to \mathbb{R}$ belongs to the class $P(I)$ if it is nonnegative and satisfies

$$f(\lambda x + (1-\lambda)y) \le f(x) + f(y)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.3 ([9])**.** For $f : [0, b] \to \mathbb{R}$ and $m \in (0, 1]$, if

$$f(\lambda x + m(1-\lambda)y) \le \lambda f(x) + m(1-\lambda)f(y)$$

is valid for all $x, y \in [0, b]$ and $\lambda \in [0, 1]$, then we say that f is an m-convex function on $[0, b]$.

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Definition 1.4 ([7]). For $f:[0,b]\to\mathbb{R}$ and $(\alpha,m)\in(0,1]\times(0,1]$, if

$$f(\lambda x+m(1-\lambda)y)\le\lambda^\alpha f(x)+m(1-\lambda^\alpha)f(y)$$

is valid for all $x,y\in[0,b]$ and $\lambda\in[0,1]$, then we say that f is an (α,m)-convex function on $[0,b]$.

Definition 1.5 ([3,4]). A function $f:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$ is said to be convex on co-ordinates in Δ for $a<b$ and $c<d$ if the partial functions

$$f_y:[a,b]\to\mathbb{R},\quad f_y(u)=f_y(u,y)\quad\text{and}\quad f_x:[c,d]\to\mathbb{R},\quad f_x(v)=f_x(x,v)$$

are convex for all $x\in(a,b)$ and $y\in(c,d)$.

Definition 1.6 ([3,4]). A function $f:\Delta=[a,b]\times[c,d]\subseteq\mathbb{R}^2\to\mathbb{R}$ is said to be convex on co-ordinates in Δ for $a<b$ and $c<d$ if the inequality

$$\begin{aligned}&f(tx+(1-t)z,\lambda y+(1-\lambda)w)\\ \le&t\lambda f(x,y)+t(1-\lambda)f(x,w)+(1-t)\lambda f(z,y)+(1-t)(1-\lambda)f(z,w)\end{aligned}$$

holds for all $t,\lambda\in[0,1]$ and $(x,y),(z,w)\in\Delta$.

Some integral inequalities of Hermite–Hadamard type for the above mentioned convex functions may be recited as follows.

Theorem 1.1 ([6]). *Let $f:\mathbb{R}_0=[0,\infty)\to\mathbb{R}$ be m-convex and $m\in(0,1]$. If $f\in L([a,b])$ for $0\le a<b<\infty$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\le\min\left\{\frac{f(a)+mf(b/m)}{2},\frac{mf(a/m)+f(b)}{2}\right\}.$$

Theorem 1.2 ([5, Theorem 3.1]). *Let $f\in P(I)$, $a,b\in I$ with $a<b$, and $f\in L([a,b])$. Then*

$$f\left(\frac{a+b}{2}\right)\le\frac{2}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\le2(f(a)+f(b)).$$

Both inequalities are the best possible.

Theorem 1.3 ([3,4, Theorem 2.2]). *Let $f:\Delta=[a,b]\times[c,d]\to\mathbb{R}$ be convex on co-ordinates in Δ for $a<b$ and $c<d$. Then*

$$\begin{aligned}&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ \le&\frac12\left[\frac{1}{b-a}\int_a^b f\left(x,\frac{c+d}{2}\right)\mathrm{d}\,x+\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},y\right)\mathrm{d}\,y\right]\\ \le&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,\mathrm{d}\,y\,\mathrm{d}\,x\\ \le&\frac14\left[\frac{1}{b-a}\left(\int_a^b f(x,c)\,\mathrm{d}\,x+\int_a^b f(x,d)\,\mathrm{d}\,x\right)+\frac{1}{d-c}\left(\int_c^d f(a,y)\,\mathrm{d}\,y\right.\right.\\ &\left.\left.+\int_c^d f(b,y)\,\mathrm{d}\,y\right)\right]\\ \le&\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}.\end{aligned}\tag{1.1}$$

For more information on new results in this topic, please refer to [1, 2, 8, 10] and plenty of references therein.

In this paper, we will introduce a new notion "$(\alpha, m; P)$-convex function on co-ordinates" and establish some new integral inequalities of Hermite–Hadamard type for $(\alpha, m; P)$-convex functions on co-ordinates in a rectangle from the plane $\mathbb{R}_0 \times \mathbb{R}$.

2. Integral inequalities of Hermite–Hadamard type

Motivated by Definitions 1.2, 1.4, and 1.6, we now introduce a new notion "$(\alpha, m; P)$-convex function on co-ordinates" as follows.

Definition 2.1. For $m, \alpha \in (0, 1]$, a mapping $f : [0, b] \times [c, d] \to \mathbb{R}_0$ is called co-ordinated $(\alpha, m; P)$-convex for $0 < b$ and $c < d$, if the inequality

$$\begin{aligned}&f(tx + m(1-t)z, \lambda y + (1-\lambda)w)\\ \le& t^\alpha[f(x, y) + f(x, w)] + m(1-t^\alpha)[f(z, y) + f(z, w)]\end{aligned}$$

holds for all $t, \lambda \in [0, 1]$ and $(x, y), (z, w) \in [0, b] \times [c, d]$.

Now we establish some new integral inequalities of Hermite–Hadamard type for $(\alpha, m; P)$-convex functions on co-ordinates in a rectangle from the plane $\mathbb{R}_0 \times \mathbb{R}$.

Theorem 2.1. *Let $f : [0, \frac{b}{m}] \times [c, d] \to \mathbb{R}_0$ be integrable for $0 \le a < b$, $c < d$, and $m \in (0, 1]$ and let $f \in L_1\big([0, \frac{b}{m}] \times [c, d]\big)$. If f is co-ordinated $(\alpha, m; P)$-convex on $[0, \frac{b}{m}] \times [c, d]$ for $\alpha \in (0, 1]$, then*

$$f\left(\frac{m(a+b)}{2}, \frac{c+d}{2}\right) \le \frac{\int_c^d \int_a^b [f(mx, y) + m(2^\alpha - 1)f(x, y)]\,\mathrm{d}\,x\,\mathrm{d}\,y}{2^{\alpha-1}(b-a)(d-c)}$$

and

$$\begin{aligned}&\frac{1}{(b-a)(d-c)}\int_c^d \int_a^b f(x, y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\ \le& \frac{1}{\alpha+1}\left[f(a, c) + f(a, d) + m\alpha f\left(\frac{b}{m}, c\right) + m\alpha f\left(\frac{b}{m}, d\right)\right].\end{aligned}$$

Proof. The $(\alpha, m; P)$-convexity of f gives

$$\begin{aligned}&f\left(\frac{m(a+b)}{2}, \frac{c+d}{2}\right)\\ =& f\left(\frac{m(ta + (1-t)b + (1-t)a + tb)}{2}, \frac{\lambda c + (1-\lambda)d + (1-\lambda)c + \lambda d}{2}\right)\\ \le& \frac{1}{2^\alpha}\big\{f\big(m[ta + (1-t)b], \lambda c + (1-\lambda)d\big) + f\big(m[ta + (1-t)b],\\ &(1-\lambda)c + \lambda d\big) + m(2^\alpha - 1)\big[f\big((1-t)a + tb, \lambda c + (1-\lambda)d\big)\\ &+ f\big((1-t)a + tb, (1-\lambda)c + \lambda d\big)\big]\big\}. \qquad (2.1)\end{aligned}$$

Putting $x = ta + (1-t)b$ and $y = \lambda c + (1-\lambda)d$ for $0 \le t, \lambda \le 1$ and integrating the inequality (2.1) on $[0,1]\times[0,1]$ over (t,λ) lead to

$$
\begin{aligned}
&f\left(\frac{m(a+b)}{2},\frac{c+d}{2}\right)\\
\le&\frac{1}{2^\alpha}\int_0^1\int_0^1\Big\{f\big(m[ta+(1-t)b],\lambda c+(1-\lambda)d\big)\\
&+f\big(m[ta+(1-t)b],(1-\lambda)c+\lambda d\big)+m(2^\alpha-1)\big[f\big((1-t)a+tb,\\
&\lambda c+(1-\lambda)d\big)+f\big((1-t)a+tb,(1-\lambda)c+\lambda d\big)\big]\Big\}\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\\
=&\frac{1}{2^{\alpha-1}(b-a)(d-c)}\int_c^d\int_a^b\big[f(mx,y)+m(2^\alpha-1)f(x,y)\big]\,\mathrm{d}\,x\,\mathrm{d}\,y. \qquad (2.2)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\
=&\int_0^1\int_0^1 f(ta+(1-t)b,\lambda c+(1-\lambda)d)\,\mathrm{d}\,t\,\mathrm{d}\,\lambda\\
\le&\int_0^1\int_0^1\left\{t^\alpha[f(a,c)+f(a,d)]+m(1-t^\alpha)\left[f\left(\frac{b}{m},c\right)+f\left(\frac{b}{m},d\right)\right]\right\}\mathrm{d}\,t\,\mathrm{d}\,\lambda\\
=&\frac{1}{\alpha+1}\left[f(a,c)+f(a,d)+m\alpha f\left(\frac{b}{m},c\right)+m\alpha f\left(\frac{b}{m},d\right)\right].
\end{aligned}
$$

Theorem 2.1 is thus proved. □

Corollary 2.1. *Under the assumptions of Theorem 2.1,*

1. *if* $m=1$, *then*

$$
\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\le\frac{2}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\
&\le\frac{2}{\alpha+1}\big[f(a,c)+f(a,d)+\alpha f(b,c)+\alpha f(b,d)\big];
\end{aligned}
$$

2. *if* $\alpha=1$, *then*

$$
f\left(\frac{m(a+b)}{2},\frac{c+d}{2}\right)\le\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b\big[f(mx,y)+mf(x,y)\big]\,\mathrm{d}\,x\,\mathrm{d}\,y
$$

and

$$
\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\
\le&\frac{1}{2}\left[f(a,c)+f(a,d)+mf\left(\frac{b}{m},c\right)+mf\left(\frac{b}{m},d\right)\right];
\end{aligned}
$$

3. *if* $m=\alpha=1$, *then*

$$
\begin{aligned}
f\left(\frac{a+b}{2},\frac{c+d}{2}\right)&\le\frac{2}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\
&\le f(a,c)+f(a,d)+f(b,c)+f(b,d).
\end{aligned}
$$

Theorem 2.2. *Let $f:[0,\frac{b}{m^2}]\times[c,d]\to\mathbb{R}_0$ be integrable for $0\le a<b$, $c<d$, and $m\in(0,1]$ and let $f\in L_1\big([0,\frac{b}{m^2}]\times[c,d]\big)$. If f is co-ordinated $(\alpha,m;P)$-convex on $[0,\frac{b}{m^2}]\times[c,d]$ for $\alpha\in(0,1]$, then*

$$
\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\
&\le \frac{1}{2^{\alpha+1}(b-a)}\int_a^b Q(x,c,d)\,\mathrm{d}\,x+\frac{1}{(\alpha+1)(d-c)}\int_c^d\left[f(a,y)+m\alpha f\left(\frac{b}{m},y\right)\right]\mathrm{d}\,y\\
&\le \frac{1}{2^{\alpha-1}(\alpha+1)}\left[Q(a,c,d)+m\alpha Q\left(\frac{b}{m},c,d\right)\right],
\end{aligned}
$$

where

$$
Q(x,c,d)=f(x,c)+f(x,d)+m(2^{\alpha}-1)\left[f\left(\frac{x}{m},c\right)+f\left(\frac{x}{m},d\right)\right]
$$

for $x\in\left[a,\frac{b}{m}\right]$.

Proof. Letting $y=\lambda c+(1-\lambda)d$ for $0\le\lambda\le1$ and empoying the $(\alpha,m;P)$-convexity of f reveal

$$
\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y=\frac{1}{b-a}\int_0^1\int_a^b f\left(\frac{x+x}{2},\lambda c+(1-\lambda)d\right)\mathrm{d}\,x\,\mathrm{d}\,\lambda\\
&\le \frac{1}{2^{\alpha}(b-a)}\int_0^1\int_a^b\left\{f(x,c)+f(x,d)+m(2^{\alpha}-1)\left[f\left(\frac{x}{m},c\right)+f\left(\frac{x}{m},d\right)\right]\right\}\mathrm{d}\,x\,\mathrm{d}\,\lambda\\
&=\frac{1}{2^{\alpha}(b-a)}\int_a^b Q(x,c,d)\,\mathrm{d}\,x.
\end{aligned}
$$

Taking $x=ta+(1-t)b$ for $0\le t\le1$ and utilizing the $(\alpha,m;P)$-convexity of f figure out

$$
\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\
&\le\frac{1}{2^{\alpha}(b-a)}\int_a^b\left\{f(x,c)+f(x,d)+m(2^{\alpha}-1)\left[f\left(\frac{x}{m},c\right)+f\left(\frac{x}{m},d\right)\right]\right\}\mathrm{d}\,x\\
&\le\frac{1}{2^{\alpha-1}}\int_0^1\left[t^{\alpha}Q(a,c,d)+m(1-t^{\alpha})Q\left(\frac{b}{m},c,d\right)\right]\mathrm{d}\,t\\
&=\frac{1}{2^{\alpha-1}(\alpha+1)}\left[Q(a,c,d)+m\alpha Q\left(\frac{b}{m},c,d\right)\right].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\
&\le\frac{2}{d-c}\int_c^d\int_0^1\left[t^{\alpha}f(a,y)+m(1-t^{\alpha})f\left(\frac{b}{m},y\right)\right]\mathrm{d}\,t\,\mathrm{d}\,y\\
&=\frac{2}{(\alpha+1)(d-c)}\int_c^d\left[f(a,y)+m\alpha f\left(\frac{b}{m},y\right)\right]\mathrm{d}\,y\\
&\le\frac{1}{2^{\alpha-1}(\alpha+1)}\int_0^1\left[Q(a,c,d)+m\alpha Q\left(\frac{b}{m},c,d\right)\right]\mathrm{d}\,\lambda\\
&=\frac{1}{2^{\alpha-1}(\alpha+1)}\left[Q(a,c,d)+m\alpha Q\left(\frac{b}{m},c,d\right)\right].
\end{aligned}
$$

The proof of Theorem 2.2 is completed. □

Corollary 2.2. *Under the conditions of Theorem 2.2,*

1. *if $m = 1$, then*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\
\le&\frac{1}{2(b-a)}\int_a^b\bigl[f(x,c)+f(x,d)\bigr]\,\mathrm{d}\,x\\
&+\frac{1}{(\alpha+1)(d-c)}\int_c^d\bigl[f(a,y)+\alpha f(b,y)\bigr]\,\mathrm{d}\,y\\
\le&\frac{2}{\alpha+1}\bigl\{f(a,c)+f(a,d)+\alpha\bigl[f(b,c)+f(b,d)\bigr]\bigr\};
\end{aligned}$$

2. *if $\alpha = 1$, then*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\
&\le\frac{1}{4(b-a)}\int_a^b Q(x,c,d)\,\mathrm{d}\,x+\frac{1}{2(d-c)}\int_c^d\biggl[f(a,y)+mf\biggl(\frac{b}{m},y\biggr)\biggr]\,\mathrm{d}\,y\\
&\le\frac{1}{2}\biggl\{f(a,c)+f(a,d)+m\biggl[f\biggl(\frac{b}{m},c\biggr)+f\biggl(\frac{b}{m},d\biggr)\biggr]\\
&\quad+m\biggl[f\biggl(\frac{a}{m},c\biggr)+f\biggl(\frac{a}{m},d\biggr)+m\biggl(f\biggl(\frac{b}{m^2},c\biggr)+f\biggl(\frac{b}{m^2},d\biggr)\biggr)\biggr]\biggr\}.
\end{aligned}$$

Theorem 2.3. *Let $f:[0,b]\times[c,d]\to\mathbb{R}_0$ be integrable for $0\le a<b$ and $c<d$ and let $f\in L_1([0,b]\times[c,d])$. If f is co-ordinated $(\alpha,m;P)$-convex on $[0,b]\times[c,d]$ for $\alpha\in(0,1]$ and $m\in(0,1]$, then*

$$\begin{aligned}
&f\biggl(\frac{m^2(a+b)}{2},\frac{c+d}{2}\biggr)\\
\le&\frac{1}{2^\alpha(b-a)}\int_a^b\biggl[f\biggl(m^2x,\frac{c+d}{2}\biggr)+m(2^\alpha-1)f\biggl(mx,\frac{c+d}{2}\biggr)\biggr]\,\mathrm{d}\,x\\
&+\frac{1}{2^\alpha(d-c)}\int_c^d\biggl[f\biggl(\frac{m^2(a+b)}{2},y\biggr)+m(2^\alpha-1)f\biggl(\frac{m(a+b)}{2},y\biggr)\biggr]\,\mathrm{d}\,y\\
\le&\frac{1}{2^{2\alpha-2}(b-a)(d-c)}\int_c^d\int_a^b\bigl[f(m^2x,y)\\
&+2m(2^\alpha-1)f(mx,y)+m^2(2^\alpha-1)^2f(x,y)\bigr]\,\mathrm{d}\,x\,\mathrm{d}\,y.
\end{aligned}$$

Proof. Similar to the proof of (2.2), by the $(\alpha,m;P)$-convexity of f, we see that

$$\begin{aligned}
f\biggl(\frac{m^2(a+b)}{2},\frac{c+d}{2}\biggr)&\le\frac{1}{2^{\alpha-1}}\int_0^1\biggl[f\biggl(m^2[ta+(1-t)b],\frac{c+d}{2}\biggr)\\
&+m(2^\alpha-1)f\biggl(m[(1-t)a+tb],\frac{c+d}{2}\biggr)\biggr]\,\mathrm{d}\,t\\
&=\frac{1}{2^{\alpha-1}(b-a)}\int_a^b\biggl[f\biggl(m^2x,\frac{c+d}{2}\biggr)+m(2^\alpha-1)f\biggl(mx,\frac{c+d}{2}\biggr)\biggr]\,\mathrm{d}\,x
\end{aligned}$$

$$\le \frac{1}{2^{2\alpha-1}(b-a)} \int_0^1 \int_a^b \big\{ f(m^2x, \lambda c + (1-\lambda)d) + f(m^2x, (1-\lambda)c + \lambda d)$$
$$+ 2m(2^\alpha - 1)\big[f(mx, \lambda c + (1-\lambda)d) + f(mx, (1-\lambda)c + \lambda d)\big]$$
$$+ m^2(2^\alpha - 1)^2\big[f(x, \lambda c + (1-\lambda)d) + f(x, (1-\lambda)c + \lambda d)\big]\big\} \,\mathrm{d}\,x \,\mathrm{d}\,\lambda$$
$$= \frac{1}{2^{2\alpha-2}(b-a)(d-c)} \int_c^d \int_a^b \big[f(m^2x, y) + 2m(2^\alpha - 1) f(mx, y)$$
$$+ m^2(2^\alpha - 1)^2 f(x, y)\big] \,\mathrm{d}\,x \,\mathrm{d}\,y$$

and

$$f\left(\frac{m^2(a+b)}{2}, \frac{c+d}{2}\right)$$
$$\le \frac{1}{2^\alpha} \int_0^1 \left\{ f\left(\frac{m^2(a+b)}{2}, \lambda c + (1-\lambda)d\right) + f\left(\frac{m^2(a+b)}{2}, (1-\lambda)c + \lambda d\right)\right.$$
$$\left. + m(2^\alpha - 1)\left[f\left(\frac{m(a+b)}{2}, \lambda c + (1-\lambda)d\right) + f\left(\frac{m(a+b)}{2}, (1-\lambda)c + \lambda d\right)\right]\right\} \,\mathrm{d}\,\lambda$$
$$= \frac{1}{2^{\alpha-1}(d-c)} \int_c^d \left[f\left(\frac{m^2(a+b)}{2}, y\right) + m(2^\alpha - 1) f\left(\frac{m(a+b)}{2}, y\right)\right] \mathrm{d}\,y$$
$$\le \frac{1}{2^{2\alpha-2}(d-c)} \int_c^d \int_0^1 \big\{ f(m^2[ta + (1-t)b], y) + m(2^\alpha - 1) f(m[ta + (1-t)b], y)$$
$$+ m(2^\alpha - 1)\big[f(m[ta + (1-t)b], y) + m(2^\alpha - 1) f(ta + (1-t)b, y)\big]\big\} \,\mathrm{d}\,t \,\mathrm{d}\,y$$
$$= \frac{1}{2^{2\alpha-2}(b-a)(d-c)} \int_c^d \int_a^b \big[f(m^2x, y) + 2m(2^\alpha - 1) f(mx, y)$$
$$+ m^2(2^\alpha - 1)^2 f(x, y)\big] \,\mathrm{d}\,x \,\mathrm{d}\,y.$$

Theorem 2.3 is thus proved. □

Corollary 2.3. *Let* $f : \left[0, \frac{b}{m^2}\right] \times [c, d] \to \mathbb{R}_0$ *be integrable for* $0 \le a < b$, $c < d$, *and* $m \in (0, 1]$ *and* $f \in L_1\left(\left[0, \frac{b}{m^2}\right] \times [c, d]\right)$. *If* f *is co-ordinated* $(\alpha, m; P)$*-convex on* $\left[0, \frac{b}{m^2}\right] \times [c, d]$ *for* $\alpha \in (0, 1]$, *then*

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$
$$\le \frac{1}{2^\alpha(b-a)} \int_a^b \left[f\left(x, \frac{c+d}{2}\right) + m(2^\alpha - 1) f\left(\frac{x}{m}, \frac{c+d}{2}\right)\right] \mathrm{d}\,x$$
$$+ \frac{1}{2^\alpha(d-c)} \int_c^d \left[f\left(\frac{a+b}{2}, y\right) + m(2^\alpha - 1) f\left(\frac{a+b}{2m}, y\right)\right] \mathrm{d}\,y$$
$$\le \frac{1}{2^{2\alpha-2}(b-a)(d-c)} \int_c^d \int_a^b \left[f(x, y) + 2m(2^\alpha - 1) f\left(\frac{x}{m}, y\right)\right.$$
$$\left. + m^2(2^\alpha - 1)^2 f\left(\frac{x}{m^2}, y\right)\right] \mathrm{d}\,x \,\mathrm{d}\,y.$$

Corollary 2.4. *Under the conditions of Theorems 2.2 and 2.3, if* $m = 1$, *then*

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \le \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) \mathrm{d}\,x + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) \mathrm{d}\,y$$

$$
\begin{aligned}
&\le \frac{4}{(b-a)(d-c)}\int_c^d\int_a^b f(x,y)\,\mathrm{d}\,x\,\mathrm{d}\,y\\
&\le \frac{2}{b-a}\int_a^b\bigl[f(x,c)+f(x,d)\bigr]\,\mathrm{d}\,x+\frac{2}{d-c}\int_c^d\bigl[f(a,y)+f(b,y)\bigr]\,\mathrm{d}\,y\\
&\le 4[f(a,c)+f(a,d)+f(b,c)+f(b,d)].
\end{aligned}
$$

Corollary 2.5. *Under the conditions of Theorem 2.3, if $\alpha = 1$, then*

$$
\begin{aligned}
&f\left(\frac{m^2(a+b)}{2},\frac{c+d}{2}\right)\\
\le&\frac{1}{2(b-a)}\int_a^b\left[f\left(m^2x,\frac{c+d}{2}\right)+mf\left(mx,\frac{c+d}{2}\right)\right]\mathrm{d}\,x\\
&+\frac{1}{2(d-c)}\int_c^d\left[f\left(\frac{m^2(a+b)}{2},y\right)+mf\left(\frac{m(a+b)}{2},y\right)\right]\mathrm{d}\,y\\
\le&\frac{1}{(b-a)(d-c)}\int_c^d\int_a^b\bigl[f(m^2x,y)+2mf(mx,y)+m^2f(x,y)\bigr]\,\mathrm{d}\,x\,\mathrm{d}\,y.
\end{aligned}
$$

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