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MATHEMATICAL MODELING TO SIMULATE THE MOVEMENT OF CONTAMINANTS IN GROUNDWATER

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Abstract The objectives of this paper are twofold. Firstly, we formulate a system of partial differential equations that models the contamination of groundwater due to migration of dissolved contaminants through unsaturated to saturated zone. A closed form solution using the singular perturbation techniques for the flow and solute transport equations in the unsaturated zone is obtained. Indeed, the solution can be used as a tool to verify the accuracy of numerical models of water flow and solute transport. The second part of this paper, deals with how the water level in a water reserve drops due to pumping water out of a well that is some distance away.

Keywords Approximate solutions, singular perturbation method, solute transport, Talbot's algorithm, Bromwich contour, Laplace transform, Inverse Laplace transform.

MSC(2010) 26A33, 35F25, 35C10.

1. Introduction

Underground water is a term used to denote all the water sites found beneath the surface of the ground. Aquifers may be classified as confined or unconfined, depending upon the absence or presence of a water table. A confined aquifer is one which is bounded from above and from below by impervious formations. Unconfined aquifer is one in which a water table serves as its upper boundary. In an aquifer, flow takes place through a complex network of interconnected pores or openings. When dealing with flow in an aquifer, we overlook the microscopic flow patterns inside the individual pores. One of the main characteristics of underground water motion is that it occurs at very slow velocities, because large specific surface area through which this motion takes place; However, large quantities of water, or contamination are transported. Analysis and prediction of solute transport in hydrogeological systems generally involve the use of some form of the advection-dispersion equation. Dispersivity (i.e., the spreading of solute carried by a fluid flowing in a porous medium) has traditionally been considered a constant for the entire medium (see, [5]). De Marsily [11] points out that the breakthrough curves monitored at different distances from the source cannot, in general be matched optimally with a single dispersivity value, and that optimal dispersivities seem to

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increase with distance of the observation point from the source. The movement of water and solute through the unsaturated zone has been of importance in traditional applications of groundwater hydrology, soil physics, and agronomy. In recent years, the need to understand the behavior of hazardous waste and toxic chemicals in soils has resulted in a renewed interest in this subject. One of the primary concerns is that dissolved contaminants may migrate through the unsaturated zone, reach the saturated zone, and contaminate the groundwater. Additionally, movement of solutes or pesticides should be identified before their application in agriculture, to prevent any pollution. Therefore, mathematical models are useful tools for a first assessment of the expected concentration of contaminants in groundwater, which may enable identification of the pesticides with the highest contamination potential. The search for solutions to model water flow and solute transport continues to be of scientific interest. Typically, water flow and solute transport in unsaturated soils result in transient phenomena, making it a challenging problem. The nature of soil hydraulic conductivity renders the governing flow equation nonlinear. In recent years, several analytical methods were developed to simulate water movement and solute transport in the unsaturated zone, for more details see [3,4,6,7,9,22–24]. Although much progress has already been made in solving the problems of transient water flow in unsaturated and saturated porous flow media, many new developments have been made by numerical investigations in recent years [17–20]. A large number of numerical solution are generally approached by a finite-difference approximation [8], or by a finite volume element method [16]. The author in [13] addresses critical issues that describe the transient transfer of groundwater flow in saturated fractured rock. The work in [14] discusses the analysis of solution of groundwater flow in inclined porous media. In [12], it is shown that groundwater reservoirs can only be linear if their thickness can be assumed independent of the hydraulic head. A mathematical model based on the transport diffusion is presented in [1]. The classical Darcy law is generalized by regarding the water flow as a function of a non-integer order derivative of the piezometric head, and a numerical solutions of time-fractional groundwater flow in obtained in [2].

The first objective of this paper is to capitalize on the features of the analytical flow model and extend its use to simulate contaminant movement in soils to that a complete, closed form approximate solution for solute transport in the unsaturated zone is achieved using singular perturbation techniques as well as using the Laplace transform method. The second objective in this paper, which will be discussed in section 6, is to study the effect of pumping water out of a well some distance away from a water reservoir. The presented methods are very efficient in obtaining closed form analytic solutions of the problems considered.

The organization of the paper is as follows. In section 2, the solute problem is stated mathematically. In section 3, the singular perturbation technique is employed to derive a closed form approximate solution of the solute problem. In section 4, we apply the Laplace transform technique to arrive at a closed form solution given by a convolution integral of the solute problem. Numerical simulations of the solution are discussed in section 5. The remaining sections, sections 6, 7 and 8, are concerned with the problem of the effect of pumping water out of a well which is some distance away from the water reserve. In section 6, the problem is stated mathematically and a closed form solution (in the Laplace domain) is derived. In section 7, we consider two approaches to invert the solution to the time domain, and in section 8, we consider three examples to simulate the obtained solution. Finally, we conclude the paper by concluding remarks.

2. Advection-dispersion equation for solute transport

The traditional approach to describing solute transport through soil is based on the advection-dispersion equation. In one dimension, the theoretical basis for modeling the liquid phase water movement in unsaturated porous media, can be described by a combination of the Darcys law, and the equation of continuity, (see, [11], Chapter 10]) as:

$$\frac{\partial}{\partial t}c(x,t) = D\frac{\partial^2}{\partial x^2}c(x,t) - v\frac{\partial}{\partial x}c(x,t) - \lambda c(x,t), \tag{2.1}$$

where t is the time, x is the horizontal distance taken zero at the soil center, and measured positive to the right of the soil center; $c(x,t)$ is the solute concentration (mass of solute over volume of solute) at time t, distance x; D is the soil-water diffusivity; v is the average velocity, and λ is the decay coefficient (1/time). The contamination in groundwater can be calculated by means of equation (2.1). We consider the behavior of contamination in a saturated zone with zero initial concentration, i.e.,

$$c(x,0) = 0, \quad x > 0, \tag{2.2}$$

and at $x = 0$ a periodic inflation rate is prescribed as:

$$c(0,t) = c_0(1 + \sin \omega t), \qquad t > 0, \tag{2.3}$$

where c_0 is the constant concentration at the entrance of the medium ($x = 0$) prescribed from $t = 0$.

3. Singular Perturbation Technique

In this section, we employ the singular perturbation technique to obtain an approximate solution in closed form for the solute concentration $c(x,t)$ satisfying (2.1) subject to (2.2) and (2.3).

We first scale the problem by selecting characteristic values for the dependent and independent variables. Consequently, we define dimensionless variables by

$$\bar{t} = \frac{t}{\lambda^{-1}}, \quad \bar{x} = \frac{x}{v/\lambda}, \quad \bar{c} = \frac{c}{c_0}.$$

We shall abuse notation and keep the symbols t, x and c to denote $\bar{t}$, $\bar{x}$ and $\bar{c}$. The reader should be aware that these are scaled. In terms of the scaled variables, equations (2.1)-(2.3) become

$$\frac{\partial}{\partial t}c(x,t) = \varepsilon\frac{\partial^2}{\partial x^2}c(x,t) - \frac{\partial}{\partial x}c(x,t) - c(x,t), \tag{3.1}$$

$$c(x,0) = 0, \quad x > 0, \tag{3.2}$$

$$c(0,t) = 1 + \sin \Omega t, \quad t > 0, \tag{3.3}$$

where $\Omega = \omega/\lambda$ and assumed to be $O(1)$ and $\varepsilon = \lambda D/v^2 \ll 1$.

Since equation (3.1) contains a small parameter ε, singular perturbation method would be suitable for obtaining an approximate solution for the problem. First, we consider the unperturbed problem, i.e., equation (3.1) with $\varepsilon = 0$,

$$\frac{\partial}{\partial t}c^o(x,t) + \frac{\partial}{\partial x}c^o(x,t) = -c^o(x,t), \tag{3.4}$$

where $c^o(x,t)$ is the *outer* solution of the unperturbed problem. Using the method of characteristics, we find that the general solution of (3.4) is

$$c(x,t) = f(t-x)e^{-x}, \tag{3.5}$$

where $f(\cdot)$ is an arbitrary function chosen so as to satisfy the initial and boundary conditions (3.2) and (3.3). To satisfy (3.2) and (3.3), we choose $f(\cdot)$ to be

$$f(\xi) = \begin{cases} 0, & \xi < 0, \\ 1 + \sin(\Omega\xi) \equiv c_b(\xi), & \xi > 0. \end{cases}$$

Thus, the outer solution is given by

$$c^o(x,t) = \begin{cases} 0, & x > t, \\ c_b(t-x)e^{-x}, & x < t. \end{cases} \tag{3.6}$$

We should remark that along the line $t = x$ the solution (3.6) is not continuous. We will focus in along $t = x$, where we will put a boundary layer and solve the *inner* problem whose solution is denoted by $c^i(x,t)$. To determine the thickness of the boundary layer, make the transformation $\tau = \frac{t-x}{\varepsilon^\nu}$, $\eta = x$ and let $c^i(\eta,\tau) = c(x,t)$ and choose $\nu > 0$ so that we obtain an $O(1)$ equation for $c^i(\eta,\tau)$. In the η-τ coordinate system, equation (3.1) becomes

$$\frac{\partial^2}{\partial\tau^2}c^i(\eta,\tau) - 2\varepsilon^\nu \frac{\partial^2}{\partial\eta\partial\tau}c^i(\eta,\tau) + \varepsilon^{2\nu}\frac{\partial^2}{\partial\eta\partial\eta}c^i(\eta,\tau) - \varepsilon^{2\nu-1}\frac{\partial}{\partial\eta}c^i(\eta,\tau) - \varepsilon^{2\nu-1}c^i(\eta,\tau) = 0$$

from which we get an $O(1)$ equation if we set $2\nu - 1 = 0$ or $\nu = 1/2$. Thus, the thickness of the boundary layer is $\sqrt{\varepsilon}$ and the transformation is $\tau = \frac{t-x}{\sqrt{\varepsilon}}$ and $\eta = x$. The order 1 equation for $c^i(\eta,\tau)$ is

$$\frac{\partial^2}{\partial\tau^2}c^i(\eta,\tau) - \frac{\partial}{\partial\eta}c^i(\eta,\tau) - c^i(\eta,\tau) = 0, \tag{3.7}$$

which can be transformed via $c^i(\eta,\tau) = u(\eta,\tau)e^{-\eta}$ into the diffusion equation

$$\frac{\partial^2}{\partial\tau^2}u(\eta,\tau) - \frac{\partial}{\partial\eta}u(\eta,\tau) = 0, \tag{3.8}$$

whose general solution is

$$u(\eta,\tau) = A + B\,\mathrm{erf}\left(\frac{\tau}{2\sqrt{\eta}}\right).$$

The general solution for the inner solution $c^i(\eta,\tau)$ is then

$$c^i(\eta,\tau) = \left[A + B\,\mathrm{erf}\left(\frac{\tau}{2\sqrt{\eta}}\right)\right]e^{-\eta},$$

where A and B are constants chosen such that the inner solution $c^i(x,t)$ decays to zero (at least exponentially) as $|\tau| \to \infty$. It is easy to see that for $A = 1/2$ and $B = 1/2$, $c^i(\eta,\tau) \to 0$ exponentially as $|\tau| \to \infty$. Thus, the inner solution is

$$c^i(x,t) = \frac{1}{2}\left\{1 + \operatorname{erf}\left(\frac{t-x}{\sqrt{4\,x\varepsilon}}\right)\right\} e^{-x}. \tag{3.9}$$

Finally, to obtain a composite expansion that is uniformly valid in the domain, we add up the outer and the inner solutions

$$c^o(x,t) + c^i(x,t) = \begin{cases} \frac{1}{2}\left\{1 + \operatorname{erf}\left(\frac{t-x}{\sqrt{4x\varepsilon}}\right)\right\} e^{-x}, & x > t, \\ \frac{1}{2}\left\{1 + \operatorname{erf}\left(\frac{t-x}{\sqrt{4x\varepsilon}}\right)\right\} e^{-x} + c_b(t-x)e^{-x}, & x < t. \end{cases} \tag{3.10}$$

Hence, by subtracting the common limit in the overlap domain, which has value 0 for $x > t$, and e^{-x} for $x < t$, we have the solution

$$c(x,t) = \begin{cases} \frac{1}{2}\left\{1 + \operatorname{erf}\left(\frac{t-x}{\sqrt{4x\varepsilon}}\right)\right\} e^{-x}, & x \geq t, \\ \frac{1}{2}\left\{1 + \operatorname{erf}\left(\frac{t-x}{\sqrt{4x\varepsilon}}\right)\right\} e^{-x} + c_b(t-x)e^{-x} - e^{-x}, & x \leq t. \end{cases} \tag{3.11}$$

Therefore, (3.11) gives a closed form solution for the concentration $c(x,t)$ valid for $x > 0,\ t > 0$, where again x, t and c are the scaled version of the original variables.

4. Laplace Transform Technique

In this section, we use the Laplace transform to find the solution to (3.1) subject to (3.2) and (3.3). Let $\hat{c}(x,s)$ be the Laplace transform of $c(x,t)$, i.e.,

$$\hat{c}(x,s) = \int_0^\infty c(x,t)e^{-st}\,dt.$$

Then equation (3.1), in the Laplace domain, becomes

$$\varepsilon \hat{c}_{xx}(x,s) - \hat{c}_x(x,s) - (1+s)\,\hat{c}(x,s) = 0, \tag{4.1}$$

whose general solution is given by

$$\hat{c}(x,s) = A(s)e^{r\,x} + B(s)e^{\bar{r}\,x}, \tag{4.2}$$

where

$$r = \frac{1 - \sqrt{1 + 4\,\varepsilon\,(1+s)}}{2\,\varepsilon}, \qquad \bar{r} = \frac{1 + \sqrt{1 + 4\,\varepsilon\,(1+s)}}{2\,\varepsilon}.$$

In addition to the boundary condition at $x = 0$, equation (3.3), we have the boundary condition at $x = \infty$, $c(\infty,t) = 0$. In the Laplace domain, this condition becomes $\hat{c}(\infty,s) = 0$. Applying this condition and the initial condition (3.2) to (4.2), we obtain $B(s) = 0$, as $\bar{r} > 0$, and $A(s) = \hat{c}(0,s) = \mathcal{L}(c(0,t)) = \left(\frac{1}{s} + \frac{\Omega}{s^2+\Omega^2}\right)$. So, $\hat{c}(x,s)$ reduces to

$$\hat{c}(x,s) = \left(\frac{1}{s} + \frac{\Omega}{s^2+\Omega^2}\right) e^{r\,x}. \tag{4.3}$$

By taking the inverse Laplace transform, we obtain $c(x,t)$ given by

$$c(x,t) = c(0,t) \star \mathcal{L}^{-1}(e^{r\,x}),$$

where '$\star$' denotes convolution. The inverse Laplace transform of $e^{r\,x}$ is given by

$$\mathcal{L}^{-1}(e^{r\,x}) = \frac{x\,e^{-t}e^{-\frac{(x-t)^2}{4\varepsilon t}}}{2\sqrt{\pi\,\varepsilon}\,t^{3/2}}.$$

Finally, we have the concentration $c(x,t)$ given by

$$c(x,t) = \int_0^t [1+\sin(\Omega(t-\tau))]\frac{x\,e^{-\tau}e^{-\frac{(x-\tau)^2}{4\varepsilon\tau}}}{2\sqrt{\pi\,\varepsilon}\,\tau^{3/2}}\,d\tau\,. \tag{4.4}$$

We note that the integral in (4.4) cannot be done analytically and a numerical integration scheme is needed. In fact, we used the software package *Mathematica* to carry out the numerical integration.

5. Numerical Implementation

In the this section, the perturbation solution obtained in section 3 is tested, The input requirement for the perturbed simulation includes, the diffusivity D, the average velocity v, the decaying coefficient λ. For the sake of illustration, we choose $D = 0.05$ m^2/yr, $v = 1$ m/r, and $\lambda = 9.29$ yr^{-1}. The boundary condition is $c(0,t) = 1 + \sin 8t$, and thus $\Omega = 0.861, \varepsilon = 0.4645$. The concentration $c(x,t)$ in equation (3.11) accounts for the pulse entering the coupled unsaturated/saturated system as $t = 0$. Figure 1 shows the concentration in the saturated zone as a function of time $(0 < t < 30)$ and distance $0 < x < 5$, calculated with equation (3.11). The results in Figure 1 indicate that the pulse input concentration decreases with increasing distance x (as expected). Also, it shows that at any point x, the variation of concentration with time is sinusoidal, similar to that at the source $x = 0$, but with smaller amplitude, and the maximum and minimum concentration do not change. Figure 2 displays similar confirming results obtained using the Laplace transform method.

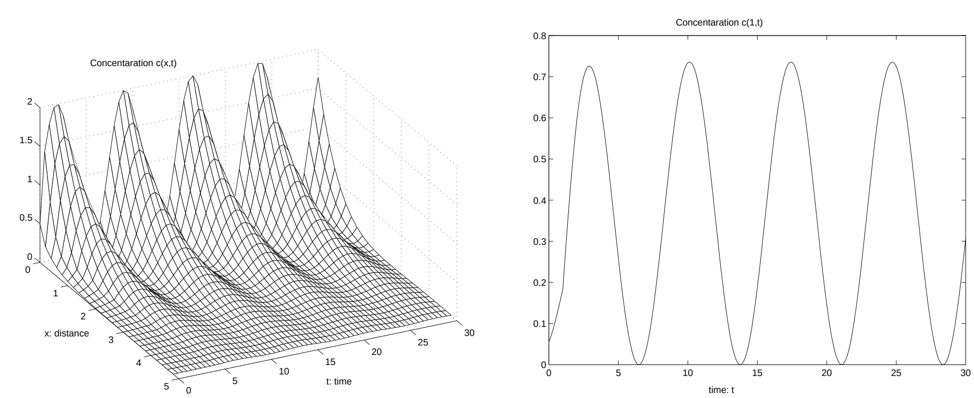

Figure 1. Left: Concentration of solute for $0 \leq t \leq 30$ and $0 \leq x \leq 5$, Right: Concentration of solute for $0 \leq t \leq 30$ at $x = 1$, using (3.11).

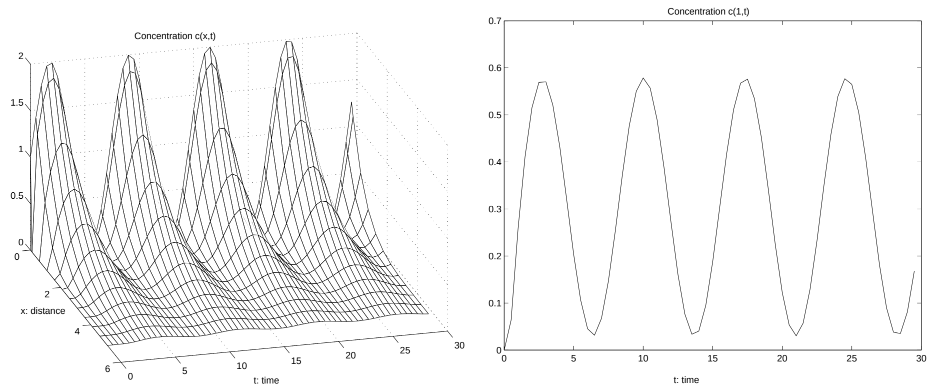

Figure 2. Left: Concentration of solute for $0 \le t \le 30$ and $0 \le x \le 5$, Right: Concentration of solute for $0 \le t \le 30$ at $x = 1$, using (4.4).

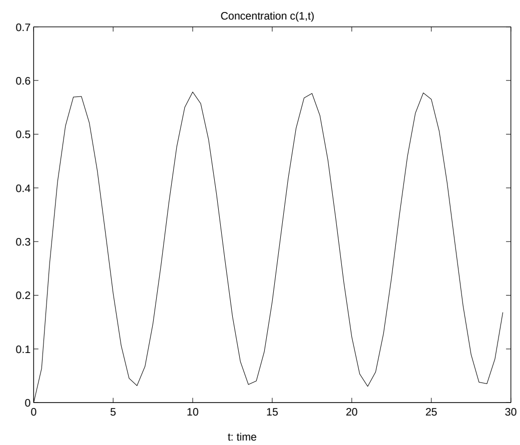

Figure 3. Concentration of solute for $0 \le t \le 30$ at $x = 1$, using (4.4).

6. Modeling Movement of Underground Water

This section deals with the effect of pumping water out of a well on the water level in a water reserve located some distance away from the well. The study of such phenomena is very important as it gives insights on effect of pumping rate and relative distance of the well on the water level in the water deposit. We will study a particular situation here in our region, where the well is in south Jordan (Desah) and the reservoir is in Saudi Arabia. It is assumed that the well is connected to the reservoir by a confined aquifer. The storativity of a confined aquifer is defined as the volume of water released from the reservoir, or added to it, per unit of horizontal area of the aquifer and per unit decline or rise of piezoroetric head. In fact, in that area the water deposit is not renewable. A cross section of the situation is shown in Figure 4. The aquifer has thickness ℓ, conductivity κ, and storativity μ. The water deposit has width $2b$. The well is a distance d from the center of the water reserve. The initial height of the head before pumping is h_0 and $h(x,t)$ is the height at any point x for $t > 0$. $H(t)$ is the height of the water in the water deposit. The pumping rate (per unit of length of the well) is a constant Q. In the model, it is assume that

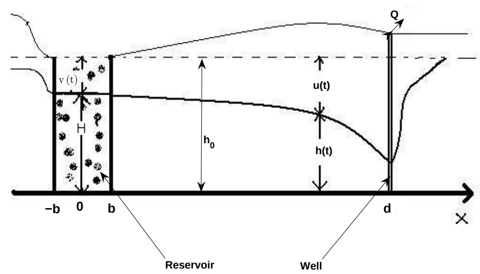

Figure 4. A cross section of the reservoir and well.

the well is narrow (very small) relative to the distance d, it may be treated as a *point sink* which is modeled mathematically by a delta function. The delta (or unit impulse) function at $x = a$ denoted by $\delta(x - a)$ is regarding as a distribution is the sense that for any function $f(x)$,

$$\int f(x)\delta(x-a)\,dx = f(a). \tag{6.1}$$

Also, $\delta(x - a)$ is regarded as the derivative of a unit step function (Heaviside function) $U(x - a)$, where

$$U(x-a) = \begin{cases} 1, & x > a, \\ 0, & x < a. \end{cases}$$

Our objective in this section and the next is to formulate a model for $H(t)$. To this end, we first determine the height of head $h(x, t)$. For more information see [5]. Using Flick's law, the governing partial differential equation which models this phenomena is ([11], Chapter 5)

$$\mu h_t(x,t) = \kappa\ell h_{xx}(x,t) - Q(x), \tag{6.2}$$

where $h(x, t), Q(x)$ are the height of the hydraulic head of the water and the pumping rate at position x, respectively. Since the well is assumed to be very narrow, the pumping rate $Q(x)$ can be modeled by

$$Q(x) = Q\,\delta(x-d),$$

where Q is a constant pumping rate. Equation (6.2) then becomes

$$\mu h_t(x,t) = \kappa\ell h_{xx}(x,t) - Q\delta(x-d). \tag{6.3}$$

The initial and boundary conditions are

$$h(x,0) = h_0 = H(0), \tag{6.4}$$

$$h(b,t) = H(t), \qquad h(\pm\infty) = h_0, \tag{6.5}$$

where $H(t)$ is given by the flux condition

$$H'(t) = \frac{\kappa\ell}{2b}[h_x(b,t) - h_x(-b,t)]. \tag{6.6}$$

The differential equation (6.3) along with the initial and boundary conditions (6.5)-(6.6) constitute the boundary value problem for the unknown function $h(x,t)$ which models the phenomena under consideration.

Let $u(x,t) = h_0 - h(x,t)$ be the drawdown in the aquifer and define $v(t) = h_0 - H(t)$. Equations (6.3) and (6.5)-(6.6), in terms of u and v, become

$$u_t(x,t) = \alpha u_{xx}(x,t) + \frac{1}{\mu} Q\,\delta(x-d), \qquad \alpha = \frac{\kappa\ell}{\mu}, \tag{6.7}$$

$$u(x,0) = 0,\ v(0) = 0, \tag{6.8}$$

$$u(b,t) = v(t),\ u(\pm\infty,t) = 0, \tag{6.9}$$

$$v'(t) = \frac{\kappa\ell}{2b}[u_x(b,t) - u_x(-b,t)]. \tag{6.10}$$

To solve (6.7) subject to (6.8) and (6.9), we use the Laplace transform. Let $\widehat{u}(x,s)$ and $\widehat{v}(s)$ be the Laplace transforms of $u(x,t)$ and $v(t)$. Then, in the Laplace domain, equations (6.7)-(6.10) become

$$\widehat{u}_{xx}(x,s) - \frac{s}{\alpha}\widehat{u}(x,s) = -\frac{1}{\mu\alpha} Q\,\delta(x-d), \tag{6.11}$$

$$\widehat{u}(b,s) = \widehat{v}(s), \quad \widehat{u}(\pm\infty,s) = 0, \tag{6.12}$$

$$\widehat{v}(s) = \frac{\kappa\ell}{2b\,s}[\widehat{u}_x(b,s) - \widehat{u}_x(-b,s)]. \tag{6.13}$$

The general solution of (6.11) is

$$\widehat{u}(x,s) = c_1 e^{-x\sqrt{s/\alpha}} + c_2 e^{x\sqrt{s/\alpha}} + \frac{Q}{2\mu\sqrt{\alpha\,s}}\left[e^{-(x-d)\sqrt{s/\alpha}} - e^{(x-d)\sqrt{s/\alpha}}\right] U(x-d). \tag{6.14}$$

Now applying, the boundary conditions at $x = b$ and $x = +\infty$, see equation (6.12), we obtain the following values for c_1 and c_2:

$$c_1 = \left(\widehat{v}(s) - \frac{Q}{2\mu\sqrt{\alpha\,s}} e^{(b-d)\sqrt{s/\alpha}}\right) e^{b\sqrt{s/\alpha}},$$

$$c_2 = \frac{Q}{2\mu\sqrt{\alpha\,s}} e^{-d\sqrt{s/\alpha}},$$

which when substituted in (6.14) give the solution for $\widehat{u}(x,s)$ valid for $x \geq b$,

$$\begin{aligned}\widehat{u}(x,s) =& \widehat{v}(s) e^{-(x-b)\sqrt{s/\alpha}} + \frac{Q}{2\mu\sqrt{\alpha\,s}}\left(e^{(x-d)\sqrt{s/\alpha}} - e^{(2b-x-d)\sqrt{s/\alpha}}\right) \\ &+ \frac{Q}{2\mu\sqrt{\alpha\,s}}\left[e^{-(x-d)\sqrt{s/\alpha}} - e^{(x-d)\sqrt{s/\alpha}}\right] U(x-d).\end{aligned} \tag{6.15}$$

Next, we find a solution for $\widehat{u}(x,s)$ valid for $x \leq -b$. We solve (6.11) with zero right hand side and using the boundary conditions $\widehat{u}(-b,s) = \widehat{v}(s)$ and $\widehat{u}(-\infty,s) = 0$, we obtain the solution

$$\widehat{u}(x,s) = \widehat{v}(s) e^{(x+b)\sqrt{s/\alpha}}. \tag{6.16}$$

Now the solution for $\widehat{v}(s)$ can be obtained by substituting solutions (6.15) and (6.16) in (6.13). We obtain

$$\widehat{v}(s) = \frac{Q}{2}\,\frac{e^{(b-d)\sqrt{s/\alpha}}}{bs + \kappa\ell\sqrt{s/\alpha}}. \tag{6.17}$$

Finally, $v(t)$ is given as the inverse Laplace transform of (6.17), from which the function $H(t)$ is obtained via

$$H(t) = h_0 - v(t).$$

Unfortunately, the inverse Laplace transform of $\widehat{v}(s)$ is not easy to find. In fact,in this case, it is impossible to obtain $v(t)$ explicitly using any of the available Laplace transform tables. We shall address the problem numerically in the next section.

7. Numerical approach to find $v(t)$

In this section we address the problem of finding the inverse Laplace transform of $\widehat{v}(s)$ given by (6.17). Two approaches are proposed. The first is to express the solution $v(t)$ as a convolution integral of two explicit time functions. The integral, as in turns out, is impossible to evaluate analytically, and hence has to be numerically approximated. The second approach is to represent the inverse Laplace transform in terms of Bromwich contour integral [10].

7.1. A closed integral form for the drawdown $v(t)$

For convenience, we rewrite $\widehat{v}(s)$ in (6.17) as

$$\widehat{v}(s) = \frac{Q}{2b} \frac{\frac{e^{\frac{b-d}{\sqrt{\alpha}}\sqrt{s}}}{\sqrt{s}}}{\frac{\kappa\ell}{b\sqrt{\alpha}} + \sqrt{s}} \equiv F_1(s)\, F_2(s),$$

with

$$F_1(s) = \frac{Q}{2b} \frac{e^{\frac{b-d}{\sqrt{\alpha}}\sqrt{s}}}{\sqrt{s}}, \quad F_2(s) = \frac{1}{\frac{\kappa\ell}{b\sqrt{\alpha}} + \sqrt{s}}.$$

Then by the convolution theorem, we have

$$v(t) = \mathcal{L}^{-1}\{F_1(s)\} \star \mathcal{L}^{-1}\{F_2(s)\},$$

where "$\star$" denotes convolution defined by

$$f(t) \star g(t) = \int_0^t f(t) g(t-\tau)\, d\tau.$$

It can be verified that

$$f_1(t) = \mathcal{L}^{-1}\{F_1(s)\} = bQ \frac{e^{-\frac{(b-d)^2}{4\alpha t}}}{\sqrt{4\pi\, t}}, \tag{7.1}$$

$$f_2(t) = \mathcal{L}^{-1}\{F_2(s)\} = \frac{1}{\sqrt{\pi}\sqrt{t}} - \frac{\kappa\ell}{b\sqrt{\alpha}} e^{\left(\frac{\kappa\ell}{b\sqrt{\alpha}}\right)^2 t} \operatorname{erfc}\left(\frac{\kappa\ell}{b\sqrt{\alpha}}\sqrt{t}\right), \tag{7.2}$$

where $\operatorname{erfc}(z) = 1 - \operatorname{erf}(z)$ is the complementary error function, and $\operatorname{erf}(z)$ is the error function:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt.$$

Finally, we find that

$$v(t) = f_1(t) \star f_2(t) = \int_0^t f_1(t-\tau) f_2(\tau)\, d\tau,$$

where $f_1(t)$ and $f_2(t)$ are given by (7.1) and (7.2), respectively.

7.2. Numerical evaluation of the drawdown using Talbot's algorithm

In this subsection, we employ Talbot's algorithm [15] to numerically find $v(t)$ whose Laplace transform is given by (6.17). For the benefit of the reader, we review Talbot's algorithm. Let $F(s) = \mathcal{L}(f(t))$, $Re(s) > \gamma_0$, be the Laplace transform of $f(t)$. The standard inversion formula of $F(s)$ is

$$f(t) = \frac{1}{2\pi i} \int_B F(s) e^{st}\, ds, \tag{7.3}$$

where B is a Bromwich contour from $\gamma - i\infty$ to $\gamma + i\infty$, where $\gamma > \gamma_0$, so that B is to the right of all singularities of $F(s)$. Direct numerical integration along B is impractical due to the oscillations of e^{st} as $Im(s) \to \pm\infty$ [15]. It is shown in [15] that integration along B can be replaced by an integration along an equivalent contour L starting and ending in the left half-plane, so that $Re(s) \to -\infty$ at each end of L. This replacement is possible if (i) L encloses all singularities of $F(s)$ and (ii) $|F(s)| \to 0$ in $Re(s) \leq \gamma_0$ as $|s| \to \infty$. Further, it was suggested in [15] that the integration along L can be equivalently replaced by an integration over the imaginary interval $M = [-2\pi\, i, 2\pi\, i]$ using an appropriate one-to-one mapping, $T(z)$, which maps M to L. Applying chain rule to (7.3), we have

$$f(t) = \frac{1}{2\pi i} \int_{-2\pi\, i}^{2\pi\, i} F(T(z)) T'(z) e^{T(z)t}\, dz = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} G(iz)\, dz, \tag{7.4}$$

where $G(z) = F(T(z))T'(z)e^{T(z)t}$. It is important to mention that though condition (ii) in the replacement of B by L is almost always satisfied for most functions of interest, condition (i) may not satisfied by a particular contour L for a given $F(s)$, as some singularities of $F(s)$ may fall outside the scope of L. If this happens, the contour L can be made to hold for a modified $\tilde{F}(s) = F(\lambda s + \sigma)$, where λ and σ are appropriate scale and shift parameters chosen so that L enclosed all singularities of $\tilde{F}(s)$. The shift and scale parameters σ and λ are chosen in such away that if s_0 is a singularity of $F(s)$ outside L, then $\tilde{s}_0 = (s_0 - \sigma)/\lambda$ (a singularity of $\tilde{F}(s)$) is inside L. Making the change of variable $s \to \lambda s + \sigma$ in (7.3), we obtain

$$f(t) = \frac{1}{2\pi i} \int_B \lambda F(\lambda s + \sigma) e^{(\lambda s + \sigma)t}\, ds,$$

and upon changing from B to L to M, we obtain

$$f(t) = \frac{1}{2\pi i} \int_{-2\pi\, i}^{2\pi\, i} \lambda F(\lambda T(z) + \sigma) T'(z) e^{(\lambda T(z) + \sigma)t}\, dz = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} \tilde{G}(iz)\, dz, \tag{7.5}$$

where $\tilde{G}(z) = \lambda F(\lambda T(z) + \sigma)T'(z)e^{(\lambda T(z)+\sigma)t}$. The integral (7.4) (or (7.5)) can be numerically approximated using the trapezoidal rule to give

$$f(t) \approx \frac{1}{n} \left[G(2\pi\, i) + 2 \sum_{k=1}^{n-1} G\left(\frac{2k - n}{n} 2\pi\, i \right) + G(2\pi i) \right].$$

Note that $G(\pm 2\pi\, i) = 0$ because $T(\pm 2\pi\, i) = s$ with Re(s)$= -\infty$ and $F(s) = 0$. In our numerical simulations, we use the one-to-one map

$$s = T(z) = \frac{z}{1 - e^{-z}}. \tag{7.6}$$

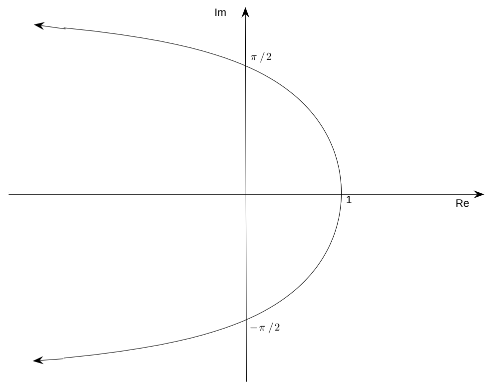

Figure 5. Plot of the contour L using the map $T(z)$ in (7.6).

A plot of the contour L using the above mapping $T(z)$ is shown on Figure 5. In our case of the previous section, we have

$$F(s) = \widehat{v}(s) = \frac{Q}{2}\,\frac{e^{(b-d)\sqrt{s/\alpha}}}{bs + \kappa\ell\sqrt{s/\alpha}}.$$

Then

$$G(z) = \frac{Q}{2}\,\frac{e^{T(z)\,t+(b-d)\sqrt{T(z)/\alpha}}}{bT(z) + \kappa\ell\sqrt{T(z)/\alpha}}\,T'(z). \tag{7.7}$$

We remark here that $G(z)$ in (7.7) has a singular point at $z = 0$, so in the trapezoidal rule we choose n to be odd.

8. Numerical simulations

In this section we consider as a case study a well in the south of Jordan (Desah), and the water deposit, which is not renewable, in Saudi-Arabia. The parameter values in the model are taken from Hydrogeological Services International [21] , and given by

$$\mu = 0.2\times 10^{-5}\text{m/sec},\ \ \ell = 1000\text{ m},\ \ \kappa = 1.5\text{ m/day},\ \ h = 120\text{ m and } b = 20\text{km}.$$

We simulate the model for different values of the distance d between the well and the center of the reservoir and of the pumping rate Q.

Example 8.1. In this example we consider the distance $d = 70$ km and different values of $Q = 80, 100, 200$ and 300 million m^3-year. The results are shown the Fig. 6.

We note that for the distance $d = 70$ km, the water level in the reservoir is not affected for the first 2500 years, after which the level starts to drop to about 117 meters after 500 years for the case $Q = 300$. As expected, the water level drops at slower rate for smaller pumping rate.

Example 8.2. In this example we consider the distance $d = 50$ km and different values of $Q = 80, 100, 200$ and 300 million m^3-year. The results are shown the Fig. 7

Figure 7 shows that the decrease in water level begins earlier in time after about 1000 years and drops with slower rate for different pumping rates.

Example 8.3. In this third example we consider the distance $d = 30$ km and different values of $Q = 80, 100, 200$ and 300 million m^3-year. The results are shown the Fig. 8, which shows that water level starts to drop much earlier than the previous examples, after about 100 years and drops with slower rate for different pumping rates.

The results of this third example are particularly interesting. They reveal that the closer the well to the reservoir the earlier the drop in water level starts. Moreover, Fig. 8 reveals that for all three different pumping rates, the water level drops to zero in a span of 45 years to 50 years. This is in contrast to the previous two examples.

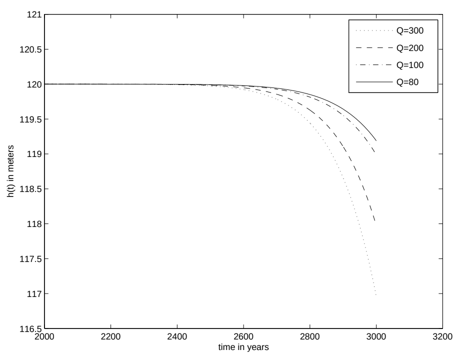

Figure 6. Plot of water level $h(t)$ versus time t in years for the case $d = 70$ km.

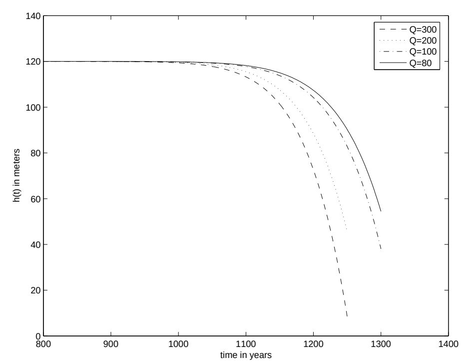

Figure 7. Plot of water level $h(t)$ versus time t in years for the case $d = 50$ km.

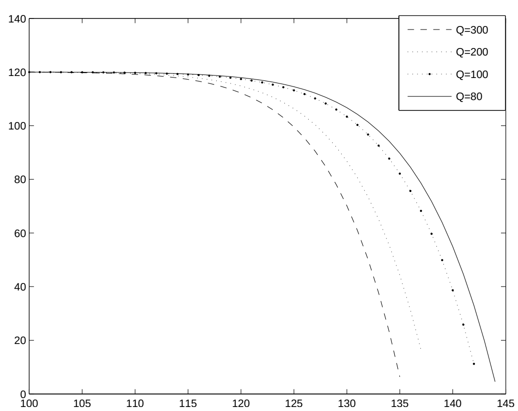

Figure 8. Plot of water level $h(t)$ versus time t in years for the case $d = 30$ km.

9. Conclusion

In this paper, two analytical methods were developed to simulate water movement and solute transport in the unsaturated zone. Namely, on the basis of the numerical results obtained in section 5, we conclude that both the Laplace transform method and the singular perturbation technique are proved to be efficient in finding a closed form approximate solution for solute transport in the unsaturated zone. The proposed methods can be used to simulate contaminant movement is soils.

The numerical simulations in section 8, show the effect of pumping water out of a well that is some distance away from the water deposit. The results show that the farther away from the water reserve the well is, the longer the water reserve lives. Also, when the water level in the reserve starts to decrease, it takes longer to decrease to zero. However, the closer the well is, the sooner the water level starts to drop and when it starts to drop, it takes shorter time to vanish. These qualitative results are, of course, a function of the pumping rate Q, that is, the higher Q is, the relatively faster the water level drops. What is interesting in the results is that for the same pumping rate Q, increasing the distance d increases the life time of the water reserve dramatically.

The results presented in this paper are in good agreement with the real situation in Jordan, and can be very useful to governments and ministries in charge when deciding on where to dig a well relative to the water reserve so as to maximize the life time of the water reserve. An interesting future research direction is the determination of optimal distance d and pumping rate Q for a given initial water reserve level h_0. In fact, this point is currently being investigated by the authors.

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