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AN OPTIMAL HOMOTOPY ANALYSIS METHOD BASED ON PARTICLE SWARM OPTIMIZATION: APPLICATION TO FRACTIONAL-ORDER DIFFERENTIAL EQUATION*

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Abstract This paper describes a new problem-solving mentality of finding optimal parameters in optimal homotopy analysis method (optimal HAM). We use particle swarm optimization (PSO) to minimize the exact square residual error in optimal HAM. All optimal convergence-control parameters can be found concurrently. This method can deal with optimal HAM which has finite convergence-control parameters. Two nonlinear fractional-order differential equations are given to illustrate the proposed algorithm. The comparison reveals that optimal HAM combined with PSO is effective and reliable. Meanwhile, we give a sufficient condition for convergence of the optimal HAM for solving fractional-order equation, and try to put forward a new calculation method for the residual error.

Keywords Optimal homotopy analysis method, particle swarm optimization, fractional-order differential equation, Caputo derivative, residual error.

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1. Introduction

Homotopy analysis method (HAM) was first proposed by Liao [15, 16], employing the concept of homotopy in topology to obtain an analytical approximate method for solving nonlinear equation. The most common and widely used methods for determining analytical approximate solutions of a nonlinear system are perturbation methods. Most of the perturbation methods unfortunately, require the inclusion of a small parameter in the equation. Unlike perturbation techniques, the HAM is not limited to any small physical parameters in the considered equation [16]. Therefore, the HAM can overcome the foregoing restrictions and limitations of perturbation techniques as it provides us with a powerful tool to analyze strongly nonlinear problems [1,16,27]. With this method, the analytical approximate solution of a nonlinear equation is expressed as an infinite series. Its convergence rate and convergence region is controlled by an auxiliary parameter $\hbar$, which is called convergence-control parameter. The effective $\hbar$ is obtained by using a $\hbar$-curve [15]. However, when the nonlinear problem contains multiple equations, there may exist two or more

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convergence-control parameters $(\hbar_1, \hbar_2, \cdots, \hbar_k)$. In this case, it is difficult to find appropriate convergence-control parameters.

To deal with the problem of convergence-control parameters, some optimal homotopy analysis (or asymptotic) approaches have been put forward by many scholars recently. Where, the minimization of the residual error of the nonlinear equation is a crucial step. Yabushita [32] draws the iso-lines of the residual error on the plane, then minimize the residual error and get the optimal convergence-control parameters. When the number of convergence-control parameters are greater than two, this approach is not appropriate. Marinca et al. [20–22] minimize the residual error based on necessary (not sufficient) conditions for extremum existence. Thus, the obtained parameters may not be optimal. In this approach, compute the partial derivative of the residual error function with respect to each convergence-control parameter, and the results are set equal to zero. However, this approach leads to a system of coupled nonlinear algebraic equations with multiple variables which becomes more and more difficult to solve if the number of the convergence-control parameters increases [6]. Liao [17] introduce a new averaged residual error function, which only contains at most three convergence-control parameters and is computationally rather efficient. When the number of convergence-control parameters is less than three, these two methods are effective. Through practical examples, Liao [17] and Marinca et al [20–22] propose that the residual error function with less than three convergence-control parameters also can obtain good results. Of course, there may exist cases where more convergence-control parameters are considered. Here, how to get a set of optimal convergence-control parameters, a new method is needed to deal with this situation. Niu and Wang [24] put forward a one-step method, which calculate the optimal convergence-control parameters step by step. But, the rationale of this method is not based on the necessary conditions for extremum existence, nor the sufficient condition. The obtained results may not be optimal convergence-control parameters.

In this paper, we use PSO to deal with minimization of the residual error and get optimal convergence-control parameters. PSO is a population-based stochastic approach for solving continuous and discrete optimization problems. When optimal HAM is combined with PSO, it can effectively find the all convergence-control parameters. Subsequently, we apply it to solve nonlinear fractional-order differential equation (FDE) that exact solutions are difficult to achieve [30]. As far as we know, this is the first attempt to deal with optimal convergence-control parameters using optimization algorithm, and apply optimal HAM with PSO to solve FDE.

The structure of this paper is organized as follows. Section 2 is devoted to the basic concepts of fractional-order integral and derivative. PSO and optimal HAM are presented in Section 3. The convergence analysis is discussed in Section 4. We apply optimal HAM and PSO to solve nonlinear FDE in Section 5. Another form of residual error is proposed in Section 6. Conclusions are stated in the last Section.

2. Preliminaries

In this section, we introduce some definitions of fractional calculus and derivative, which will be used later [4, 12].

Definition 2.1. The Euler gamma function $\Gamma(z)$ is defined by

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt \quad (\Re(z) > 0). \tag{2.1}$$

This integral is convergent for all complex $z \in \mathbb{C}$ and $\Gamma(z+1) = z\Gamma(z)$.

Definition 2.2. A real function $f(t)$, $t > 0$, is said to be in space $C_\alpha, \alpha \in \mathbb{R}$ if there exists a real number $p(> \alpha)$, such that $f(t) = t^p f_1(t)$ where $f_1(t) \in C[0, \infty]$.

Definition 2.3. A real function $f(t), t > 0$, is said to be in space $C_\alpha^m, m \in \mathbb{N} \bigcup \{0\}$ iff $f^{(m)}(t) \in C_\alpha$.

Definition 2.4. The Riemann-Liouville integral operator of order $\beta > 0$ is defined as

$$J^\beta z(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} z(\tau) \mathrm{d}\tau, \tag{2.2}$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.5. The Caputo fractional derivative of $y(t)$, $y(t) \in C_{-1}^m, m \in \mathbb{N} \bigcup \{0\}$, is defined as

$$D_*^\alpha y(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} y^{(m)}(\tau) \mathrm{d}\tau, \ \alpha > 0, \tag{2.3}$$

where $m - 1 < \alpha < m$ $(m \in \mathbb{N})$, $y^{(m)}(t)$ is the ordinary mth derivative of $y(t)$, J is the Riemann-Liouville integral operator.

3. Combination of optimal HAM and PSO

3.1. Particle Swarm Optimization

PSO is an effective computation technique like genetic algorithms, simulated annealing, etc [13, 14]. In PSO, candidate solutions of a specific optimization problem are called particles. Each particle in the searching space (n-dimension) is characterized by two factors, i.e., position $x_i = (x_{i1}, x_{i2}, \cdots, x_{in})$ and velocity $v_i = (v_{i1}, v_{i2}, \cdots, v_{in})$, where i denote the ith particle in the swarm. The fitness of each particle can be evaluated according to the objective function of optimization problem. PSO starts with the random initialization of a swarm of particles in the search space. Let $pbest_i(k)$ is the best position found by particle i within k iteration steps. $gbest$ denotes the best position for all particles so far. Then all particles update their velocities and positions based on their own experience $pbest_i(k)$ and experience of all particles $gbest$. The updating rules of velocity and position are given by (3.1) and (3.2) respectively.

$$\begin{cases} v_i^{k+1} = w \times v_i^k + c_1 \times r_1 \times (pbest_i(k) - x_i^k) + c_2 \times r_2 \times (gbest - x_i^k), & (3.1) \\ x_i^{k+1} = x_i^k + v_i^{k+1}, i = 1, 2, \cdots, n, & (3.2) \end{cases}$$

where n is the swarm size, v_i^{k+1} and x_i^{k+1} represent the velocity and position of particle i at kth iteration step respectively; r_1 and r_2 are two independent random numbers in the range of $[0, 1]$; c_1 and c_2 are acceleration constant, usually $c_1 = c_2 =$

2.0; w is called the inertia weight factor, which take a random number. Generally, the value of each component in v_i can be limited to a range $[v_{min}, v_{max}]$ to control excessive roaming of particles outside the searching space. With the Eqs.(3.1) and (3.2), all particles find their new positions and apply these new positions to update their individual best positions and global best position of the swarm. This process is repeated until a user-defined stopping criterion, usually maximum iteration number t_{max} is reached. For more details, one can consult the references [13, 14, 33, 34].

3.2. Optimal Homotopy Analysis Method

Consider the following nonlinear differential equation

$$\mathcal{N}[y(t)] = 0, \quad t \in \Omega, \tag{3.3}$$

where $\mathcal{N}$ is a nonlinear differential operator and $y(t)$ is an unknown function. t is independent time variable. Liao [18] in 1999 introduced the following zeroth-order deformation equation:

$$[1 - B(q)]\,\mathcal{L}[\phi(t;q) - y_0(t)] = c_0 A(q)\mathcal{N}[\phi(t;q)], \tag{3.4}$$

where $q \in [0,1]$, $A(q)$ and $B(q)$ are the so-called deformation functions satisfying

$$A(0) = B(0) = 0, \quad A(1) = B(1) = 1, \tag{3.5}$$

whose Taylor series

$$A(q) = \sum_{k=1}^{+\infty} \mu_k q^k, \quad B(q) = \sum_{k=1}^{+\infty} \sigma_k q^k \tag{3.6}$$

exist and are convergent for $|q| \leq 1$ [15, 16]. As given by Liao [16, 17], there are one special parameter deformation functions which are given as

$$\begin{cases} A(q;c_1) = \displaystyle\sum_{k=1}^{+\infty} \mu_k(c_1) q^k, & (3.7) \\ B(q;c_2) = \displaystyle\sum_{k=1}^{+\infty} \sigma_k(c_2) q^k, & (3.8) \end{cases}$$

where $|c_1| \leq 1$ and $|c_2| \leq 1$ are constants, called the convergence-control parameters. In Liao [16,17], one can define $\mu_1(c_1) = 1 - c_1, \sigma_1(c_2) = 1 - c_2, \mu_k(c_1) = (1-c_1)c_1^{k-1}$ and $\sigma_k(c_2) = (1 - c_2)c_2^{k-1}$, where $k \geq 2$. Thus there only exist three convergence-control parameters c_0, c_1, c_2.

In the following, we assume $B(q) = q$ and $c_0 A(q) = q\hbar(q)$, then construct the zeroth-order deformation equation [6]

$$(1 - q)\mathcal{L}[\phi(t;q) - y_0(t)] = q\hbar(q)\mathcal{N}[\phi(t;q)], \tag{3.9}$$

where $\hbar(q)$ is convergence-control(auxiliary) function with $\hbar(1) \neq 0$. When q increases from 0 to 1 continuously, $\phi(t;q)$ varies (or deforms) from the initial guess solution $y_0(t)$ (i.e., $\phi(t;0)$) to the exact solution $y(t)$ (i.e., $\phi(t;1)$) of Eq.(3.3). $q \in [0,1]$ is an embedding parameter. $\mathcal{L}$ is an auxiliary linear operator. $y_0(t)$ is the solution of initial guess. The Eq.(3.9) is so-called zeroth-order deformation equation of optimal HAM [9, 16, 24, 28]. Now, expand $\phi(t;q)$ and $\hbar(q)$ to the Maclaurin series with respect to q as follows:

$$\begin{cases} \phi(t;q) = \sum_{m=0}^{+\infty} y_m(t) q^m, \\ \hbar(q) = \sum_{k=0}^{+\infty} h_k q^k, \end{cases} \tag{3.10}$$

where $y_m(t) = \frac{1}{m!} \frac{\partial^m \phi(t;q)}{\partial q^m}\Big|_{q=0}$, $h_k = \frac{1}{k!} \frac{\partial^k \hbar(q)}{\partial q^k}\Big|_{q=0}$. If the two series (3.10) are convergence at $q = 1$, then the solution of the nonlinear Eq. (3.3) is

$$y(t) = y_0(t) + \sum_{m=1}^{\infty} y_m(t). \tag{3.11}$$

Similar to homotopy analysis method [15–18], we can construct the following high-order deformation equation. Differentiating the Eq.(3.9) $m(\geq 2)-$times with respect to the parameter q, dividing the resulting equation by $m!$ and setting $q = 0$, we get mth-order deformation equation

$$\mathcal{L}\left[y_m(t) - \chi_m y_{m-1}(t)\right] = \sum_{k=0}^{m-1} h_k R_{m-1-k}(t), \tag{3.12}$$

where

$$R_i(t) = \frac{1}{i!} \frac{\partial^i \mathcal{N}(\phi(t;q))}{\partial q^i}\Big|_{q=0}, \quad i = 0, 1, 2, \cdots, m-1, \tag{3.13}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{3.14}$$

Applying the inverse operator $\mathcal{L}^{-1}$ on both sides of the Eq.(3.12), we get the recursive equation

$$y_m(t) = \chi_m y_{m-1}(t) + \mathcal{L}^{-1}\left[\sum_{k=0}^{m-1} h_k R_{m-1-k}(t)\right]. \tag{3.15}$$

Combine (3.11) with (3.15), we obtain mth-order approximate analytical solution of the nonlinear Eq. (3.3).

$$\widetilde{y}_m(t) = \sum_{k=0}^{m} y_k(t). \tag{3.16}$$

The approximate analytical solution (3.16) only depends on control parameters $h_0, h_1, \cdots, h_{m-1}$. Substituting the Eq.(3.16) into the Eq.(3.3), if $\mathcal{N}(\widetilde{y}_m(t)) = 0$, then $\widetilde{y}_m(t)$ happens to be the exact solution of the Eq. (3.3). Generally such a case will not arise for the nonlinear differential Eq. (3.3), but we define the exact square residual error as [20–22]

$$E_m = \int_\Omega \left\{ N\left[\sum_{k=0}^{m} y_k(t)\right]\right\}^2 \mathrm{d}\Omega. \tag{3.17}$$

In theory, if $E_m \to 0$, then $\sum\limits_{k=0}^{+\infty} y_k(t)$ is the solution of the nonlinear Eq.(3.3). Usually, we only can make E_m minimization by choosing optimal values of convergence-control parameters $h_0, h_1, \cdots, h_{m-1}$, i.e.,

$$\min_{(h_0,h_1,\cdots,h_{m-1})\in R^m} \int_\Omega \left\{ \mathcal{N}\left[\sum_{k=0}^{m} y_k(t) \right] \right\}^2 \mathrm{d}\Omega. \tag{3.18}$$

E_m is nonlinear functions with variables $h_0, h_1, \cdots, h_{m-1}$, which corresponds to the following set of m algebraic equations

$$\begin{cases} \dfrac{\partial E_m}{\partial h_0} = 0, \\ \dfrac{\partial E_m}{\partial h_1} = 0, \\ \quad\vdots \\ \dfrac{\partial E_m}{\partial h_{m-1}} = 0. \end{cases} \tag{3.19}$$

The optimal values of h_0, h_1, $\cdots$, h_{m-1} are obtained. These optimal convergence-control parameters can be substituted into the Eq.(3.16). Thus we get the m-order approximate analytical solution of the Eq. (3.3) [20]. For higher-order approximation the Mathematica is difficult to use for the solution of the obtained nonlinear algebraic Eq.(3.19). For more details on the optimal HAM see also the references([6, 16, 17, 20–22, 24, 28] and references cited therein).

Remark 3.1. Combining Liao's HAM [16] and the exact square residual error method [20–22, 32], one can get an optimal HAM, which minimize the square residual error: at the mth-order approximation, including a set of nonlinear algebraic equation and convergence-control parameters $h_0, h_1, \cdots, h_{m-1}$. The optimal solution $(h_0, h_1, \cdots, h_{m-1})$ of the Eqs.(3.19) is stationary point. It is necessary condition for extreme value problem (3.18), not sufficient condition. Thus, the solution $(h_0, h_1, \cdots, h_{m-1})$ of the Eqs.(3.19) may be maximum, not be the minimum, also may not be an extreme point. See the following example. Moreover, it is difficult to find all roots of the nonlinear Eqs.(3.19) when the number of equations is large (greater than or equal to three).

Example 3.1. Assume that $E_m = h_0^3 + h_1^3 - 9h_0h_1 + 27$, then

$$\begin{cases} \dfrac{\partial E_m}{\partial h_0} = 3h_0^2 - 9h_1 = 0, \\ \dfrac{\partial E_m}{\partial h_1} = 3h_1^2 - 9h_0 = 0. \end{cases} \tag{3.20}$$

Thus, we get two stationary points $A_1(0, 0)$ and $A_2(3, 3)$. But, it is easy to prove that $A_1(0, 0)$ is not extreme point (minimum or maximum) and $E_m(0, 0) = 27$. However, $A_2(3, 3)$ is minimum and $E_m(3, 3) = 0$.

Remark 3.2. Liao [16,17] developed an optimal HAM with only three convergence-control parameters. There exist at most only three unknown convergence-control parameters c_0, c_1 and c_2 at any order of approximations, and adopt the averaged residual error. In essence, the present method ((3.9) − (3.19)) belongs to the infinite-parameter optimal HAM and there exist an infinite number of unknown convergence-control parameters (see Chapter 3 of [16]). In practice, we adopt the mth-order homotopy-approximation which contains finite unknown convergence-control parameters.

Since the Remark 3.1, here we propose the PSO method to deal with the optimal problem (3.18) in the optimal HAM. Thus we can avoid solving algebraic Eqs.(3.19) and get the optimal convergence-control parameters.

4. Convergence Analysis of HAM for FDE

Consider the general form of FDE

$$\begin{cases} D_*^\alpha y(t) = f(t, y(t)), \quad 0 < \alpha \le 1, \\ \quad y(0) = y_0. \end{cases} \tag{4.1}$$

Rewrite (4.1) as $D_*^\alpha y(t) - f(t, y(t)) = 0$, then the corresponding Eq.(3.3) becomes

$$\mathcal{N}[y(t)] = D_*^\alpha y(t) - f(t, y(t)) = 0. \tag{4.2}$$

Thus we construct linear operator and its inverse operator are $\mathcal{L} = D_*^\alpha$ and $\mathcal{L}^{-1} = J^\alpha$ respectively. In the following, we prove that if the series solution (3.11) is convergent, it converges to the correct solution of the nonlinear Eq.(3.3).

Theorem 4.1. *As long as the series solution (3.11) converges,*

$$\sum_{m=0}^{+\infty} R_m(t) = 0. \tag{4.3}$$

Proof. This proof is similar to Theorem 1 in [18] and Theorem 2 in [31]. Since the series (3.11) $y(t) = y_0(t) + \sum\limits_{m=1}^{\infty} y_m(t)$ converges, it can be written as $S(t) = \sum\limits_{m=0}^{+\infty} y_m(t)$ and by necessary condition for the convergence of the Series, it holds that $\lim\limits_{m\to+\infty} y_m(t) = 0$. Summation of Eq.(3.12) from $m = 1$ to $+\infty$ gives

$$\sum_{m=1}^{+\infty} D_*^\alpha \left[y_m(t) - \chi_m y_{m-1}(t)\right] = \sum_{m=1}^{+\infty} \sum_{k=0}^{m-1} h_k R_{m-1-k}(t). \tag{4.4}$$

The left-hand side can be rearranged to give

$$D_*^\alpha \left\{ \sum_{m=1}^{+\infty} \left[y_m(t) - \chi_m y_{m-1}(t)\right] \right\} = \sum_{m=1}^{+\infty} \sum_{k=0}^{m-1} h_k R_{m-1-k}(t), \tag{4.5}$$

which becomes

$$D_*^\alpha \left\{ \lim_{m\to+\infty} y_m(t) \right\} = \sum_{m=1}^{+\infty} \sum_{k=0}^{m-1} h_k R_{m-1-k}(t). \tag{4.6}$$

With $D_*^\alpha[0] = 0$, we have

$$\sum_{m=1}^{+\infty} \sum_{k=0}^{m-1} h_k R_{m-1-k}(t) = 0. \tag{4.7}$$

Since the second series of (3.10) is convergence at $q = 1$, we denote $\hbar(1) = \sum_{k=0}^{+\infty} h_k$. Rewrite (4.7) as

$$\sum_{k=0}^{+\infty} h_k R_0(t) + \sum_{k=0}^{+\infty} h_k R_1(t) + \sum_{k=0}^{+\infty} h_k R_2(t) + \cdots + \sum_{k=0}^{+\infty} h_k R_n(t) + \cdots = 0, \tag{4.8}$$

i.e., $\hbar(1) \sum_{m=0}^{+\infty} R_m(t) = 0$, With $\hbar(1) \neq 0$, we get $\sum_{m=0}^{+\infty} R_m(t) = 0$. □

Theorem 4.2. *If the series solution (3.11) converges, it must be a solution of the nonlinear equation (3.3).*

Proof. Let $\varepsilon(t; q) = N[\phi(t; q)]$ denote the residual error of Eq.(3.3). The residual error at $q = 1$ can be expanded by a Taylor series at $q = 0$ to give

$$\begin{aligned} \varepsilon(t; q = 1) &= \sum_{m=0}^{+\infty} \frac{1}{m!} \frac{\partial^m N[\phi(t; q)]}{\partial q^m} \bigg|_{q=0} \\ &= \sum_{m=0}^{+\infty} R_m(t) \\ &= 0. \end{aligned} \tag{4.9}$$

Thus, as long as the series solution (3.11) converges, it is a solution of Eq.(3.3). □

5. Approximate analytical solution of FDE

In general, there exists no method that yields an exact solution for nonlinear FDE, so approximation and numerical techniques must be used [5]. For example, the HAM [2, 8, 10], the homotopy perturbation method (HPM) [19, 26] and the variational iteration method (VIM) [25,29] have been used to provide analytical approximation to FDE. In this section, we employ optimal HAM and PSO to find out approximate analytical solutions of FDE.

Test Problem 1:

Consider the fractional-order modified logistic equation [3]

$$\begin{cases} D_*^\alpha y(t) = ry(t)(1 - \dfrac{y(t)}{k})(y(t) - n), & t \geq 0, \\ \quad y(0) = y_0, \end{cases} \tag{5.1}$$

where $0 < \alpha \leq 1$ and y_0, r, n, k are all positive constants.

We define the nonlinear operator

$$\mathcal{N}(y(t, q)) = D_*^\alpha y(t, q) + nry(t, q) - (r + \frac{nr}{k})y^2(t, q) + \frac{r}{k}y^3(t, q), \tag{5.2}$$

linear operator $\mathcal{L} = D_*^\alpha$ and its corresponding inverse operator $\mathcal{L}^{-1} = J^\alpha$. According to (3.9)–(3.16), we have

$$\begin{cases} y_m(t) = \chi_m y_{m-1}(t) + J^\alpha \left[\sum_{k=0}^{m-1} h_k R_{m-1-k}(t) \right], \\ R_{m-1-k}(t) = D_*^\alpha y_{m-1-k}(t) + nry_{m-1-k}(t) - \dfrac{kr+nr}{k} \sum_{i=0}^{m-1-k} y_i(t) y_{m-1-k-i}(t) \\ \qquad\qquad + \dfrac{r}{k} \sum_{i=0}^{m-1-k} y_{m-1-k-i}(t) \sum_{j=0}^{i} y_j(t) y_{i-j}(t). \end{cases} \tag{5.3}$$

Beginning with $y_0(t) = y_0$, by the iteration formulation (5.3), we can obtain $y_1(t) = \frac{h_0 A}{\Gamma(1+\alpha)} t^\alpha$, $y_2(t) = \frac{B}{\Gamma(1+\alpha)} t^\alpha + \frac{C}{\Gamma(1+2\alpha)} t^{2\alpha}$, $y_3(t) = \frac{D}{\Gamma(1+\alpha)} t^\alpha + \frac{E}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{F}{\Gamma(1+3\alpha)} t^{3\alpha}$, $y_4(t) = \frac{G}{\Gamma(1+\alpha)} t^\alpha + \frac{H}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{J}{\Gamma(1+3\alpha)} t^{3\alpha} + \frac{M}{\Gamma(1+4\alpha)} t^{4\alpha}, \cdots$, where $A = nry_0 - \left(r + \frac{nr}{k}\right) y_0^2 + \frac{r}{k} y_0^3$, $B = (h_0^2 + h_0 + h_1) A$, $C = \left[nr - \frac{kr+nr}{k/2} y_0 + \frac{3r}{k} y_0^2 \right] h_0^2 A$, $D = (h_0+1)B + (h_0 h_1 + h_2) A$, $E = C(1 + \frac{h_1}{h_0} + h_0 + \frac{CB}{h_0 A})$, $F = \frac{C^2}{h_0 A} + \left[3y_0 \frac{r}{k} - (r + \frac{nr}{k}) \right] \frac{h_0^3 A^2 \Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}$, $G = D + Dh_0 + Bh_1 + Ah_0 h_2 + Ah_3$, $H = E + Eh_0 + Dnrh_0 + Ch_1 + \frac{BCh_1 + C^2 h_1}{Ah_0^2} + Anrh_0 h_2 - 2Drh_0 y_0 - 2Arh_0 h_2 y_0 - \frac{2Dnrh_0 y_0 + 2Anrh_0 h_2 y_0 - 3Drh_0 y_0^2 - 3Arh_0 h_2 y_0^2}{k}$, $J = F(1 + h_0) + Enrh_0 - 2Erh_0 y_0 - \frac{2ABr\Gamma(1+2\alpha) h_0^2 + A^2 r \Gamma(1+2\alpha) h_0^2 h_1}{\Gamma^2(1+\alpha)} - \frac{2ABnr\Gamma(1+2\alpha) h_0^2 + A^2 nr \Gamma(1+2\alpha) h_0^2 h_1 - 6ABr\Gamma(1+2\alpha) h_0^2 y_0}{k \Gamma^2(1+\alpha)} + \frac{3A^2 r h_0^2 h_1 y_0}{k \frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)}} - \frac{2Enrh_0 y_0 - 3Erh_0 y_0^2}{k}$, $M = Fnrh_0 - 2Frh_0 y_0 - \frac{2ACr\Gamma(1+3\alpha) h_0^2}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} - \frac{2ACnr\Gamma(1+3\alpha) h_0^2}{k\Gamma(1+\alpha)\Gamma(1+2\alpha)} + \frac{6ACr\Gamma(1+3\alpha) h_0^2 y_0}{k\Gamma(1+\alpha)\Gamma(1+2\alpha)} - \frac{2Fnrh_0 y_0 - 3Frh_0 y_0^2}{k} + \frac{A^3 r \Gamma(1+3\alpha) h_0^4}{k \Gamma^3(1+\alpha)}$. Thus we get the 4th-order approximate analytical solution of the Eq.(5.1) as following.

$$\widetilde{y}_m(t) = y_0 + \frac{h_0 A + B + D + G}{\Gamma(1+\alpha)} t^\alpha + \frac{C + E + H}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{F + J}{\Gamma(1+3\alpha)} t^{3\alpha} + \frac{M}{\Gamma(1+4\alpha)} t^{4\alpha}. \tag{5.4}$$

With the exact square residual error (3.17), we obtain

$$F_2 = \int_{t_0}^{t} \left[D_*^\alpha \widetilde{y}_m(t) + nr\widetilde{y}_m(t) - \left(r + \frac{nr}{k}\right) \widetilde{y}_m^2(t) + \frac{r}{k} \widetilde{y}_m^3(t) \right]^2 \mathrm{d}t, \tag{5.5}$$

where $D_*^\alpha \widetilde{y}_m(t) = h_0 A + B + D + G + \frac{C+E+H}{\Gamma(1+\alpha)} t^\alpha + \frac{F+J}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{M}{\Gamma(1+3\alpha)} t^{3\alpha}$. Based on (5.5), we get the following optimization problem

$$\min_{(h_0, h_1, h_2, h_3) \in R^4} F_2. \tag{5.6}$$

Now, fix the parameters $r = 0.5, n = 1, k = 10$, $\alpha = 0.98, y_0 = 0.8$, $t \in [0, 3]$, we use PSO to deal with the optimization problem (5.6) and find the optimal convergence-control parameters h_0, h_1, h_2 and h_3. The whole design steps can be summarized as follows (Matlab software).

Step 1. Define global variables: swarm size, inertia weight factor, maximum iteration number, etc.

Step 2. Construct initial subfunction (initial.m), it contains maximum number of iterations, maximal velocity, c_1, c_2, swarm size n and swarm *pop*, and so on.

$pop =$

$$\begin{pmatrix} h_{01} & \dots & h_{31} & v_{01} & \dots & v_{31} & pbesth_{01} & \dots & pbesth_{31} & fbest_1 & fit_1 \\ h_{02} & \dots & h_{32} & v_{02} & \dots & v_{32} & pbesth_{02} & \dots & pbesth_{32} & fbest_2 & fit_2 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\ h_{0n} & \dots & h_{3n} & v_{0n} & \dots & v_{3n} & pbesth_{0n} & \dots & pbesth_{3n} & fbest_n & fit_n \end{pmatrix}_{n\times 14}.$$

In the matrix *pop*, the first four columns $pop(:,1) \sim pop(:,4)$ storage positions of particles $h_0 \sim h_3$ respectively. Their initial values are random numbers which lie in interval $[-1,1]$; The columns $pop(:,5) \sim pop(:,8)$ represent velocity of particles $h_0 \sim h_3$; The columns $pop(:,9) \sim pop(:,12)$ respectively represent best positions of particles $h_0 \sim h_3$ in history; The $pop(:,13)$ represents the best fitness value of every particles. The column $pop(:,14)$ is fitness value of the all current particles. Then, at the kth-times iteration, global best fitness value of F_2 is $bestfitness(k) = \min(pop(:,13)) = \min fbest_i, \quad i = 1, 2 \cdots, n$.

Step 3. Construct a subfunction (adapting.m), including fitness function based on the Eq.(5.5). Calculating fitness value of every particle, it is stored in $pop(:,14)$. Comparison of $pop(:,14)$ and $pop(:,13)$, updating $pop(:,9) \sim pop(:,13)$, then we get global best fitness value $bestfitness(k) = \min(pop(:,13))$ at kth-times iteration, and obtain situation of $min(pop(:,13))$, i.e., $(i,13)$. Then, the optimal convergence-control parameters $h_0 = pop(i,9), h_1 = pop(i,10), h_2 = pop(i,11), h_3 = pop(i,12)$ in all k iterations. We save them in matrices $trace$, $trace2$, $trace3$, $trace4$, and save $bestfitness(k)$ in $besthistory$.

Step 4. Based on the Eqs. (3.1) and (3.2), construct updating function of velocity and situation, i.e., updatepop.m.

Step 5. Repeat steps 1-4, until stop criterion (Maximum number of iterations) is reached, get optimal convergence-control parameters: $h_0 = -0.85241709290217$, $h_1 = -0.31797308168261$, $h_2 = -0.05745525388485$, $h_3 = -0.04168853741609$. At this time, the optimal fitness value is $\min\limits_{(h_0,h_1,h_2,h_3)} F_2 = 7.364986400000000 \times 10^{-6}$, $\widetilde{y}_m(t) = 0.8 - 0.7257098306 \times 10^{-1} t^{\frac{49}{50}} - 0.1141039014 \times 10^{-1} t^{\frac{49}{25}} + 0.2050324723 \times 10^{-3} t^{\frac{147}{50}} + 0.8869896610 \times 10^{-4} t^{\frac{98}{25}}$.

When $\alpha = 1$, $r = 0.5$, $n = 1$, $k = 10$, $y_0 = 0.8$, the exact solution of Eq.(5.1) is

$$\frac{1}{10}\ln|y| + \frac{1}{90}\ln|y-10| - \frac{1}{9}\ln|y-1| + \frac{1}{20}t = c_0, \tag{5.7}$$

where $c_0 = \frac{1}{10}\ln|y_0| + \frac{1}{90}\ln|y_0-10| - \frac{1}{9}\ln|y_0-1|$.

Enlarge $bestfitness$ with 10^5, plot $trace$, $trace2$, $trace3$, $trace4$ (enlarge all of them with 10) and $besthistory$, the results are as showing in Figure 1. Approximate analytical solution of Optimal HAM using the 4-term, the numerical solution and the exact solution have been plotted in Figure 2. The exact solution (5.7) is implicit form. It has two branches (labeled with $--$) as shown in Figure 2. The upper branch does not satisfy the initial condition ($y_0 = 0.8$). Thus we adopt the lower branch. It is explicitly shown that an agreement between the derived solution, numerical solution and exact solution is excellent. An almost complete overlap can be observed.

Remark 5.1. In this method, we used the objective function value ($\min F_2$) to measure the extent of optimal convergence-control parameters ($h_0 \sim h_3$). If the $\min F_2$ is approximately equal to zero, then the corresponding $h_0 \sim h_3$ are better.

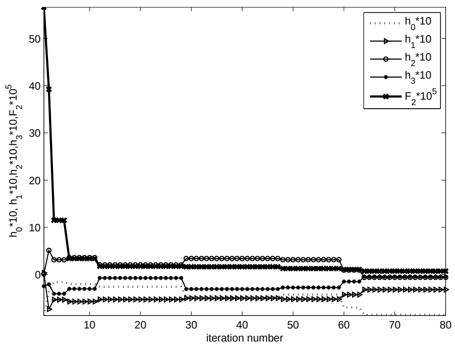

Figure 1. Convergence process of h_0-h_3 and fitness value F_2

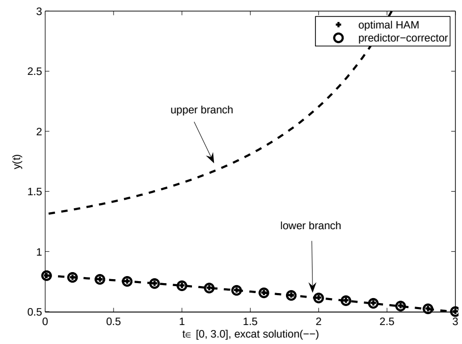

Figure 2. Comparison between optimal HAM solution, numerical solution and exact solution

If $\min F_2 \geq 0.1$, then we can relax the initial range of $h_0 \sim h_3$ and re-optimized procedure is necessary in practice.

Test Problem 2:

Consider the following fractional-order Riccati differential equation [11]

$$\begin{cases} D_*^\alpha y(t) = -y^2(t) + 1, \quad 0 < \alpha \leq 1, \\ y(0) = 0. \end{cases} \tag{5.8}$$

Note that the exact solution of (5.8) is $y(t) = \frac{e^{2t}-1}{e^{2t}+1}$ when $\alpha = 1$.

Construct the nonlinear operator

$$\mathcal{N}(y(t,q)) = D_*^\alpha y(t,q) + y^2(t,q) - 1, \tag{5.9}$$

linear operator $\mathcal{L} = D_*^\alpha$ and its corresponding inverse operator $\mathcal{L}^{-1} = J^\alpha$. According to (3.9)–(3.16) and beginning with $y_0(t) = y_0$, we get the 4th-order approximate analytical solution of the Eq.(5.8).

$$\widetilde{y}_m(t) = y_0 + \frac{h_0A+B+D+G}{\Gamma(1+\alpha)}t^\alpha + \frac{C+E+H}{\Gamma(1+2\alpha)}t^{2\alpha} + \frac{F+J}{\Gamma(1+3\alpha)}t^{3\alpha} + \frac{M}{\Gamma(1+4\alpha)}t^{4\alpha}, \tag{5.10}$$

where $A = y_0^2 - 1, B = (h_0^2 + h_0 + h_1)A, C = 2y_0h_0^2A,\ D = B + h_1h_0A + h_2A + h_0B,$ $E = C + 2y_0h_1h_0A + h_0C + 2h_0y_0B,\ F = 2h_0y_0C + h_0^3A^2\Gamma(1+2\alpha)/\Gamma^2(1+\alpha),$ $G = (1+h_0)D + h_1B + h_2h_0A + h_3A, H = (1+h_0)E - 2h_0y_0D + h_1C + 2h_1y_0B +$ $2y_0h_2h_0A, J = (1+h_0)F - 2h_0y_0E - \frac{2h_0^2AB\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)} + 2h_1y_0C + \frac{h_1h_0^2A^2\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)}, M =$ $-2h_0y_0F - \frac{2h_0^2AC\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}$. With the exact square residual error (3.17), we obtain

$$F_2 = \int_{t_0}^{t} \left[D_*^\alpha \widetilde{y}_m(t) + \widetilde{y}_m^2(t) - 1\right]^2 \mathrm{d}t, \tag{5.11}$$

where $D_*^\alpha \widetilde{y}_m(t) = h_0A + B + D + G + \frac{C+E+H}{\Gamma(1+\alpha)}t^\alpha + \frac{F+J}{\Gamma(1+2\alpha)}t^{2\alpha} + \frac{M}{\Gamma(1+3\alpha)}t^{3\alpha}$. Based on the Eq. (5.11), we get the following optimization problem

$$\min_{(h_0,h_1,h_2,h_3)\in R^4} F_2. \tag{5.12}$$

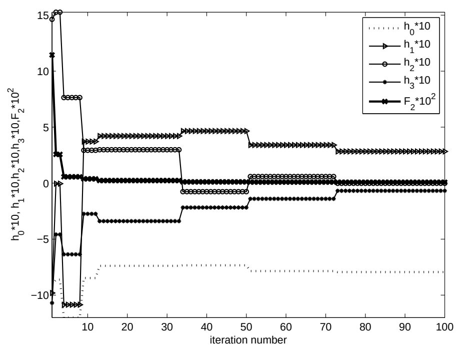

Figure 3. Convergence process of h_0-h_3 and fitness value F_2

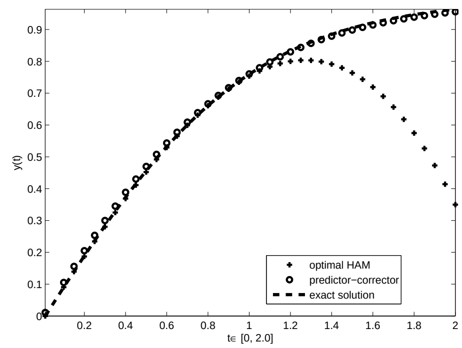

Figure 4. Comparison between optimal HAM solution, numerical solution and exact solution

Now, fix the order $\alpha = 0.98$ and $t \in [0, 2]$, we use PSO algorithm to deal with this optimization problem (5.12) and get optimal convergence-control parameters: $h_0 = -0.79423340098725, h_1 = 0.28262642620470$, $h_2 = -0.00217053743085, h_3 = -0.06866543558425$. At this time, the optimal fitness value is $\min_{(h_0,h_1,h_2,h_3)} F_2 = 7.893640106000000 \times 10^{-4}$ and $\widetilde{y}_m(t) = 0.9599487046 \times t^{\frac{49}{50}} - 0.2029082450 \times t^{\frac{147}{50}}$. Now, enlarge F_2 with 10^2, plot h_0, h_1, h_2, h_3(enlarge all of them with 10), the results are as showing in Figure 3. Approximate analytical solution of Optimal HAM using the 4-term, the numerical solution and the exact solution have been plotted in Figure 4. It is explicitly shown that an agreement between the derived solution, numerical solution and exact solution is excellent.

Remark 5.2. During the operation of each procedure, the value of $h_0 \sim h_3$ may be different, but $\min F_2$ is stable in the vicinity of 10^{-6} in Test Problem 1 and 10^{-4} in Test Problem 2 respectively. The smaller value of $\min F_2$ ensure that the parameters $h_0 \sim h_3$ are more efficient.

Remark 5.3. Theoretically, PSO can deal with the optimal problem $\min F_2$ with finite parameters $h_0 \sim h_n$. But the iteration process of optimal HAM is complicated when n is large. Thus, the objective function F_2 is complex and the process of finding the optimal parameters may be time-consuming. The obtained results (Figure 4) are consistent with the Example 4.1 in [11].

6. Another form of residual error

Firstly, we give the following mean value theorem for multiple integral.

Theorem 6.1 ([7]). *Let D is bounded closed region in $\mathbb{R}^n$, $m(\partial D) = 0$, $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is continuous in D, $g : D \subset \mathbb{R}^n \to \mathbb{R}$ is Lebesgue integral in D, and $g(P)$ is non-negative almost everywhere in D, then there exists at least an interior point P_0 in D, such that*

$$\int_D f(P)g(P)\mathrm{d}\sigma = f(P_0)\int_D g(P)\mathrm{d}\sigma, \tag{6.1}$$

where $m(\partial D) = 0$ is Lebesgue measure of boundary ∂D. $\int_D g(P)d\sigma$ is Lebesgue integral in D.

When $g(P) = 1$, (6.1) becomes

$$\int_D f(P)d\sigma = f(P_0)m(D). \tag{6.2}$$

Based on the Theorem 6.1, the special case (6.2) and (3.18), we can construct the following another form of residual error.

$$\min_{(h_0,\cdots,h_{m-1})\in R^m} \int_\Omega \left|\mathcal{N}\left[\sum_{k=0}^{m} y_k(t)\right]\right| \mathrm{d}\Omega = \min_{(h_0,\cdots,h_{m-1})\in R^m \cup t_0} \left|\mathcal{N}\left[\sum_{k=0}^{m} y_k(t_0)\right]\right| m(\Omega), \tag{6.3}$$

where $t_0 \in \Omega$. In practice, $m(\Omega)$ is fixed, thus we only need to handle the following optimization problem with optimization algorithm.

$$\min_{(h_0,\cdots,h_{m-1})\in R^m \cup t_0 \in \Omega} \left|\mathcal{N}\left[\sum_{k=0}^{m} y_k(t_0)\right]\right|. \tag{6.4}$$

Test Problem 3:

Consider the following fractional-order logistic differential equation [23]

$$\begin{cases} D_*^\alpha y(t) = \rho y(t)(1 - y(t)), \\ y(0) = y_0, \end{cases} \tag{6.5}$$

where $t > 0, 0 < \alpha \leq 1, \rho > 0$. Note that the exact solution of (6.5) is $y(t) = \frac{e^{\rho t}}{1+e^{\rho t}}$ when $\alpha = 1$. The first three terms of HAM series solution are as follows: $y_1(t) = \frac{A}{\Gamma(\alpha+1)}t^\alpha$, $y_2(t) = \frac{B}{\Gamma(\alpha+1)}t^\alpha + \frac{C}{\Gamma(2\alpha+1)}t^{2\alpha}$, $y_3(t) = \frac{D}{\Gamma(\alpha+1)}t^\alpha + \frac{E}{\Gamma(2\alpha+1)}t^{2\alpha} + \frac{F}{\Gamma(3\alpha+1)}t^{3\alpha}$, where $A = \rho h(y_0^2 - y_0)$, $B = (1 + h)A$, $C = \rho^2 h^2 (y_0^2 - y_0)(2y_0 - 1)$, $D = (1 + h)B$, $E = (1 + h)C + \rho^2 h^2 (1 + h)(y_0^2 - y_0)(2y_0 - 1)$, $F = \rho^3 h^3 (y_0^2 - y_0)(2y_0 - 1)^2 + \rho^2 h^2 (y_0^2 - y_0)\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}$. Hence, the 3-order approximate solution of problem (6.5) with one homotopy parameter h can be given by

$$\widetilde{y}_m(t) = y_0 + \frac{(A + B + D)t^\alpha}{\Gamma(\alpha + 1)} + \frac{(C + E)t^{2\alpha}}{\Gamma(2\alpha + 1)} + \frac{Ft^{3\alpha}}{\Gamma(3\alpha + 1)} \tag{6.6}$$

and $D_*^\alpha \widetilde{y}_m(t) = A+B+D+\frac{C+E}{\Gamma(\alpha+1)}t^\alpha + \frac{F}{\Gamma(2\alpha+1)}t^{2\alpha}$. Construct $F_2 = \left|\mathcal{N}\left[\sum_{i=0}^{3} y_i(t)\right]\right|$ and get the optimization problem

$$\min_{h\in R\cup t\in\Omega} F_2, \tag{6.7}$$

where $\mathcal{N}(y(t)) = D_*^\alpha y(t) - \rho y(t)(1 - y(t))$. Now, fix the order $\alpha = 0.98$, $\Omega = (0, 2]$ and $\rho = y_0 = 0.5$, we use PSO algorithm to deal with this optimization problem (6.7) and get optimal convergence-control parameters: $h = -0.99999809125743, t = 9.083730616909948 \times 10^{-10}$, $F_2 = 0$. The numerical results are shown in Figures 5 and 6.

Remark 6.1. During the operation of each procedure, the value of h and t may be different, but F_2 and h are stable in the vicinity of 0 and -1 respectively. The results obtained in this section are consistent with ones of h-curve [23]. Where, the valid region of h is a horizontal line segment. It's a rough range. But in this method, we get a specific value of h. This is more likely to be chosen.

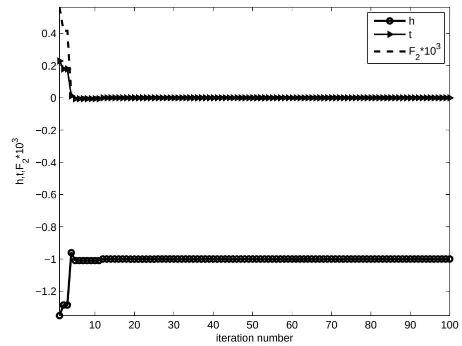

Figure 5. Convergence process of h, t and fitness value F_2

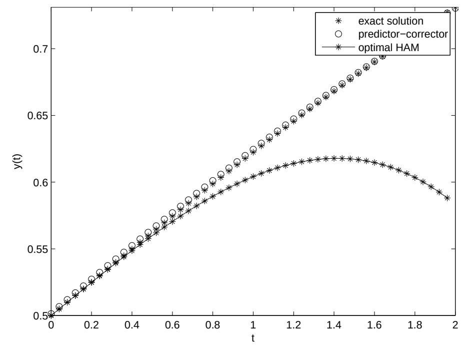

Figure 6. Comparison between optimal HAM solution, numerical solution and exact solution

7. Conclusions

There are two important goals that we have achieved in the present paper. First one is employing the PSO to deal with the optimal problem of optimal HAM. This method is different from the previous literatures which are based on the necessary conditions for the existence of extremum. The PSO algorithm avoids solving nonlinear equations. It is directly applied to deal with optimal problem $\min F_2$, then get optimal convergence-control parameters. Meanwhile, finding better global optimization algorithm for this problem can be an interesting topic for future research work. Another important part of the study is to present optimal HAM to solving approximate analytical solutions of nonlinear FDE. It is usually difficult to obtain the exact solution. The presented examples show that the results (approximate analytical solution) of the proposed method are in excellent agreement with numerical solution and exact solution. In addition, the problem of convergence of HAM for FDE has also been studied. All of these indicate that the combination of optimal HAM and PSO is very effective. Finally, the new form of residual error proposed here requires further study.

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References

[1] O. Abdulaziz, A. Bataineh and I. Hashim, *On convergence of homotopy analysis method and its modification for fractional modified KdV equations*, J. Appl. Math. Comput., 33(2010), 61–81.

[2] A. Alomari, M. Noorani, R. Nazar and C. Li, *Homotopy analysis method for solving fractional Lorenz system*, Commun. Nonlinear Sci., 15(2010), 1864–1872.

[3] S. Abbas, M. Banerjee and S. Momani, *Dynamical analysis of fractional-order modified logistic model*, Comput. Math. Appl., 62(2011), 1098–1104.

[4] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer Press, New York, 2010.

[5] K. Diethelm, N. Ford and A. Freed, *A predictor-corrector approach for the numerical solution of fractional differential equation*, Nonlinear Dynam., 29(2002), 3–22.

[6] T. Fan and X. You, *Optimal homotopy analysis method for nonlinear differential equations in the boundary layer*, Numer. Algorithms, 62(2013), 337–354.

[7] J. Fan and B. Yang, *The mean value theorem for multiple integral*, Math. Pract. Theory, 37(2007), 197–200. (in Chinese)

[8] I. Hashim, O. Abdulaziz and S. Momani, *Homotopy analysis method for fractional IVPs*, Commun. Nonlinear Sci., 14(2009), 674–684.

[9] J. Izadian, M. MohammadzadeAttar and M. Jalili, *Numerical solution of deformation equations in homotopy analysis method*, Appl. Math. Sci., 6(2012), 357–367.

[10] H. Jafari, K. Sayevand, H. Tajadodi and D. Baleanu, *Homotopy analysis method for solving Abel differential equation of fractional order*, Cent. Eur. J. Phys., 11(2013), 1523–1527.

[11] H. Jafari, H. Tajadodi and D. Baleanu, *A modified variational iteration method for solving fractional riccati differential equation by adomian polynomials*, Fract. Calc. Appl. Anal., 16(2013), 109–122.

[12] A. Kilbas, H.Srivastava and J. Trujillo, *Theory and Application of Fractional Differential Equations*, Elsevier Science, Netherlands, 2006.

[13] J. Kennedy and R. Eberhart, *Particle swarm optimization*, Proceedings IEEE International Conference on Neural Networks, 4(1995), 1942–1948.

[14] J. Kennedy, R. Eberhart and Y. Shi, *Swarm Intelligence*, CA: Morgan Kaufman, San Francisco, 2001.

[15] S. Liao, *Beyond perturbation: introduction to the homotopy analysis method*, Chapman&Hall/CRC Press, Boca Raton, 2003.

[16] S. Liao, *Homotopy analysis method in nonlinear differential equations*, Higher Education Press, Beijing, 2012.

[17] S. Liao, *An optimal homotopy-analysis approach for strongly nonlinear differential equations*, Commun. Nonlinear Sci., 15(2010), 2003–2016.

[18] S. Liao, *An explicit, totally analytic approximation of Blasius viscous flow problems*, Int. J. Nonlin. Mech., 34(1999), 759–778.

[19] Y. Liu, Z. Li and Y. Zhang, *Homotopy perturbatioin method to fractional biological population equation*, Fract. Diff. Calc., 1(2011), 117–124.

[20] V. Marinca, N. Herisanu and I. Nemes, *Optimal homotopy asymptotic method with application to thin film flow*, Cent. Eur. J. Phys., 6(2008), 648–653.

[21] V. Marinca and N. Herisanu, *Application of optimal homotopy asymptotic method for solving nonlinear equations arising in heat transfer*, Int. Commun. Heat Mass, 35(2008), 710–715.

[22] V. Marinca and N. Herisanu, *An optimal homotopy asymptotic method applied to the steady flow of a fourth-grade fluid past a porous plate*, Appl. Math. Lett., 22(2009), 245–251.

[23] M. Mohamed, *Application of optimal HAM for solving the fractional order logistic equation*, Appl. Comput. Math., 3(2014), 27–31.

[24] Z. Niu and C. Wang, *A one-step optimal homotopy analysis method for nonlinear differential equations*, Commun. Nonlinear Sci., 15(2008), 2026–2036.

[25] Y. Nawaz, *Variational iteration method and homotopy perturbation method for fourth-order fractional integro-differential equations*, Comput. Math. Appl., 61(2011), 2330–2341.

[26] Z. Odibat and S. Momani, *Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order*, Chaos, Soliton Fract., 36(2008), 167–174.

[27] N. Shaheed and R.Said, *Approximate solutions of nonlinear partial differential equations by modified-homotopy analysis method*, J. Appl. Math., (2013). DOI:10.1155/2013/569674.

[28] N. Saberi and S. Shateyi, *Application of Optimal HAM for finding feedback control of optimal control problems*, Math. Probl. Eng., (2013). DOI: 10.1155/2013/914741.

[29] N. Sweilam, M. Khader and A. Mahdy, *Numerical studies for solving fractional-order logistic equation*, Int. J. Pure Appl. Math., 78(2012), 1199–1210.

[30] V. Uchaikin, *Fractional Derivatives for Physicits and Engineers*, Higher Educatioin Press, Beijin, 2013.

[31] Y. Wu and K. Cheung, *Homotopy solution for nonlinear differential equations in wave propagation problems*, Wave Motion, 46(2009), 1–14.

[32] K. Yabushita, M. Yamashita and K. Tsuboi, *An analytic solution of projectile motion with the quadratic resistance law using the homotopy analysis method*, J. Phys. A: Math. Theor., 40(2007), 8403–8416.

[33] L. Yuan, Q. Yang and C. Zeng, *Chaos detection and parameter identification in fractional-order chaotic systems with delay*, Nonlinear Dynam., 73(2013), 439–448.

[34] X. Zhou, C. Yang, W. Gui and T. Dong, *A particle swarm optimization algorithm with variable random functions and mutation*, Acta Automatica Sinica, 40(2014), 1339–1347.