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COUNTING SPANNING TREES IN PRISM AND ANTI-PRISM GRAPHS*

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Abstract In this paper, we calculate the number of spanning trees in prism and antiprism graphs corresponding to the skeleton of a prism and an antiprism. By the electrically equivalent transformations and rules of weighted generating function, we obtain a relationship for the weighted number of spanning trees at the successive two generations. Using the knowledge of difference equations, we derive the analytical expressions for enumeration of spanning trees. In addition, we again calculate the number of spanning trees in Apollonian networks, which shows that this method is simple and effective. Finally we compare the entropy of our networks with other studied networks and find that the entropy of the antiprism graph is larger.

Keywords Electrically equivalent transformation, spanning trees, prism graph, anti-prism graph.

MSC(2010) 05C30, 05C63, 97K30.

1. Introduction

Spanning trees have been widely studied in many aspects of mathematics, such as algebra [17], combinatorics [12], and theoretical computer science [1]. An interesting issue is calculation of the number of spanning trees, which has a lot of connections with networks, such as dimer coverings [19], Potts model [14, 23], random walks [5,15], the origin of fractality for scale-free networks [9,11] and many others [24,25]. In view of its wide range of applications, the enumeration of spanning trees has received considerable attention in the scientific community [3,20]. For example, the number of spanning trees is a crucial measurement of the network reliability.

However, counting spanning trees in a network is challenging. It is known that the number of spanning trees of any finite graph is calculated by Kirchhoff's matrix-tree theorem [10], and this is a demanding and difficult task, in particular for larger graphs. Presently there has been much interest in finding the effective methods to obtain exact expressions for the number of spanning trees of some deterministic graphs, such as grids [16], lattices [18], Sierpinski gaskets [4,6], and Petersen graphs [8]. The methods of vertices decimation [26], recursive enumeration of subgraphs [27], dual graphs [28], Laplacian spectrum [7], and matrix-tree theorem [2], have often been used for calculating the number of spanning trees.

In our previous article [7], we apply the Laplacian spectrum to calculate the number of spanning trees, while the involved calculations are complicated and this

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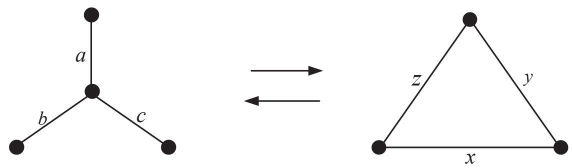

Figure 1. Wye-Delta-Wye transformations.

method is not valid for calculating the number of spanning trees of an antiprism graph. The prism and antiprism graphs corresponding to the skeleton of a prism and an antiprism belong to a family of generalized Petersen graphs. In this paper, we apply the knowledge of electrical networks, where an edge-weighted graph is regarded as an electrical network with the weights being the conductances of the corresponding edges. By the electrically equivalent transformation and the rules of the weighted number of spanning trees proposed in [21, 22], we obtain a relationship for the weighted number of spanning trees in the adjacent transformations and derive the analytical solutions of the number of spanning trees in prism and antiprism graphs. As an example, we again calculate the number of spanning trees of Apollonian networks, which is consistent with that obtained in [13, 27, 28]. Finally, we compare the entropy [12] of our models with other studied networks and show that the entropy of the antiprism graph is larger.

2. Preliminaries

In this section, we provide the relationships between electrical networks and spanning trees. Let X be a graph (edge-weighted), $X^{'}$ be the corresponding electrically equivalent graph, $\tau(X)$ denotes the weighted number of spanning trees of X. Using the obtained results in [21, 22], there have the following transformation rules:

- Parallel edges: If two parallel edges with conductances a and b in X are merged into a single edge with conductance $a+b$ in $X^{'}$, then the weighted number of spanning trees remains same, i.e., $\tau(X^{'}) = \tau(X)$.
- Serial edges: If two serial edges with conductances a and b are merged into a single edge with conductance $\frac{ab}{a+b}$, then $\tau(X^{'}) = \frac{1}{a+b} \cdot \tau(X)$.
- Wye-Delta transformation: If a star graph with conductances a, b, c (see Fig. 1) is transformed into an electrically equivalent triangle with conductances $x = \frac{bc}{a+b+c}, y = \frac{ac}{a+b+c}$ and $z = \frac{ab}{a+b+c}$, then $\tau(X^{'}) = \frac{1}{a+b+c} \cdot \tau(X)$.
- Delta-Wye transformation: If a triangle with conductances x, y, z (see Fig. 1) is changed into an electrically equivalent star graph with conductances $a = \frac{xy+yz+zx}{x}, b = \frac{xy+yz+zx}{y}$ and $c = \frac{xy+yz+zx}{z}$, then $\tau(X^{'}) = \frac{(xy+yz+zx)^2}{xyz} \cdot \tau(X)$.

3. Enumeration of spanning trees in 3-prism graph

3.1. Constructions of 3-prism graph

The algorithms of the 3-prism graph denoted by G_n are as follows: at generation $n = 1$, its initial state is a triangle. For $n \geq 2$, every existing node of the innermost

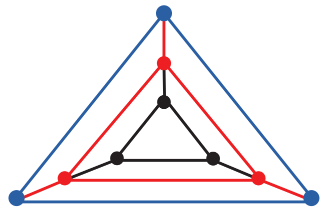

Figure 2. The first three generations of G_n.

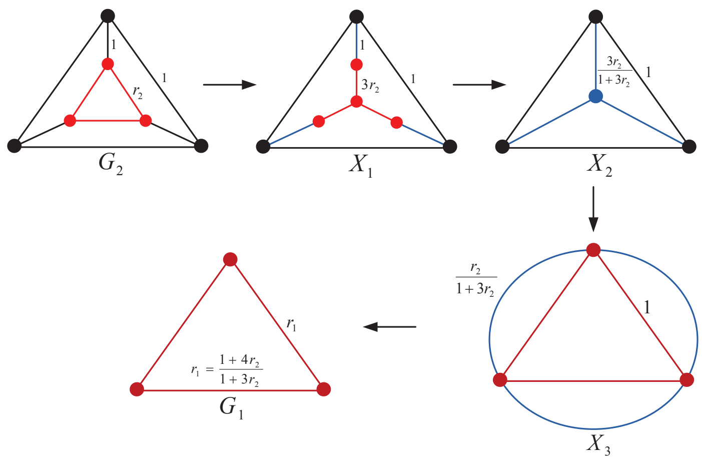

Figure 3. The transformations from G_2 to G_1.

triangle in G_{n-1} produces a new node and the three new nodes form a new triangle. The first three generations of G_n are shown in Fig. 2.

3.2. Calculations of the number of spanning trees

In order to calculate the number of spanning trees, we need to find a relationship between G_n and G_{n-1} $(n \geq 2)$. By the electrically equivalent transformation, we denote G_k as a graph consisting of k triangles and its innermost triangle has the weights r_k. Using the properties given in Section 2, we transform G_k to G_{k-1} and change the weights of the innermost triangle from r_k to r_{k-1}. For $k = 2$, we provide the transformation process from G_2 to G_1, see Fig. 3.

By the Delta-Wye transformations, we obtain $\tau(X_1) = 9r_2\tau(G_2)$. Merging two serial edges into a single edge, then $\tau(X_2) = (\frac{1}{1+3r_2})^3\tau(X_1)$. Through the Wye-Delta transformation, we obtain $\tau(X_3) = \frac{1+3r_2}{9r_2}\tau(X_2)$. Finally we merge parallel edges into a single edge and derive $\tau(G_1) = \tau(X_3)$.

Combining the above four transformations, we obtain the relationship between $\tau(G_2)$ and $\tau(G_1)$, that is,

$$\tau(G_2) = (1 + 3r_2)^2\tau(G_1).$$

Further,

$$\tau(G_n) \quad = \quad \prod_{k=2}^{n}(1 + 3r_k)^2 \cdot \tau(G_1) = 3r_1^2[\prod_{k=2}^{n}(1 + 3r_k)]^2, \tag{3.1}$$

where $r_{k-1}=\frac{4r_k+1}{3r_k+1}(k=2,3,\cdots,n)$. Its characteristic equation reads

$$3x^2-3x-1=0,$$

with two roots being $x_1=\frac{3-\sqrt{21}}{6}$ and $x_2=\frac{3+\sqrt{21}}{6}$. Subtracting these two roots into both sides of $r_{k-1}=\frac{4r_k+1}{3r_k+1}$ yields

$$\begin{aligned} r_{k-1}-\frac{3-\sqrt{21}}{6} &= \frac{4r_k+1}{3r_k+1}-\frac{3-\sqrt{21}}{6} \\ &= \frac{5+\sqrt{21}}{2}\cdot\frac{r_k-\frac{3-\sqrt{21}}{6}}{3r_k+1}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} r_{k-1}-\frac{3+\sqrt{21}}{6} &= \frac{4r_k+1}{3r_k+1}-\frac{3+\sqrt{21}}{6} \\ &= \frac{5-\sqrt{21}}{2}\cdot\frac{r_k-\frac{3+\sqrt{21}}{6}}{3r_k+1}. \end{aligned} \tag{3.3}$$

Let $a_k=\frac{r_k-\frac{3-\sqrt{21}}{6}}{r_k-\frac{3+\sqrt{21}}{6}}$ and by Eqs. (3.2) and (3.3), we obtain

$$a_{k-1}=\frac{23+5\sqrt{21}}{2}a_k$$

and

$$a_k \quad = \quad \frac{r_k-\frac{3-\sqrt{21}}{6}}{r_k-\frac{3+\sqrt{21}}{6}}=(\frac{23+5\sqrt{21}}{2})^{n-k}\cdot a_n.$$

Hence, the expression of r_k reads

$$r_k=\frac{(\frac{23+5\sqrt{21}}{2})^{n-k}\cdot\frac{3+\sqrt{21}}{6}a_n-\frac{3-\sqrt{21}}{6}}{(\frac{23+5\sqrt{21}}{2})^{n-k}a_n-1},$$

where

$$r_1=\frac{(\frac{23+5\sqrt{21}}{2})^{n-1}\cdot\frac{3+\sqrt{21}}{6}a_n-\frac{3-\sqrt{21}}{6}}{(\frac{23+5\sqrt{21}}{2})^{n-1}a_n-1}. \tag{3.4}$$

Using this expression $r_{n-1}=\frac{4r_n+1}{3r_n+1}$ and denoting A_n and B_n as the coefficients of $4r_n+1$ and $3r_n+1$, we obtain

$$\begin{aligned} 3r_n+1 &= A_0(4r_n+1)+B_0(3r_n+1), \\ 3r_{n-1}+1 &= \frac{A_1(4r_n+1)+B_1(3r_n+1)}{A_0(4r_n+1)+B_0(3r_n+1)}, \\ \vdots \qquad & \qquad \vdots \end{aligned}$$

$$3r_{n-k}+1=\frac{A_k(4r_n+1)+B_k(3r_n+1)}{A_{k-1}(4r_n+1)+B_{k-1}(3r_n+1)}, \tag{3.5}$$

$$3r_{n-(k+1)}+1=\frac{A_{k+1}(4r_n+1)+B_{k+1}(3r_n+1)}{A_k(4r_n+1)+B_k(3r_n+1)}, \tag{3.6}$$

$$\vdots \qquad\qquad \vdots$$

$$3r_2 + 1 = \frac{A_{n-2}(4r_n + 1) + B_{n-2}(3r_n + 1)}{A_{n-3}(4r_n + 1) + B_{n-3}(3r_n + 1)},$$

$$\tau(G_n) = 3r_1^2[A_{n-2}(4r_n + 1) + B_{n-2}(3r_n + 1)]^2, \tag{3.7}$$

where $A_0 = 0, B_0 = 1; A_1 = 3, B_1 = 1$. By the relationship between r_n and r_{n-1} and Eqs. (3.5) and (3.6), we obtain

$$A_{k+1} = 5A_k - A_{k-1};\ B_{k+1} = 5B_k - B_{k-1}. \tag{3.8}$$

The characteristic equation of Eq. (3.8) is

$$\lambda^2 - 5\lambda + 1 = 0,$$

with two roots being $\lambda_1 = \frac{5+\sqrt{21}}{2}$ and $\lambda_2 = \frac{5-\sqrt{21}}{2}$. Then, the general solutions of Eq. (3.8) are

$$A_k = a_1\lambda_1^k + a_2\lambda_2^k;\ B_k = b_1\lambda_1^k + b_2\lambda_2^k.$$

Substituting the initial conditions $A_0 = 0, B_0 = 1$ and $A_1 = 3, B_1 = 1$ gives

$$A_k = \frac{\sqrt{21}}{7}\lambda_1^k - \frac{\sqrt{21}}{7}\lambda_2^k;\ B_k = \frac{7-\sqrt{21}}{14}\lambda_1^k + \frac{7+\sqrt{21}}{14}\lambda_2^k. \tag{3.9}$$

If $r_n = 1$, it means that G_n without any electrically equivalent transformation. Inserting Eqs. (3.4) and (3.9) into Eq. (3.7), we finally obtain

$$\tau(G_n) = 3r_1^2(\frac{14+3\sqrt{21}}{7}\lambda_1^{n-2} + \frac{14-3\sqrt{21}}{7}\lambda_2^{n-2})^2 \quad (n \geq 2).$$

When $n = 1$, the number of spanning trees of a triangle is 3, which satisfies the above equation. Hence, we obtain the number of spanning trees in 3-prism graph, that is,

$$\tau(G_n) = 3r_1^2(\frac{14+3\sqrt{21}}{7}\lambda_1^{n-2} + \frac{14-3\sqrt{21}}{7}\lambda_2^{n-2})^2 \quad (n \geq 1), \tag{3.10}$$

where $r_1 = \frac{(18+4\sqrt{21})(\frac{23+5\sqrt{21}}{2})^{n-1}+3-\sqrt{21}}{(15+3\sqrt{21})(\frac{23+5\sqrt{21}}{2})^{n-1}+6}$, $\lambda_1 = \frac{5+\sqrt{21}}{2}$ and $\lambda_2 = \frac{5-\sqrt{21}}{2}$.

4. Enumeration of spanning trees of 3-antiprism graph

4.1. Construction

As 3-prism graph, this antiprism graph F_n is also built in an iterative way. At generation $n = 1$, F_1 is a triangle. For $n \geq 2$, F_n is obtained from F_{n-1}, where every existing node of the innermost triangle in F_{n-1} gives birth to a new node and these three new nodes form a new triangle. Compared to the innermost triangle in F_{n-1}, this new triangle is invertible, see Fig. 4.

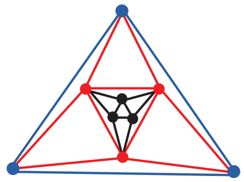

Figure 4. The first three generations of F_n.

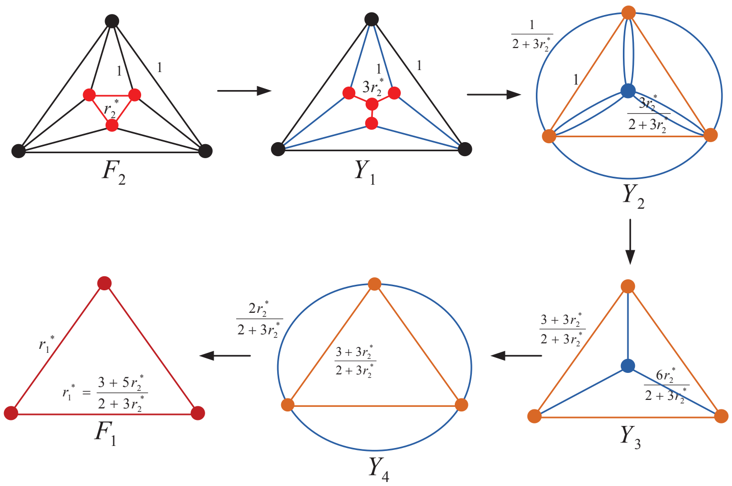

Figure 5. The transformations from F_2 to F_1.

4.2. Enumeration of spanning trees

Following the same method on calculating the number of spanning trees in 3-prism graph, we use the electrically equivalent transformation to transform F_k to F_{k-1}. The transformation between F_2 and F_1 is shown in Fig. 5.

By the Delta-Wye transformation, we have $\tau(Y_1) = 9r_2^* \cdot \tau(F_2)$. Using the Wye-Delta transformation, we obtain $\tau(Y_2) = (\frac{1}{2+3r_2^*})^3 \cdot \tau(Y_1)$. Merging parallel edges into a single edge, then $\tau(Y_3) = \tau(Y_2)$. Using the Wye-Delta transformation again, then $\tau(Y_4) = \frac{2+3r_2^*}{18r_2^*} \cdot \tau(Y_3)$. Finally we merge parallel edges into a single edge and derive $\tau(F_{2-1}) = \tau(Y_4)$.

Combing the above transformations gives

$$\tau(F_2) = 2(2+3r_2^*)^2 \cdot \tau(F_{2-1}).$$

Then,

$$\tau(F_n) \quad = \quad 2^{n-1}\prod_{k=2}^{n}(2+3r_k^*)^2\tau(F_1) = 3\cdot 2^{n-1}{r_1^*}^2[\prod_{k=2}^{n}(2+3r_k^*)]^2, \qquad (4.1)$$

where $r_{k-1}^* = \frac{5r_k^*+3}{3r_k^*+2}(k = 2,3,\cdots,n)$. Its characteristic equation is

$$x^2 - x - 1 = 0,$$

whose two roots are $x_1^* = \frac{1+\sqrt{5}}{2}, x_2^* = \frac{1-\sqrt{5}}{2}$. Subtracting these two roots into this equation, we obtain

$$\begin{aligned} r_{k-1}^* - \frac{1+\sqrt{5}}{2} &= \frac{5r_k^*+3}{3r_k^*+2} - \frac{1+\sqrt{5}}{2} \\ &= \frac{7-3\sqrt{5}}{2} \cdot \frac{r_k^* - \frac{1+\sqrt{5}}{2}}{3r_k^*+2}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} r_{k-1}^* - \frac{1-\sqrt{5}}{2} &= \frac{5r_k^*+3}{3r_k^*+2} - \frac{1-\sqrt{5}}{2} \\ &= \frac{7+3\sqrt{5}}{2} \cdot \frac{r_k^* - \frac{1-\sqrt{5}}{2}}{3r_k^*+2}. \end{aligned} \tag{4.3}$$

Let $b_k = \frac{r_k^* - \frac{1+\sqrt{5}}{2}}{r_k^* - \frac{1-\sqrt{5}}{2}}$. From Eqs. (4.2) and(4.3), we obtain

$$b_{k-1} = \frac{47-21\sqrt{5}}{2} b_k$$

and

$$b_k = \frac{r_k^* - \frac{1+\sqrt{5}}{2}}{r_k^* - \frac{1-\sqrt{5}}{2}} = (\frac{47-21\sqrt{5}}{2})^{n-k} b_n.$$

Finally we obtain the expression of r_k^*,

$$r_k^* = \frac{(1-\sqrt{5})(\frac{47-21\sqrt{5}}{2})^{n-k} b_n - \sqrt{5} - 1}{2(\frac{47-21\sqrt{5}}{2})^{n-k} b_n - 2},$$

where

$$r_1^* = \frac{(1-\sqrt{5})(\frac{47-21\sqrt{5}}{2})^{n-1} b_n - \sqrt{5} - 1}{2(\frac{47-21\sqrt{5}}{2})^{n-1} b_n - 2}. \tag{4.4}$$

Denoting C_k and D_k be the coefficients of $5r_n^*+3$ and $3r_n^*+2$, we have

$$\begin{aligned} 3r_n^*+2 &= C_0(5r_n+3) + D_0(3r_n^*+2), \\ 3r_{n-1}^*+2 &= \frac{C_1(5r_n^*+3) + D_1(3r_n^*+2)}{C_0(5r_n^*+3) + D_0(2+3r_n^*)}, \\ &\vdots \qquad\qquad \vdots \\ 3r_{n-k}^*+2 &= \frac{C_k(5r_n^*+3) + D_k(3r_n^*+2)}{C_{k-1}(5r_n^*+3) + D_{k-1}(3r_n^*+2)}, && (4.5) \\ 3r_{n-(k+1)}^*+2 &= \frac{C_{k+1}(5r_n^*+3) + D_{k+1}(3r_n^*+2)}{C_k(5r_n^*+3) + D_k(3r_n^*+2)}, && (4.6) \\ &\vdots \qquad\qquad \vdots \\ 3r_2^*+2 &= \frac{C_{n-2}(5r_n^*+3) + D_{n-2}(3r_n^*+2)}{C_{n-3}(5r_n^*+3) + D_{n-3}(2+3r_n^*)}, \\ \tau(F_n) &= 3 \cdot 2^{n-1} r_1^{*2} [C_{n-2}(5r_n^*+3) + D_{n-2}(3r_n^*+2)]^2, && (4.7) \end{aligned}$$

where $C_0 = 0, D_0 = 1; C_1 = 3, D_1 = 2$. Using Eqs. (4.5) and (4.6) and $r_{k-1}^* = \frac{5r_k^*+3}{3r_k^*+2}$, we obtain the relationships of the coefficients C_k and D_k,

$$C_{k+1} = 7C_k - C_{k-1}; D_{k+1} = 7D_k - D_{k-1}, \tag{4.8}$$

where the roots of the characteristic equation $\lambda^2 - 7\lambda + 1 = 0$ are $\lambda_1^* = \frac{7+3\sqrt{5}}{2}$ and $\lambda_2^* = \frac{7-3\sqrt{5}}{2}$.

Then, the general solution of Eq. (4.8) are

$$C_k = c_1 \lambda_1^{*k} + c_2 \lambda_2^{*k}; D_k = d_1 \lambda_1^{*k} + d_2 \lambda_2^{*k}.$$

Inserting the initial conditions of $C_0 = 0, D_0 = 1$ and $C_1 = 3, D_1 = 2$ into this equation yields

$$C_k = \frac{\sqrt{5}}{5}\lambda_1^{*k} - \frac{\sqrt{5}}{5}\lambda_2^{*k}; D_k = \frac{5-\sqrt{5}}{10}\lambda_1^{*k} + \frac{5+\sqrt{5}}{10}\lambda_2^{*k}. \tag{4.9}$$

Setting $r_n^* = 1$ and substituting Eqs. (4.4) and (4.9) into (4.7) gives

$$\tau(F_n) = 3 \cdot 2^{n-1} r_1^{*2} (\frac{25 + 11\sqrt{5}}{10}\lambda_1^{*n-2} + \frac{25 - 11\sqrt{5}}{10}\lambda_2^{*n-2})^2 \quad (n \geq 2).$$

When $n = 1$, we have $\tau(F_1) = 3$, which is same as the number of spanning trees in a triangle. Therefore, we obtain the analytical expreesion of the number of spanning trees in 3-antiprism graph,

$$\tau(F_n) = 3 \cdot 2^{n-1} r_1^{*2} (\frac{25 + 11\sqrt{5}}{10}\lambda_1^{*n-2} + \frac{25 - 11\sqrt{5}}{10}\lambda_2^{*n-2})^2 \quad (n \geq 1), \tag{4.10}$$

where $r_1^* = \frac{(2\sqrt{5}-4)(\frac{47-21\sqrt{5}}{2})^{n-1}-1-\sqrt{5}}{(\sqrt{5}-3)(\frac{47-21\sqrt{5}}{2})^{n-1}-2}$, $\lambda_1^* = \frac{7+3\sqrt{5}}{2}$ and $\lambda_2^* = \frac{7-3\sqrt{5}}{2}$.

5. Enumeration of spanning trees in Apollonian networks

The number of spanning trees in Apollonian networks have been calculated in [13,27, 28]. In this section, we use the electrically equivalent transformation to enumerate spanning trees. Let a_k be the weight of an edge in the outmost triangles of $P_k (k = 0, 1, \cdots, n)$ and b_k be the weights of other edges. According to the transformation illustrated in Fig. 6, we obtain

$$\tau(P_k) = (3b_k)^{3^{k-1}} \tau(P_{k-1}) \tag{5.1}$$

and

$$a_k = a_{k+1} + \frac{1}{3}b_{k+1}, b_k = \frac{5}{3}b_{k+1}. \tag{5.2}$$

Setting $a_n = b_n = 1$ gives

$$\tau(P_n) = \prod_{k=1}^{n} (3b_k)^{3^{k-1}} \tau(P_0) = 3^{\frac{3^n-1}{2}} \prod_{k=1}^{n} b_k^{3^{k-1}} \tau(P_0),$$

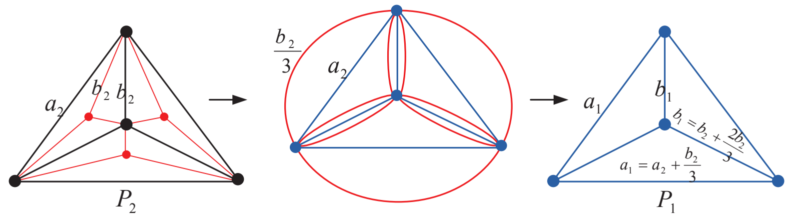

Figure 6. The transformations from P_2 to P_1.

where $a_k = \frac{1}{2}(\frac{5}{3})^{n-k} + \frac{1}{2}$, $b_k = (\frac{5}{3})^{n-k}$. Then the number of spanning trees in Apollonian networks is

$$
\begin{aligned}
\tau(P_n) &= 3^{\frac{3^n-1}{2}}(b_1^{3^0} b_2^{3^1} \cdots b_n^{3^{n-1}}) \cdot \tau(P_0) \\
&= 3^{\frac{3^n-1}{2}} \cdot (\frac{5}{3})^{[n-1+3(n-2)+\cdots+3^{n-2}]} \cdot \tau(P_0) \\
&= 3^{\frac{3^n-1}{2}} \cdot (\frac{5}{3})^{\frac{3^n-2n-1}{4}} \cdot \tau(P_0) \\
&= 3^{\frac{3^n+2n-1}{4}} \cdot 5^{\frac{3^n-2n-1}{4}} \cdot \tau(P_0), \qquad (5.3)
\end{aligned}
$$

where

$$
\tau(P_0) = 3a_0^2 \quad = \quad 3(a_1 + \frac{b_1}{3})^2 = \frac{3}{4}[(\frac{5}{3})^n + 1]^2. \qquad (5.4)
$$

Finally we obtain the number of spanning trees in Apollonian networks, i.e.,

$$
\tau(P_n) = \frac{1}{4} \cdot 3^{\frac{3^n+2n+3}{4}} \cdot 5^{\frac{3^n-2n-1}{4}} \cdot [(\frac{5}{3})^n + 1]^2.
$$

Compared to the methods used in [13, 27, 28], this method is simple and effective.

6. The entropy of spanning trees

Since the number of spanning trees grows exponentially, we can calculate the entropy [12] of spanning trees, denoted by $E(G_n)$, which is given by

$$
E(G_n) = \lim_{n\to\infty} \frac{\ln \tau(G_n)}{V_n},
$$

where V_n is the number of nodes in G_n. For the 3-prism and 3-antiprism graphs, we have $V_n = 3n$. From Eqs. (3.10) and (4.10), we obtain

$$
\begin{aligned}
E(G_n) &= \frac{2}{3}\ln(\frac{5+\sqrt{21}}{2}) \approx 1.0445, \\
E(F_n) &= \frac{\ln 2}{3} + \frac{2}{3}\ln(\frac{7+3\sqrt{5}}{2}) \approx 1.5143, \\
E(P_n) &= \frac{1}{2}(\ln 3 + \ln 2) \approx 1.3540.
\end{aligned}
$$

It is noted that the entropy of 3-antiprism graph is larger than that of 3-prism graph. In addition, the entropy of 3-prism graph is almost same as that of two-dimensional Sierpinski gasket [4] with same average degree of nodes; while the entropy of 3-antiprism graph is larger than that of Apollonian network [28] with same average degree. Therefore studying the influence of other topological quantities on the entropy remains open, e.g., the exponent of degree distribution in scale-free networks.

7. Conclusions

In this paper, we calculate the number of spanning trees in some self-similar networks by electrically equivalent transformations, which avoids the computational complexity of Laplacian spectrum. For the 3-prism and 3-antiprism graphs, we obtain the exact solutions for the number of spanning trees verified by numerical simulations. Compared to the existing methods, this method is more effective, specially for the difficulty in calculating Laplacian spectrum of some graphs. We further calculate and compare the entropy of spanning trees of 3-prism graph and 3-antiprism graph. Future work regarding weighted generating function of spanning trees in weighted networks is underway.

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