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IMPULSIVE SYNCHRONIZATION OF TIME-VARYING DYNAMICAL NETWORK*

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Abstract Synchronization of time-varying dynamical network is investigated via impulsive control. Based on the Lyapunov function method and stability theory of impulsive differential equation, a synchronization criterion with respect to the system parameters and the impulsive gains and intervals is analytically derived. Further, an adaptive strategy is introduced for designing unified impulsive controllers, with a corresponding synchronization criterion derived. In this proposed adaptive control scheme, the impulsive instants adjust themselves to the needed values as time goes on, and an algorithm for determining the impulsive instants is provided and evaluated. The derived theoretical results are illustrated to be effective by several numerical examples.

Keywords Synchronization, time-varying network, impulsive control, adaptive strategy.

MSC(2010) 34D20, 34C28, 93D21.

1. Introduction

Many large-scale real systems consisting of interactive individuals are usually modeled and studied by complex dynamical networks [1, 2, 4]. In dynamical networks, the nodes denote the individuals and the edges denote the interactions among the individuals. For better describing the real systems, many kinds of network models are introduced, such as, weighted networks [12, 13], directed networks [10, 16], colored networks [17,18], time-varying networks [3,5,11,14,19], and so on. As a typical collective dynamical behavior of complex networks, synchronization has been found and studied in many fields from biology to human society to Internet. In practical applications, synchronization can be beneficial. For example, in computer science, especially in parallel computing, synchronization means the coordination of simultaneous threads or processes to complete a task of obtaining a correct runtime order while avoiding unexpected race conditions [4]. On the other hand, many real systems cannot achieve synchronization themselves without external control due to their complexity. Therefore, how to design effective controllers is an important issue.

Impulsive control is a typical discrete control scheme, in which the controllers are applied on the nodes only at a sequence of discrete instants. That is, the impulsive controllers have a relatively simple structure and are easy to implement

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and low-cost. Therefore, many valuable results about synchronization of dynamical networks via impulsive control have been obtained [6, 7, 9, 15, 20]. The key point in designing impulsive controllers is to derive the conditions for estimating the impulsive gains and intervals. Usually, the impulsive gains and intervals are affected by some constants, such as, the Lipschitz-like constant with respect to the node dynamics and the largest eigenvalue with respect to the coupling matrices. For any given dynamical network, one can easy to choose proper impulsive gains and intervals when the constants are known. However, different dynamical networks may have totally different system parameters, i.e., the impulsive controllers with fixed impulsive gains and intervals are not unified. Further, in some cases, the constants may be difficult to calculate. For example, in time-varying networks, the outer coupling matrix changes along time and its largest eigenvalue is hard to estimated. Naturally, how to design effective and unified impulsive controllers is a challenging problem and deserves further studies.

Motivated by the above discussions, this paper consider the synchronization problem of time-varying network via impulsive control. In Section 2, the time-varying network model and some preliminaries are introduced. In Section 3, the synchronization of time-varying network is studied through designing proper impulsive controllers. Firstly, some synchronization conditions with respect to the system parameters and the impulsive gains and intervals are analytically derived. Secondly, proper adaptive strategy is introduced in impulsive controllers for designing unified controllers. In this control scheme, a parameter with adaptive updating law is introduced to estimate the constants with respect to node dynamics and coupling matrices. That is, the constants need not to be calculated beforehand and the impulsive gains or instants can be chosen or estimated according to the proposed adaptive strategy. In Section 4, some numerical simulations are performed to verify the correctness and effectiveness of the obtained results. In Section 5, this paper is concluded.

2. Model description and preliminaries

Consider a time-varying dynamical network consisting of N nodes, described by

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1, j\neq i}^{N} c_{ij}(t)H(x_j(t) - x_i(t)),\ i = 1, 2, \cdots, N, \tag{2.1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T \in R^n$ is the state vector of node i, $f : R^n \to R^n$ is a nonlinear vector-valued function, $H = diag(h_1, h_2, \cdots, h_n) \in R^{n\times n}$ is the inner coupling matrix, $C(t) = (c_{ij}(t))_{N\times N}$ is the zero-row-sum outer coupling matrix at time t, defined as: if there is a connection between nodes i and j at time t, then $c_{ij}(t) = c_{ji}(t) \neq 0$ $(i \neq j)$; otherwise, $c_{ij}(t) = 0$, and the diagonal elements of matrix $C(t)$ are defined by $c_{ii}(t) = -\sum_{j=1,j\neq i}^{N} c_{ij}(t)$. The objective here is to synchronize network (2.1) with a desired orbit $s(t)$ through designing proper impulsive controllers, where $s(t)$ is a solution of an isolated node satisfying $\dot{s}(t) = f(s(t))$. The controlled network can be described by

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} c_{ij}(t)Hx_j(t), \qquad t \neq t_k,$$

$$x_i(t_k^+) = x_i(t_k^-) + b(t_k)(x_i(t_k^-) - s(t_k)), \qquad t = t_k, \tag{2.2}$$

where t_k $(k = 1, 2, \cdots)$ are the impulsive instants and satisfy $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, and $t_k \to \infty$ as $k \to \infty$, $x_i(t_k^+) = \lim_{t \to t_k^+} x_i(t)$, $x_i(t_k^-) = \lim_{t \to t_k^-} x_i(t)$, any solution of (2.2) is assumed to be left continuous at each t_k, i.e., $x_i(t_k^-) = x_i(t_k)$, $b(t_k)$ is the impulsive gain at $t = t_k$ and $b(t) = 0$ for $t \neq t_k$.

Let $e_i(t) = x_i(t) - s(t)$ be the synchronization error, one has the following error system

$$\begin{aligned} \dot{e}_i(t) &= f(x_i(t)) - f(s(t)) + \sum_{j=1}^{N} c_{ij}(t) H e_j(t), && t \neq t_k, \\ e_i(t_k^+) &= e_i(t_k^-) + b(t_k) e_i(t_k^-), && t = t_k. \end{aligned} \tag{2.3}$$

Assumption 2.1. Suppose that there exists a positive constant L such that

$$(y(t) - x(t))^T (f(y(t)) - f(x(t))) \leq L(y(t) - x(t))^T (y(t) - x(t)),$$

hold for any $x(t), y(t) \in R^n$ and $t > 0$.

Remark 2.1. It is easy to check that many typical chaotic systems, such as the Chen, Lü and Lorenz systems, all satisfy the Assumption 2.1.

3. Main results

Let $e(t) = (e_1^T(t), e_2^T(t), \cdots, e_N^T(t))^T$, $\tau_k = t_k - t_{k-1}$ denote impulsive intervals, $\beta(t_k) = (1 + b(t_k))^2$ and $\beta(t) = 1$ for $t \neq t_k$, λ is the largest eigenvalue of $LI_N \otimes I_n + C(t) \otimes H$ for any $t > 0$. Throughout this paper, the entries of matrix $C(t)$ are assumed to be bounded, i.e., the largest eigenvalue λ is bounded.

Theorem 3.1. *Suppose that Assumption 2.1 holds. If there exists a positive constant $\alpha > 0$ such that the following conditions*

$$\ln \beta(t_k) + \alpha + 2\lambda \tau_k < 0, \ \ k = 1, 2, 3, \cdots, \tag{3.1}$$

hold, then the synchronization of controlled network (2.2) is achieved.

Proof. Consider the following Lyapunov function:

$$V(t) = \frac{1}{2} \sum_{i=1}^{N} e_i^T(t) e_i(t),$$

for $t \in (t_{k-1}, t_k]$, $k = 1, 2, \cdots$.

When $t \in (t_{k-1}, t_k)$, the derivative of $V(t)$ along the solution of (2.3) gives

$$\begin{aligned} \dot{V}(t) &= \sum_{i=1}^{N} e_i^T(t) \dot{e}_i(t) \\ &= \sum_{i=1}^{N} e_i^T(t)(f(x_i(t)) - f(s(t))) + \sum_{i=1}^{N} \sum_{j=1}^{N} c_{ij}(t) e_i^T(t) H e_j(t) \\ &\leq e^T(t)(LI_N \otimes I_n + C(t) \otimes H) e^T(t) e(t) \\ &\leq \lambda e^T(t) e(t) \\ &= 2\lambda V(t), \end{aligned}$$

which gives

$$V(t) \le V(t_{k-1}^{+}) \exp(2\lambda(t - t_k)),\ t \in (t_{k-1}, t_k). \tag{3.2}$$

When $t = t_k$, one has

$$\begin{aligned} V(t_k^{+}) &= \frac{1}{2} \sum_{i=1}^{N} e_i^T(t_k^{+}) e_i(t_k^{+}) \\ &= \frac{(1 + b(t_k))^2}{2} \sum_{i=1}^{N} e_i^T(t_k^{-}) e_i(t_k^{-}) \\ &= \beta(t_k) V(t_k^{-}). \end{aligned} \tag{3.3}$$

For $k = 1$, from inequalities (3.2) and (3.3), one has

$$\begin{aligned} &V(t_1^{-}) \le V(t_0) \exp(2\lambda\tau_1), \\ &V(t_1^{+}) \le \beta(t_1) V(t_1^{-}) \le \beta(t_1) V(t_0) \exp(2\lambda\tau_1). \end{aligned}$$

For $k = 2$, we have

$$\begin{aligned} V(t_2^{-}) &\le V(t_1^{+}) \exp(2\lambda\tau_2) \\ &\le \beta(t_1) V(t_0) \exp(2\lambda\tau_2 + 2\lambda\tau_1), \\ V(t_2^{+}) &\le \beta(t_2) V(t_2^{-}) \\ &\le \beta(t_2)\beta(t_1) V(t_0) \exp(2\lambda\tau_2 + 2\lambda\tau_1) \\ &= V(t_0) \prod_{j=1}^{2} (\beta(t_j) \exp(2\lambda\tau_j)). \end{aligned}$$

By mathematical induction, one has

$$V(t_k^{+}) \le V(t_0) \prod_{j=1}^{k} \beta(t_j) \exp(2\lambda\tau_j).$$

If condition (3.1) holds, one has

$$\beta(t_j) \exp(2\lambda\tau_j) \le \exp(-\alpha),\ j = 1, 2, \cdots,$$

and

$$V(t_k^{+}) \le V(t_0) \exp(-k\alpha),$$

which implies

$$\lim_{k\to\infty} V(t_k^{+}) = 0.$$

Then, for $t \in (t_k, t_{k+1}]$, one has

$$V(t) \le V(t_k^{+}) \exp(2\lambda(t - t_k)),$$

which gives $V(t) \to 0$ as $t \to \infty$, i.e., the synchronization is achieved. This completes the proof. $\square$

Remark 3.1. From conditions (3.1) in Theorem 3.1, for any given time-varying dynamical network, one can easily choose proper impulsive gains $b(t_k)$ and impulsive intervals τ_k for achieving the synchronization when the largest eigenvalue of $LI_N \otimes I_n + C(t) \otimes H$ is calculated.

Remark 3.2. Clearly, the largest eigenvalue λ has great effect on the impulsive gains and intervals and is not easy to be calculated. Thus, how to introduce proper adaptive strategy into impulsive controllers for estimating the largest eigenvalue λ is a key issue. Further, different dynamical networks may have totally different system parameters, i.e., the impulsive controllers with fixed impulsive gains and intervals are not valid for different networks.

Theorem 3.2. *Suppose that Assumption 2.1 holds. If there exists a positive constant $\alpha > 0$ such that the following conditions*

$$\ln \beta(t_k) + \alpha + 2\widehat{\lambda}(t_k)\tau_k < 0, \;\; k = 1, 2, 3, \cdots, \tag{3.4}$$

hold, where $\widehat{\lambda}(t)$ is the estimation of λ, $\dot{\widehat{\lambda}}(t) = \eta \sum_{i=1}^{N} e_i^T(t)e_i(t)$ and $\eta > 0$ is a positive constant, then the synchronization of controlled network (2.2) is achieved.

Proof. Consider the following Lyapunov function:

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)e_i(t) + \frac{\beta(t)}{2\eta}(\widehat{\lambda}(t) - \lambda)^2$$

for $t \in (t_{k-1}, t_k]$, $k = 1, 2, \cdots$.

When $t \in (t_{k-1}, t_k)$, the derivative of $V(t)$ along the solution of (2.3) gives

$$\begin{aligned}
\dot{V}(t) =& \sum_{i=1}^{N} e_i^T(t)\dot{e}_i(t) + \frac{1}{\eta}(\widehat{\lambda}(t) - \lambda)\dot{\widehat{\lambda}}(t) \\
=& \sum_{i=1}^{N} e_i^T(t)(f(x_i(t)) - f(s(t))) + \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij}(t)e_i^T(t)He_j(t) \\
& + (\widehat{\lambda}(t) - \lambda)\sum_{i=1}^{N} e_i^T(t)e_i(t) \\
\leq& \widehat{\lambda}(t)\sum_{i=1}^{N} e_i^T(t)e_i(t) \\
\leq& 2\widehat{\lambda}(t)V(t) \\
\leq& 2\widehat{\lambda}(t_k)V(t),
\end{aligned}$$

which gives

$$V(t) \leq V(t_{k-1}^{+})\exp(2\widehat{\lambda}(t_k)(t - t_k)), \;\; t \in (t_{k-1}, t_k).$$

When $t = t_k$, one has

$$\begin{aligned} V(t_k^+) =& \frac{1}{2}\sum_{i=1}^{N} e_i^T(t_k^+)e_i(t_k^+) + \frac{\beta(t_k^+)}{2\eta}(\widehat{\lambda}(t_k^+) - \lambda)^2 \\ =& \frac{(1+b(t_k))^2}{2}\sum_{i=1}^{N} e_i^T(t_k^-)e_i(t_k^-) + \frac{\beta(t_k^-)}{2\eta}(\widehat{\lambda}(t_k^-) - \lambda)^2 \\ =& \beta(t_k)V(t_k^-). \end{aligned}$$

Similar to the proof of Theorem 3.1, the proof can be completed. □

Remark 3.3. From the conditions (3.4) in Theorem 3.2, it is easy to see that the largest eigenvalue λ need not to be calculated beforehand and is estimated by $\widehat{\lambda}(t)$. That is, the adaptive impulsive controllers designed in Theorem 3.2 are unified for different dynamical networks.

Remark 3.4. From conditions (3.4), it is clear that the largest eigenvalue λ need not be known beforehand, which can be estimated by $\widehat{\lambda}(t)$. If the impulsive intervals τ_k and α are fixed, one can choose

$$-\exp(-\frac{\alpha + 2\widehat{\lambda}(t_k)\tau_k}{2}) - 1 + \varepsilon \le b(t_k) \le \exp(-\frac{\alpha + 2\widehat{\lambda}(t_k)\tau_k}{2}) - 1 - \varepsilon$$

such that conditions (3.4) in Theorem 3.2 hold, where ε is a small positive constant.

Remark 3.5. If the impulsive gains $b(t_k)$ and α are fixed, one can estimate the control instants t_k by solving a sequence of maximum value problems subject to

$$t_k \le t_{k-1} - (\ln\beta(t_k) + \alpha)\widehat{\lambda}^{-1}(t_k)/2, \; k = 1, 2, \cdots .$$

4. Numerical simulations

Consider a dynamical network consisting of 3 nodes with time-varying topology. Choose the node dynamics as Chen system [8]

$$\begin{aligned} \dot{x}_1 &= 35(x_2 - x_1), \\ \dot{x}_2 &= (28 - 35)x_1 - x_1x_3 + 28x_2, \\ \dot{x}_3 &= x_1x_2 - 3x_3, \end{aligned}$$

the inner coupling matrix as identity matrix and the time-varying outer coupling matrix as

$$C(t) = \begin{bmatrix} \sin(t) + \exp(-t) - 2 & 1 - \sin(t) & 1 - \exp(-t) \\ 1 - \sin(t) & \sin(t) + \cos(t) - 2 & 1 - \cos(t) \\ 1 - \exp(-t) & 1 - \cos(t) & \cos(t) + \exp(-t) - 2 \end{bmatrix}.$$

Example 4.1. According to Remark 3.4, choose $\tau_k = 0.5$, $\alpha = 0.001$, $\varepsilon = 0.001$, $\eta = 0.02$, $\widehat{\lambda}(0) = 0.5$, $b(t_k) = \exp(-\frac{\alpha + 2\widehat{\lambda}(t_k)\tau_k}{2}) - 1 - \varepsilon$ and the initial values of $x_i(t)$ and $s(t)$ randomly. Figures 1 and 2 show the orbits of synchronization errors $e_{ij}(t)$ and the impulsive gains $b(t_k)$ versus k.

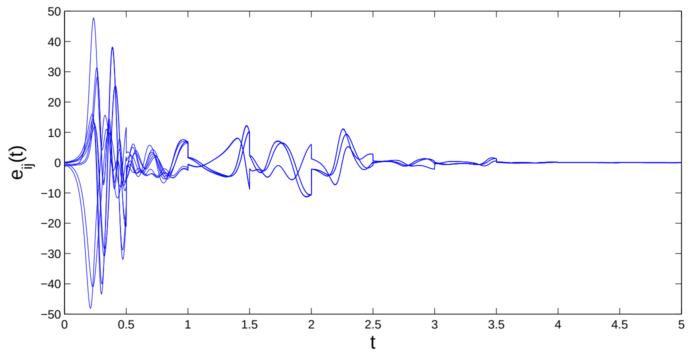

Figure 1. The orbits of synchronization errors $e_{ij}(t)$, $i, j = 1, 2, 3$.

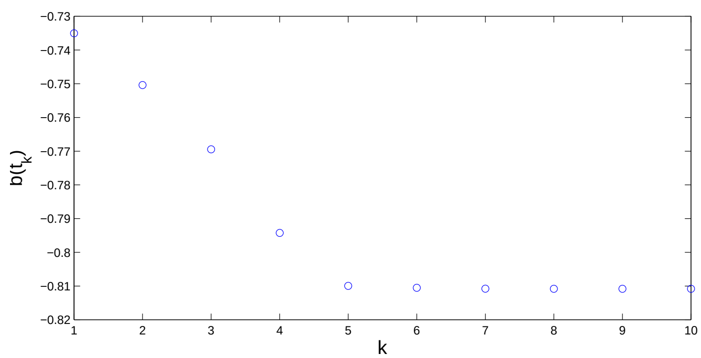

Figure 2. The impulsive intervals $b(t_k)$ versus k.

Example 4.2. Choose $b(t_k) = -0.9$, $\alpha = 0.001$, $\varepsilon = 0.001$, $\eta = 0.02$, $\widehat{\lambda}(0) = 0.5$ and the initial values of $x_i(t)$ and $s(t)$ randomly. The impulsive instants (or the impulsive intervals) are estimated by solving the maximum value problems in Remark 3.5. Figures 3 and 4 show the orbits of synchronization errors $e_{ij}(t)$ and the impulsive intervals τ_k versus k.

5. Conclusions

In this paper, synchronization of dynamical network with time-varying outer coupling matrix is considered. Impulsive control scheme is adopted to design proper controllers for achieving synchronization. Some synchronization conditions are first analytically derived based on the Lyapunov function method and stability theory of impulsive differential equation. From these conditions, one can choose the needed values of impulsive gains and intervals when the constant with respect to system parameters are calculated. Further, some unified impulsive controllers are designed through introducing proper adaptive strategy. In the adaptive impulsive control scheme, the constant need not to be calculated beforehand and is estimated by a parameter with adaptive updating law. According to Remarks 3.4 and 3.5, the impulsive gains can adjust themselves to proper values and the impulsive instants

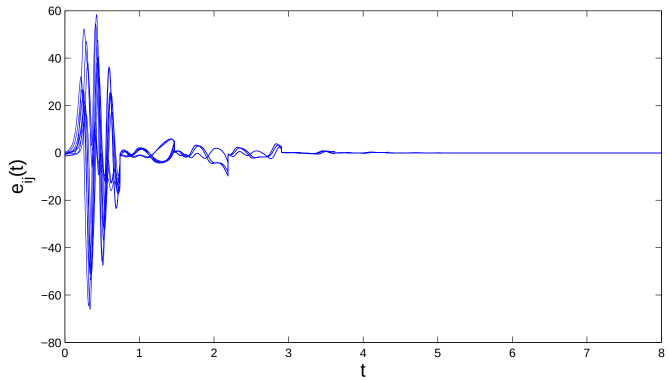

Figure 3. The orbits of synchronization errors $e_{ij}(t)$, $i, j = 1, 2, 3$.

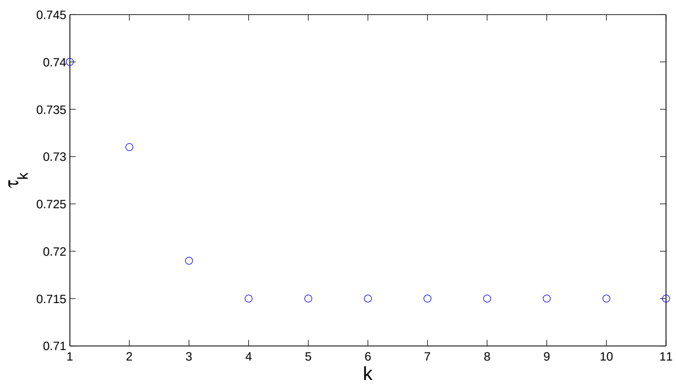

Figure 4. The impulsive intervals τ_k versus k.

can be estimated. All the derived results are illustrated to be effective by several numerical simulations.

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