APPLICATIONS OF FRACTIONAL COMPLEX TRANSFORM AND \( \left( \frac{G'}{G} \right) \)-EXPANSION METHOD FOR TIME-FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract In this paper, the fractional complex transform and the \( \left( \frac{G'}{G} \right) \)-expansion method are employed to solve the time-fractional modified Korteweg-de Vries equation (fmKdV), Sharma-Tasso-Olver, Fitzhugh-Nagumo equations, where \( G \) satisfies a second order linear ordinary differential equation. Exact solutions are expressed in terms of hyperbolic, trigonometric and rational functions. These solutions may be useful and desirable to explain some nonlinear physical phenomena in genuinely nonlinear fractional calculus.

Keywords The \( \left( \frac{G'}{G} \right) \)-expansion method, exact solutions, fractional differential equation, modified Riemann–Liouville derivative.

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1. Introduction

The fractional calculus (fractional derivatives and fractional integrals) has been recognized as an effective modeling methodology for researchers. Fractional differential equations are generalization of ordinary differential equations to arbitrary (noninteger) order. In recent decades, fractional differential equations capture nonlocal relations in space and time with power law memory kernels. Due to extensive applications in engineering and science, research in fractional differential equations has become intense around the world. Some aspects of the fractional differential equations have been investigated by many authors [24,32,35]. Among the investigations for fractional differential equations, research for seeking exact solutions solutions of time-fractional differential equations is an important topic, which can also provide valuable reference for other related research. In recently, some effective methods for fractional calculus were appeared in open literature, such as the exp-function method [5,45], the fractional sub-equation method [14,16,38,44], the \( (G'/G) \)-expansion method [4,9,46], the fractional homotopy analysis method [10–12], the Jacobi elliptic function method [13] and the first integral method [26]. Based on these methods, a variety of fractional differential equations have been investigated.

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and solved. However, there are quite a few direct approaches to exact solutions of nonlinear equations. For example, the transformed rational function method and the multiple exp-function approach provide the most powerful approaches to traveling wave and multiple wave solutions.

The fractional complex transform \([17,18]\) is the simplest approach, it is to convert the fractional differential equations into ordinary differential equations, making the solution procedure extremely simple. Recently, the fractional complex transform has been suggested to convert fractional order differential equations with modified Riemann-Liouville derivatives into integer order differential equations, and the reduced equations can be solved by symbolic computation. The \((G'/G)-\)expansion method \([20-23]\) can be used to construct the exact solutions for fractional differential equations. The present paper investigates for the applicability and efficiency of the \((G'/G)-\)expansion method on time-fractional differential equations. The aim of this paper is to extend the application of the \((G'/G)-\)expansion method to obtain exact solutions to some fractional differential equations in broad science and technology area.

In this paper, we will apply the \((G'/G)-\)expansion method for solving fractional partial differential equations in the sense of modified Riemann–Liouville derivative by Jumarie \([20,21]\). The Jumarie’s modified Riemann–Liouville derivative of order \(\alpha\) is defined by the following expression:

\[
D^{\alpha}_t f(t) = \begin{cases} 
\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int^t_0 (t-\xi)^{-\alpha} (f(\xi) - f(0))d\xi, & 0 < \alpha < 1, \\
(f^{(n)}(t))^{(\alpha-n)}, & n \leq \alpha < n+1, \quad n \geq 1.
\end{cases}
\]  

We list some important properties for the modified Riemann–Liouville derivative as follows:

\[
D^{\alpha}_t x^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha}, \gamma > 0, 
\]  

\[
D^{\alpha}_t (cf(t)) = c D^{\alpha}_t f(t), \quad c = \text{constant}, 
\]  

\[
D^{\alpha}_t \{af(t) + bg(t)\} = a D^{\alpha}_t f(t) + b D^{\alpha}_t g(t),
\]  

where \(a\) and \(b\) constant.

\[
D^{\alpha}_t c = 0, \quad c = \text{constant}.
\]

The rest of this paper is organized as follows. In Section 2, we give the definition of the \((G'/G)-\)expansion method for solving time-fractional partial differential equations. Then in Section 3-5, we use this method to construct exact solutions for the time-fractional fmKdV, Sharma-Tasso-Olver, Fitzhugh-Nagumo equations. Some conclusions are presented in Section 6.

2. The \((\frac{G'}{G})\)-expansion method for Fractional Partial Differential Equations

In the following, we give the main steps of the \((\frac{G'}{G})\)-expansion method.

1. Suppose that a fractional partial differential equation, say in the independent
variables $t, x_1, x_2, ..., x_n$ are given by

$$P(u_1, ..., u_k, \frac{\partial u_1}{\partial t}, ..., \frac{\partial u_k}{\partial t}, \frac{\partial u_1}{\partial x_1}, ..., \frac{\partial u_k}{\partial x_1}, ..., \frac{\partial u_1}{\partial x_n}, ..., \frac{\partial u_k}{\partial x_n}) = 0,$$

where $u_i = u_i(t, x_1, x_2, ..., x_n)$, $i = 1, 2, 3, ..., k$ are unknown functions, $P$ is a polynomial in $u_i$ and their various partial derivatives including fractional derivatives.

2. Li and He [17] proposed a fractional complex transform to convert fractional differential equations into ordinary differential equations (ODE), so all analytical methods devoted to the advanced calculus can be easily applied to the fractional calculus. The traveling wave variable

$$U_i(\xi) = u_i(t, x_1, x_2, ..., x_n), \quad \xi = \xi(t, x_1, x_2, ..., x_n),$$

$$\xi = \frac{ct^{\alpha}}{\Gamma(1 + \alpha)} + \frac{\tau x_1^{\beta}}{\Gamma(1 + \beta)} + \frac{\delta x_2^{\gamma}}{\Gamma(1 + \gamma)} + ... + \frac{\psi x_n^{\phi}}{\Gamma(1 + \phi)},$$

where $c, \tau, \delta, ..., \psi$ are non zero arbitrary constants.

By using the chain rule

$$D_t^\alpha u = \sigma_t^i \frac{dU}{d\xi} D_t^\alpha \xi,$$

$$D_x^\alpha u = \sigma_x^i \frac{dU}{d\xi} D_x^\alpha \xi,$$

where $\sigma_t^i$ and $\sigma_x^i$ are called the sigma indexes see [7], without loss of generality we can take $\sigma_t^i = \sigma_x^i = l$, where $l$ is a constant.

Substituting (2.2) with (2.3) and (1.2) into (2.2), equation (2.2) can be reduced into an ODE;

$$Q(U_1, ..., U_k, U'_1, ..., U'_k, U''_1, ..., U''_k, ......) = 0,$$

where the prime denotes the derivation with respect to $\xi$. If possible, we should integrate Eq. (2.4) term by term one or more times.

3. Assume that the solution of equation (2.4) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$u(\xi) = \sum_{i=-m}^{m} a_i \left(\frac{G'}{G}\right)^i, \quad a_m \neq 0,$$

where $a_i$ ($i = 0, \pm1, \pm2, ..., \pm m$) are constants, while $G(\xi)$ satisfies the following second order linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0,$$

with $\lambda$ and $\mu$ are being constants.

4. The positive integer $m$ can be determined by considering the homogeneous balance between the highest order derivaives and the nonlinar terms appearing in equation (2.4).
5. Substituting equation (2.5) into equation (2.4) and using equation (2.6) collecting all terms with the same order of \((G'/G)\) together. Then equating each coefficient of the resulting polynomial to zero, we obtain a set of algebraic equations for \(a_i (i = 0, \pm 1, \pm 2, \ldots, \pm m), c, \tau, \delta,...,\psi, \mu, \text{ and } \lambda\).

6. Solving the equations system in step 5, and using equation then substituting \(a_i (i = 0, \pm 1, \pm 2, \ldots, \pm m), c, \tau, \delta,...,\psi, \mu, \lambda\) and the general solutions of equation (2.6) into equation (2.5), we can get a variety of exact solutions of equation (2.1).

The adopted \((G'/G)\)-expansion method is also actually the expansion method using a special Riccati equation, since \(G'/G\) satisfies a Riccati equation if \(G\) solves (2.6) in the manuscript. All explicit and exact solutions to general Riccati equations are presented in [29]. More generally, Frobenius integrable decompositions [30] and the invariant subspace method [28] will help in solving nonlinear equations.

3. Time-fractional fmKdV equation

Firstly, we consider the following time fractional fmKdV equation [25]

\[
D_\alpha^t u + u^2u_x + u_{xxx} = 0, \quad t > 0, \quad 0 < \alpha \leq 1,
\]

with the initial conditions as

\[
u(x, 0) = \frac{4\sqrt{2}k \sin^2(kx)}{3 - \sin^2(kx)},
\]

where \(k\) is arbitrary constant, \(\alpha\) is a parameter describing the order of the fractional time-derivative. By using homotopy perturbation method (HPM), Hashim et al. [1] have found approximate analytical solutions for fmKdV. By using differential transform method, Kurulay and Bayram [25] have found new approximate analytical solutions for fmKdV. In recent years, Guner and Cevikel applied the exp-function method to this equation and obtained new exact solutions [15]. When \(\alpha = 1\), the fractional fmKdV equation reduces to the mKdV equation. The mKdV equation appears in many fields such as acoustic waves in certain anharmonic lattices, Alfvén waves in a collisionless plasma, transmission lines in Schottky barrier, models of traffic congestion, ion acoustic solitons, elastic media, shallow water model, plasma science, biophysics etc [40]. By using the variational iteration method, Inc [19] has found exact and numerical solutions and compared with those obtained by Adomian decomposition method. Lastly, the extended tanh method was successfully used to establish solitary wave solutions to this equation [2].

For our purpose, we introduce the following transformations;

\[
u(x, t) = U(\xi), \quad \xi = \nu x - \frac{ct^\alpha}{\Gamma(1 + \alpha)},
\]

where \(c\) and \(\nu\) are non-zero constants.

Substituting (3.3) with (2.3) and (1.2) into (3.1), we can know that (3.1) reduced into an ODE

\[
-cU' + \nu U^2U' + \nu^3U''' = 0,
\]

where \(U' = \frac{dU}{d\xi}\). By once time integrating we find

\[
\xi_0 - cU + \nu \frac{U^3}{3} + \nu^3U'' = 0,
\]

(3.5)
where $\xi_0$ is a integration constant.

Using the ansatz (3.5), for the linear term of highest order $U''$ with the highest order nonlinear term $U^3$. By simple calculation, we have balancing $U''$ with $U^3$ in (3.5) gives

$$m + 2 = 3m,$$

so that

$$m = 1.$$  

(3.7)

Suppose that the solutions of (3.5) can be expressed by a polynomial in $(G'/G)$ as follows:

$$U(\xi) = a_0 + a_1 \left( \frac{G'}{G} \right), \quad a_1 \neq 0.$$  

(3.8)

By using Eq. (2.6), from Eq. (3.8) we have

$$U''(\xi) = 2a_1 \left( \frac{G'}{G} \right)^3 + 3a_1 \lambda \left( \frac{G'}{G} \right)^2 + (2a_1 \mu + a_1 \lambda^2) \left( \frac{G'}{G} \right) + a_1 \lambda \mu,$$  

(3.9)

and

$$U^3(\xi) = a_1^3 \left( \frac{G'}{G} \right)^3 + 3a_0 a_1^2 \left( \frac{G'}{G} \right)^2 + 3a_0^2 a_1 \left( \frac{G'}{G} \right) + a_0^3.$$  

(3.10)

Substituting Eq. (3.8)-(3.10) into Eq. (3.5), collecting the coefficients of $(G'/G)^i$ $(i = 0, ..., 3)$ and set it to zero we obtain the system

$$\frac{1}{3} \nu a_1^3 + 2 \nu^3 a_1 = 0,$$

$$\nu a_0 a_1^2 + 3 \nu^3 a_1 \lambda = 0,$$

$$\nu a_0^2 a_1 - \nu^3 a_1 \lambda^2 + 2 \nu^3 a_1 \mu = 0,$$

$$-\nu a_0 + \frac{1}{3} \nu a_0^3 + \nu^3 a_1 \lambda \mu + \xi_0 = 0.$$  

(3.11)

Solving this system by simple calculation gives

$$a_0 = \pm \frac{1}{2} i \nu \sqrt{6}, \quad a_1 = \pm i \nu \sqrt{6}, \quad c = \frac{\nu^2}{2} (-\lambda^2 + 4 \mu), \quad \xi_0 = 0,$$  

(3.12)

where $\lambda$ and $\mu$ are arbitrary constants.

By using Eq. (3.12), expression (3.8) can be written as

$$U(\xi) = \pm \frac{1}{2} i \nu \sqrt{6} \pm i \nu \sqrt{6} \left( \frac{G'}{G} \right).$$  

(3.13)

Substituting general solutions of Eq. (2.6) into Eq. (3.13) we have two types of exact solutions of the time fractional fmKdV equation as follows:

When $\lambda^2 - 4 \mu > 0$,

$$U_{1,2}(\xi) = \pm i \nu \sqrt{6} \frac{\sqrt{\lambda^2 - 4 \mu}}{2} \left( C_1 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4 \mu} \xi + C_2 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4 \mu} \xi \right),$$  

(3.14)

where $\xi = x - \frac{\nu^2 (-\lambda^2 + 4 \mu)}{2(1+\alpha)} t^\alpha$ and $C_1, C_2$ are arbitrary constants.
When $\lambda^2 - 4\mu < 0$,

$$U_{3,4}(\xi) = \pm iv \sqrt{-6\lambda^2 + 24\mu} \left( -C_1 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi \right),$$

(3.15)

where $\xi = x - \nu \frac{(\lambda^2 + 4\mu)^{\alpha}}{2(1+\alpha)}$ and $C_1, C_2$ are arbitrary constants.

In particular, if $C_1 \neq 0, C_2 = 0, \lambda > 0, \mu = 0$, then $U_{1,2}$ becomes

$$u_{1,2}(x,t) = \pm iv \frac{\sqrt{6}}{2} \coth \frac{\lambda}{2} \left( x - \frac{\nu^3 (-\lambda^2 + 4\mu)}{2\Gamma(1+\alpha)} t^\alpha \right),$$

(3.16)

and $U_{3,4}$ becomes

$$u_{3,4}(x,t) = \pm iv \frac{\sqrt{6}}{2} \cot \frac{\lambda}{2} \left( x - \frac{\nu^3 (-\lambda^2 + 4\mu)}{2\Gamma(1+\alpha)} t^\alpha \right),$$

(3.17)

which are the solitary wave solutions of the time fractional fmKdV equation.

**Remark 3.1.** Comparing our results to the Guner’s results [15] it can be seen that these results are new.

### 4. Time-fractional Sharma-Tasso-Olver equation

Secondly, we consider the nonlinear fractional Sharma-Tasso-Olver equation [36]

$$D_t^\alpha u + 3a u_x^2 + 3a u_{xx} + 3a u_{xxx} + au_{xxxx} = 0, \quad t > 0, \quad 0 < \alpha \leq 1,$$

(4.1)

where $a$ is a arbitrary constant and subject to the initial condition

$$u(x,0) = -\sqrt{2B_0} \tan \left( \frac{\sqrt{2B_0} x}{2} \right),$$

(4.2)

where $a$ and $B_0$ are arbitrary constants, $\alpha$ is a parameter describing the order of the fractional time-derivative. The function $u(x,t)$ is assumed to be a causal function of time. Song et al. have obtained the approximate analytical solutions of eq.(4.1) with the variational iteration method, the adomian decomposition method and the homotopy perturbation method. Esen et al. [6] have obtained the approximate analytical solutions of this equation with the homotopy analysis method (HAM). Guner and Cevikel [15] have used the Exp-function method to find the traveling wave solutions of (4.1). Equation (4.1) has been investigated in [47] using the fractional sub-equation method. By the improved $(G'/G)$-expansion method, Zayed et al. [41] obtained abundant new exact solutions for the fractional STO equations. In the case of $\alpha = 1$, Eq. (4.1) reduces to the classical nonlinear STO equation.

We use the following transformations:

$$u(x,t) = U(\xi), \quad \xi = x - \frac{ct^\alpha}{\Gamma(1+\alpha)},$$

(4.3)

where $c \neq 0$ is a constant.
Substituting (4.3) with (2.3) and (1.2) into (4.1), equation (4.1) can be reduced into an ODE,

\[-cU' + 3a(U')^2 + 3aU^2U' + 3aUU'' + aU''' = 0, \tag{4.4}\]

where \(U' = \frac{dU}{d\xi}\).

Integrating equation (4.4) with respect to \(\xi\) yields

\[\xi_0 - cU + 3aUU' + aU^3 + aU''' = 0 \tag{4.5}\]

where \(\xi_0\) is a constant of integration.

By the same procedure as illustrated in the section 3, we can determine value of \(m\) by balancing \(U^3\) and \(U''\) in Eq.(4.3). We find \(m = 1\). We can suppose that the solutions of Eq. (4.1) is of the form

\[U(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right), \quad a_1 \neq 0. \tag{4.5}\]

By using Eq. (4.5) and (2.6) it is derived that

\[U'(\xi) = -a_1 \left(\frac{G'}{G}\right)^2 - a_1 \lambda \left(\frac{G'}{G}\right) - a_1 \mu, \tag{4.6}\]

and

\[U^3(\xi) = a_1^3 \left(\frac{G'}{G}\right)^3 + 3a_0 a_1^2 \left(\frac{G'}{G}\right)^2 + 3a_0^2 a_1 \left(\frac{G'}{G}\right) + a_0^3. \tag{4.7}\]

Substituting Eq. (4.5)-(4.7) into Eq. (4.1), collecting the coefficients of \(\left(\frac{G'}{G}\right)^i\) \((i = 0, \ldots, 3)\) and set it to zero we obtain the system

\[2aa_1 + aa_1^3 - 3aa_1 = 0, \tag{4.8}\]

\[-3aa_0 a_1 + 3aa_1 \lambda + 3aa_0 a_1^2 - 3aa_1^2 \lambda = 0, \tag{4.8}\]

\[-3aa_0 a_1 \lambda - 3aa_1^2 \mu - ca_1 + 3aa_0^2 a_1 + 2aa_1 \mu + aa_1 \lambda^2 = 0, \tag{4.8}\]

\[-3aa_0 a_1 \mu + \xi_0 - ca_1 + aa_1^3 + aa_1^2 \lambda \mu = 0. \tag{4.8}\]

We can solve this system by symbolic computation get sets of solutions.

**Case 1:**

\[a_0 = \lambda, \quad a_1 = 2, \quad a = a, \quad c = a\lambda^2 - 4a\mu, \quad \xi_0 = 0, \tag{4.9}\]

where \(\lambda\) and \(\mu\) are arbitrary constants. By using Eq. (4.9), expression (4.5) can be written as

\[U(\xi) = \lambda + 2 \left(\frac{G'}{G}\right). \tag{4.10}\]

Substituting general solutions of Eq. (2.6) into Eq. (4.10) we have three types of exact solutions of the time-fractional Sharma-Tasso-Olver equation as follows:

When \(\lambda^2 - 4\mu > 0\),

\[U_1(\xi) = \sqrt{\lambda^2 - 4\mu} \left(\frac{C_1 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi}{C_1 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi}\right), \tag{4.11}\]
where \( \xi = x - \frac{a\lambda^2 - 4a\mu}{\Gamma(1+\alpha)} t^\alpha \) and \( C_1, C_2 \) are arbitrary constants. When \( \lambda^2 - 4\mu < 0 \),

\[
U_2(\xi) = \sqrt{4\mu - \lambda^2} \left( -C_1 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi \right),
\]

where \( \xi = x - \frac{a\lambda^2 - 4a\mu}{\Gamma(1+\alpha)} t^\alpha \) and \( C_1, C_2 \) are arbitrary constants

In particular, if \( C_1 \neq 0, C_2 = 0, \lambda > 0, \mu = 0 \), then \( U_1 \) becomes

\[
u_1(x, t) = \lambda \tanh \frac{\lambda}{2} \left( x - \frac{a\lambda^2 - 4a\mu}{\Gamma(1+\alpha)} t^\alpha \right),
\]

and \( U_2 \) becomes

\[
u_2(x, t) = -i\lambda \tan \frac{\lambda}{2} \left( x - \frac{a\lambda^2 - 4a\mu}{\Gamma(1+\alpha)} t^\alpha \right).
\]

which are the solitary wave solutions of the time-fractional Sharma-Tasso-Olver equation.

**Case 2:**

\[
a_0 = a_0, a_1 = 1, \quad c = -3aa_0\lambda - a\mu + 3aa_0^3 + a\lambda^2,
\]

\[
a = a, \quad \xi_0 = 2aa_0\mu - 3aa_0^2\lambda + 2a_0a\lambda^2 - a\lambda\mu,
\]

where \( \lambda \) and \( \mu \) are arbitrary constants.

Substituting Eq. (4.9) into Eq. (4.5) yields

\[
U(\xi) = a_0 + \left( \frac{G'}{G} \right).
\]

Substituting general solutions of Eq. (2.6) into Eq. (4.10) we have three types of exact solutions of the time-fractional Sharma-Tasso-Olver equation as follows:

When \( \lambda^2 - 4\mu > 0 \),

\[
U_1(\xi) = a_0 - \frac{\lambda}{2} + \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \left( \frac{C_1 \sin \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \cos \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi}{C_1 \sin \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \cos \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi} \right),
\]

where \( \xi = x + \frac{3aa_0\lambda + a\mu - 3aa_0^3 - a\lambda^2}{\Gamma(1+\alpha)} t^\alpha \).

When \( \lambda^2 - 4\mu < 0 \),

\[
U_2(\xi) = a_0 - \frac{\lambda}{2} + \frac{1}{2} \sqrt{4\mu - \lambda^2} \left( -\frac{C_1 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi}{C_1 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi} \right),
\]

where \( \xi = x + \frac{3aa_0\lambda + a\mu - 3aa_0^3 - a\lambda^2}{\Gamma(1+\alpha)} t^\alpha \).

When \( \lambda^2 - 4\mu = 0 \),

\[
U_3(\xi) = a_0 - \frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 \xi}.
\]
In particular, if $C_1 \neq 0, C_2 = 0, \lambda > 0, \mu = 0$, then $U_1$ becomes
\begin{equation}
    u_1(x,t) = a_0 - \frac{\lambda}{2} + \frac{\lambda}{2} \tanh \frac{\lambda}{2} \left( x + \frac{3aa_0\lambda - 3aa_0^2 - a\lambda^2}{\Gamma(1 + \alpha)} t^\alpha \right),
\end{equation}
and $U_2$ becomes
\begin{equation}
    u_2(x,t) = a_0 - \frac{\lambda}{2} - i \frac{\lambda}{2} \tan \frac{\lambda}{2} \left( x + \frac{3aa_0\lambda - 3aa_0^2 - a\lambda^2}{\Gamma(1 + \alpha)} t^\alpha \right),
\end{equation}
which are the solitary wave solutions of the time-fractional Sharma-Tasso-Olver equation.

**Remark 4.1.** We note that the exact solutions established in (4.13), (4.14), (4.19), (4.20) and (4.21) are new exact solutions to the time-fractional Sharma-Tasso-Olver equation.

### 5. Time fractional Fitzhugh-Nagumo equation

Thirdly, we take into account the fractional Fitzhugh-Nagumo equation
\begin{equation}
    \frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-\psi), \quad t > 0, \quad 0 < \alpha \leq 1, \quad x \in \mathbb{R},
\end{equation}
which is an important nonlinear reaction-diffusion equation, applied to model the transmission of nerve impulses [8,33], and also used in biology and the area of population genetics in circuit theory [37]. Pandir and Tandoğan, applied the modified trial equation method to obtain analytical solutions of the time-fractional Fitzhugh-Nagumo equation [34]. Khan et al. employed the homotopy perturbation method (HPM) to obtain approximate analytical solutions of the time-fractional reaction-diffusion equation of the Fisher type [22]. Merdan is using He’s variational iteration method, obtain the analytical solutions of nonlinear time-fractional reaction-diffusion equations of the Fisher type [31]. For $\alpha = 1$, Eq. (5.1) reduces to the classical nonlinear Fitzhugh-Nagumo equation. By using some methods, many researchers have tried to obtain the exact solutions of this equation. For example; by using Hirota method, Kawahara and Tanaka [23] have found new exact solutions of Eq. (5.1); by using the first integral method, Li and Guo [27] have obtained a series of new exact solutions of the Fitzhugh-Nagumo equation. When $\psi = -1$, the Fitzhugh-Nagumo equation reduces to the real Newell-Whitehead equation.

For our goal, we present the following transformation
\begin{equation}
    u(x,t) = U(\xi), \quad \xi = cx - \frac{mt^\alpha}{\Gamma(1 + \alpha)},
\end{equation}
where $c$ and $m$ are non zero constants.

Then by use of Eq. (5.2) with (2.3) and (1.2), Eq. (5.1) can be turned into an ODE
\begin{equation}
    mU' + c^2 U'' + U(1-U)(U-\psi) = 0,
\end{equation}
where $U' = \frac{dU}{d\xi}$. 

Using the ansatz (5.3), for the linear term of highest order \( U'' \) with the highest order nonlinear term \( U^3 \). By simple calculation, we have balancing \( U'' \) with \( U^3 \) in (5.3) gives

\[ n + 2 = 3n, \tag{5.4} \]

so that

\[ n = 1. \tag{5.5} \]

Suppose that the solutions of (5.3) can be expressed by a polynomial in \( \left( \frac{G'}{G} \right) \) as follows:

\[ U(\xi) = a_0 + a_1 \left( \frac{G'}{G} \right), \quad a_1 \neq 0. \tag{5.6} \]

By using Eq. (2.6), from Eq. (5.6) we have

\[ U'(\xi) = -a_1 \left( \frac{G'}{G} \right)^2 - a_1 \lambda \left( \frac{G'}{G} \right) - a_1 \mu, \tag{5.7} \]

\[ U''(\xi) = 2a_1 \left( \frac{G'}{G} \right)^3 + 3a_1 \lambda \left( \frac{G'}{G} \right)^2 + (2a_1 \mu + a_1 \lambda^2) \left( \frac{G'}{G} \right) + a_1 \lambda \mu, \tag{5.8} \]

and

\[ U^3(\xi) = a_1^3 \left( \frac{G'}{G} \right)^3 + 3a_0 a_1^2 \left( \frac{G'}{G} \right)^2 + 3a_0^2 a_1 \left( \frac{G'}{G} \right) + a_0^3. \tag{5.9} \]

Substituting Eq. (5.6)-(5.9) into Eq. (5.3), collecting the coefficients of \( \left( \frac{G'}{G} \right)^i \) \( (i = 0, ..., 3) \) and set it to zero we obtain the system

\[-ma_1 \mu + c^2 a_1 \lambda + a_0^2 - a_0 \psi - a_0^3 + a_0^2 \psi = 0, \]

\[-ma_1 \lambda + c^2 a_1 \lambda^2 + 2c^2 a_1 \mu + 2a_0 a_1 - 3a_0^2 a_1 + 2a_0 a_1 \psi - a_1 \psi = 0, \]

\[-ma_1 + 3c^2 a_1 \lambda - 3a_0 a_1^2 + a_1^2 + a_1^2 \psi = 0, \]

\[2c^2 a_1 - a_1^3 = 0. \tag{5.10} \]

Solving this system by symbolic computation gives

\[ a_0 = \frac{4\mu - \lambda^2 \pm \sqrt{\lambda^4 - 4\lambda^2 \mu}}{2(4\mu - \lambda^2)}, \quad a_1 = \frac{\mp 3\lambda^2 + (1 - 2\psi)\sqrt{\lambda^4 - 4\lambda^2 \mu}}{\lambda(4\mu - \lambda^2 \pm 2\lambda^2 \psi + 8\mu \psi + 3\sqrt{\lambda^4 - 4\lambda^2 \mu})}, \]

\[ c = \pm \sqrt{\frac{1}{2\lambda^2 - 8\mu}}, \quad m = \pm \sqrt{\frac{\lambda^4 - 4\lambda^2 \mu(2\psi - 1)}{2\lambda(4\mu - \lambda^2)}}, \tag{5.11} \]

where \( \lambda \) and \( \mu \) are arbitrary constants.

By using Eq. (5.11), expression (5.7) can be written as

\[ U(\xi) = \frac{4\mu - \lambda^2 \pm \sqrt{\lambda^4 - 4\lambda^2 \mu}}{2(4\mu - \lambda^2)} \left( \frac{G'}{G} \right) + \frac{\mp 3\lambda^2 + (1 - 2\psi)\sqrt{\lambda^4 - 4\lambda^2 \mu}}{\lambda(4\mu - \lambda^2 \pm 2\lambda^2 \psi + 8\mu \psi + 3\sqrt{\lambda^4 - 4\lambda^2 \mu})} \left( \frac{G'}{G} \right). \tag{5.12} \]

Substituting general solutions of Eq. (2.6) into Eqs. (3.13) we have two types of exact solutions of the time fractional FitzHugh-Nagumo equation as follows:
Comparing our results to the Pandir’s results [O. Abdulaziz, I. Hashim and E.S. Ismail, and fractional complex transform with help of (G̃-function solutions and rational solutions. The work emphasized our belief that the solutions are obtained including the hyperbolic function solutions, trigonometric fmKdV, Sharma-Tasso-Olver, Fitzhugh-Nagumo equations. As a result, some exact solutions are arbitrary constants. In particular, if $C_1 \neq 0, C_2 = 0, \lambda > 0, \mu = 0$, then $U_{1,2}$ becomes

$$u_{1,2}(x, t) = -\frac{\lambda^2 \pm \lambda^2}{2(-\lambda^2)} + i \frac{\pm 3\lambda^2 + (1 - 2\psi)\lambda^2}{2(\pm \lambda^2 \pm 2\lambda^2 \psi + 33\lambda^2)} \left( \tan \frac{\lambda}{2} \left( \pm \sqrt{\frac{1}{2\lambda^2} x - \frac{\pm \lambda^2 (2\psi - 1)t^\alpha}{2\lambda(-\lambda^2) \Gamma(1 + \alpha)}} \right) + i \right).$$

(5.16)

which are the solitary wave solutions of the time fractional Fitzhugh-Nagumo equation.

**Remark 5.1.** Comparing our results to the Pandir’s results [34] it can be seen that these results are new.

### 6. Conclusion

In this work, $(G'/G)$-expansion method is extended to solve the time-fractional fmKdV, Sharma-Tasso-Olver, Fitzhugh-Nagumo equations. As a result, some exact solutions are obtained including the hyperbolic function solutions, trigonometric function solutions and rational solutions. The work emphasized our belief that the method is a reliable technique to handle nonlinear fractional differential equations and fractional complex transform with help of $(G'/G)$-expansion method offer significant advantages in terms of its straightforward applicability, its computational effectiveness and its powerful. This method has more advantages: it is direct and concise. Therefore, we deduce that the proposed method can be extended to solve many systems of nonlinear fractional differential equations. This is our task in the future.

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