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A CLASS OF LYAPUNOV FUNCTIONS AND THE GLOBAL STABILITY OF SOME EPIDEMIC MODELS WITH NONLINEAR INCIDENCE*

Jianquan Li[1,†], Yali Yang[1,2,3], Yanni Xiao[4] and Shuo Liu[5]

Abstract In this paper, by investigating an SIR epidemic model with nonlinear incidence, we present a new technique for proving the global stability of the endemic equilibrium, which consists of introducing a variable transformation and constructing a more general Lyapunov function. For the model we obtain the following results. The disease-free equilibrium is globally stable in the feasible region as the basic reproduction number is less than or equal to unity, and the endemic equilibrium is globally stable in the feasible region as the basic reproduction number is greater than unity. The generality of the technique is illustrated by considering certain nonlinear incidences and SIS and SIRS epidemic models.

Keywords Epidemic model, nonlinear incidence, global stability, Lyapunov function, equilibrium.

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1. Introduction

Bilinear and standard incidences have been frequently used in classical epidemic models [15]. Several different forms of incidences have been proposed by some researchers. Let $S(t)$ and $I(t)$ be the numbers of susceptible and infective individuals at time t, respectively, Capasso and Serio [1] introduced a saturated incidence $Sf(I)$ into epidemic models to study of the cholera epidemic spread in Bari in 1973. The nonlinear incidences of the forms βI^pS^q and $\beta I^pS/(1+\alpha I^q)$ were proposed by Liu et al. [14]. Epidemic models with the incidence βI^pS^q had also been studied in [4,6,11,13]. An SEIRS epidemic model with the saturation incidence $\beta SI/(1+aS)$ was examined in [2]. Epidemic models with the incidence $\beta I^pS/(1+aI^q)$ had been investigated in [16, 20]. The nonlinear incidences of the form $\beta(I+\nu I^p)S$ proposed

†the corresponding author. Email address: jianq_li@263.net(J. Li)
[1]Science College, Air Force Engineering University, 710051 Xi'an, P.R. China
[2]College of Mathematics and Information Science, Shaanxi Normal University, 710062, Xi'an, P.R. China
[3]Centre for Disease Modelling, York University, M3J 1P3, Toronto, Canada
[4]Department of Applied Mathematics, Xi'an Jiaotong University, 710049 Xi'an, P. R. China
[5]School of Biomedical Engineering, the Fourth Military Medical University, 710032 Xian, P.R. China

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by van den Driessche & Watmough [18] was used in [10, 12, 21]. The more general forms of nonlinear incidence were considered in [3, 7, 8, 17, 23].

For the nonlinear incidence $Sf(I)$, there are often two kinds of assumptions about function $f(I)$. One is that the number of effective contacts between infective and susceptible individuals may be saturated at high infective levels due to crowding of infective individuals or due to the protection measures by the susceptible individuals. The other is that the number of effective contacts is decreasing when the number of infective individuals is large. This can be used to interpret the "psychological" effect: for a very large number of infectives the infection force may decrease as the number of infective individuals increases, because in the presence of large number of infectives the population may tend to reduce the number of contacts per unit time. For example, function $f(I) = \beta I^p/(1 + aI^q)$ corresponds to the former case as $p = q$ and the latter one as $p < q$.

For an epidemic dynamical model, global analysis on the feasible region is an important issue for understanding the transmission mechanism of the infection. The most common method to prove the global stability of epidemic models is the Lyapunov's direct method. The key to applying the method is to construct an appropriate Lyapunov function and prove the negative or seminegative definiteness of the associated derivative. However, when an equilibrium of the dynamical system is indeed globally stable on the region, it usually may not be easy to find such an appropriate Lyapunov function, or we can not determine if the constructed Lyapunov function is suitable for the system since the negative or seminegative definiteness of the associated derivative can not be proved. The main objective of this paper is to present a new technique to prove global stability of the system by considering an SIR epidemic model with the nonlinear incidence $Sf(I)$. This technique consists of two steps, the first is to introduce a new variable for equivalently making transformation, the second is to propose a class of novel Lyapunov function. In contrast to the previous results obtained in [3, 7, 8, 17, 23], the condition obtained in this paper is necessary and sufficient to ensure the global stability of the endemic equilibrium in the feasible region under the assumptions.

The organization of this paper is as follows. In the next section, the idea of constructing a Lyapunov function is introduced, and some reasonable assumptions on the nonlinear incidence are given. In Section 3, an SIR epidemic model with nonlinear incidence is investigated, the global stability of the model is considered by introducing a new variable and constructing a new class of Lyapunov function, and the advantages of the technique of the new class of Lyapunov function are shown by two remarks. Finally, we illustrate the generality and benefit of the technique presented here by considering two different epidemic models.

2. Lyapunov function and nonlinear incidence

Lyapunov functions of the integral form

$$\int_{x^*}^{x} \frac{f(u) - f(x^*)}{f(u)} du \tag{2.1}$$

have been used in many epidemic models with nonlinear incidence [3, 7, 8, 17, 23], which is generalization of the logarithmic form $x - x^* - x^* \ln(x/x^*)$ since

$$x - x^* - x^* \ln \frac{x}{x^*} = \int_{x^*}^{x} \frac{u - x^*}{u} du. \tag{2.2}$$

However, for some epidemic models, applying the Lyapunov functions of form (2.1) can only give the sufficient instead of the unnecessary condition on the global stability of the endemic equilibrium [3,7,8,17,23]. In [5,22], the Lyapunov functions of the linear combination form of two types of functions, the perfect square expression $(x - x^*)^2$ and the logarithmic one (2.2), were used to examine global stability of the endemic states.

Note that the integral expression

$$\int_{x^*}^{x} \frac{u - x^*}{f(u)} du, \tag{2.3}$$

which is different from form (2.1), is the other extended form of form (2.2), sine (2.3) can become (2.2) as $f(u) = u$. So we may extend the Lypunov functions of the linear combination form of the perfect square and the logarithmic expressions into a new form, i.e., the linear combination of the perfect square expression and the integral one (2.3), and apply them to prove the global stability of SIS, SIR and SIRS epidemic models with the nonlinear incidence $Sf(I)$.

For the nonlinear incidence $Sf(I)$, we assume that $f(I)$ is a real locally Lipschitz function at least on $[0, +\infty)$ which satisfies the following conditions:

(i) $f(0) = 0, f(I) > 0$ for $I > 0$;

(ii) $f(I)/I$ is continuous and monotonely nonincreasing for $I > 0$, and $\lim_{I\to 0+} f(I)/I$ exists, denoted by $\beta(0 < \beta < +\infty)$;

(iii) $\int_{0^+}^{1} 1/f(u)du = +\infty$.

Here, the first condition is obvious. Since the incidence $Sf(I)$ can be rewritten as $[f(I)/I]SI$, function $f(I)/I$ refers to effective contact rate between an infective and a susceptible individuals. The second condition implies that the effective contact rate between infective and susceptible individuals is nonincreasing with increase of infectives, and the existence of the limit of function $f(I)/I$ as $I \to 0+$ shows that the effective contact rate is bounded above, that is, $f(I)/I \le \beta$ for $I > 0$. Functions satisfying the conditions (i) and (ii) may accord with the assumptions about function $f(I)$ stated in the previous section. Obviously, function $f(I) = \beta I/(1 + aI^p)(a \ge 0, \beta > 0, p > 0)$ satisfies the above conditions. On the basis of the conditions (i) and (ii), the third condition is used to ensure that the integration $\int_{I^*}^{I}(u - I^*)/f(u)du$ is positive definite and radially unbounded in the interval $(0, +\infty)$.

3. SIR model and its global stability

In this section, we consider an SIR epidemic model with the nonlinear incidence $Sf(I)$,

$$\begin{aligned} \frac{dS}{dt} &= \mu A - \mu S - Sf(I), \\ \frac{dI}{dt} &= Sf(I) - (\mu + \gamma + \alpha)I, \\ \frac{dR}{dt} &= \gamma I - \mu R, \end{aligned} \tag{3.1}$$

where function $f(I)$ satisfies the conditions proposed in Section 2. And, $S = S(t)$, $I = I(t)$ and $R = R(t)$ represent the numbers of individuals in the susceptible,

infected and removed compartments at time t, respectively. μ denotes the per capita natural death rate, μA the recruitment of susceptible individuals, γ the recovery rate of an infected individual, α the per capita disease-induced death rate.

Since the variable R in SIR model (3.1) does not appear in the equations of S and I, for model (3.1) we only need to consider the subsystem

$$\begin{aligned} \frac{dS}{dt} &= \mu A - \mu S - Sf(I), \\ \frac{dI}{dt} &= Sf(I) - (\mu + \gamma + \alpha)I. \end{aligned} \tag{3.2}$$

From model (3.2) we have

$$\frac{d(S+I)}{dt} = \mu A - \mu(S+I) - (\gamma+\alpha)I \le \mu A - \mu(S+I) \text{ for } I \ge 0,$$

then it follows that $\limsup_{t\to\infty}(S+I) \le A$. Therefore, the region

$$\Omega = \left\{(S,I) \in \mathbb{R}^2_+ : S + I \le A\right\}$$

is a positively invariant attractive set for model (3.2).

Obviously, system (3.2) always has the disease-free equilibrium $E_0(A,0)$. Again, if model (3.2) has a positive (endemic) equilibrium $E^*(S^*,I^*)$ ($I^* > 0$), S^* and I^* satisfy the following equations

$$\begin{aligned} &\mu A - \mu S - Sf(I) = 0, \\ &Sf(I) - (\mu + \gamma + \alpha)I = 0. \end{aligned} \tag{3.3}$$

From the last equation of equations (3.3) we have $S = (\mu+\gamma+\alpha)I/f(I)$. Substituting it into the first equation of (3.3) yields the following equation

$$H(I) \triangleq \mu(\mu+\gamma+\alpha)\frac{I}{f(I)} + (\mu+\gamma+\alpha)\, I - \mu A = 0. \tag{3.4}$$

According to condition (ii) in Section 2, function $H(I)$ is strictly increasing, and

$$\lim_{t\to 0+} H(I) = \mu\left(\frac{\mu+\gamma+\alpha}{\beta} - A\right) = \frac{\mu(\mu+\gamma+\alpha)}{\beta}\left(1 - \frac{\beta A}{\mu+\gamma+\alpha}\right).$$

Notice that $H(A) > 0$, then, equation (3.4) has no root in the interval $(0,A)$ as $\beta A/(\mu+\gamma+\alpha) \le 1$ and has a unique root in the interval $(0,A)$ as $\beta A/(\mu+\gamma+\alpha) > 1$.

For model (3.2), we easily get that $1/(\mu+\gamma+\alpha)$ is the average infectious period, A is the number of susceptible individuals at steady state in the absence of infection, and $\lim_{I\to 0^+} f(I)/I = \beta$ for the effective contact rate $f(I)/I$, then their product $\beta A/(\mu+\gamma+\alpha)$ is the basic reproduction number of model (3.2), that is, the average number of secondary infections produced when one infective individual is introduced into a completely susceptible population. We denote the basic reproduction number as R_0, i.e., $R_0 = \beta A/(\mu+\gamma+\alpha)$. Consequently, we have the following results with respect to the existence of equilibria of model (3.2).

Theorem 3.1. *When $R_0 \le 1$, model (3.2) has only the disease-free equilibrium $E_0(A,0)$; when $R_0 > 1$, besides the disease-free equilibrium E_0, model (3.2) also has a unique endemic equilibrium $E^*(S^*,I^*)$ where $S^* = (\mu+\gamma+\alpha)I^*/f(I^*)$, and I^* is the positive root of equation (3.4) in the interval $(0,A)$.*

The following theorem provides the global properties of model (3.2) on the feasible region Ω.

Theorem 3.2. *For model* (3.2)*, the disease-free equilibrium E_0 is globally stable on the feasible region Ω as $R_0 \leq 1$, the endemic equilibrium E^* is globally stable in the feasible region Ω as $R_0 > 1$.*

Proof. We first consider the global stability of the disease-free equilibrium E_0.

Define a function $V = I$, then the derivative of function V along solutions of model (3.2) is

$$\begin{aligned}\frac{dV}{dt} =& Sf(I) - (\mu+\gamma+\alpha)I \leq I\left[\beta S - (\mu+\gamma+\alpha)\right]\\ \leq& I\left[\beta A - (\mu+\gamma+\alpha)\right] = (\mu+\gamma+\alpha)(R_0-1)V,\end{aligned}$$

where condition (ii) in Section 2 and $S \leq A$ are used. Thus, $dV/dt \leq 0$ as $R_0 \leq 1$.

It is obvious that $dV/dt = 0$ is equivalent to $S = (\mu+\gamma+\alpha)I/f(I)$. According to condition (ii) in Section 2,

$$\frac{(\mu+\gamma+\alpha)I}{f(I)} \geq \frac{\mu+\gamma+\alpha}{\beta} = \frac{A}{R_0} \quad \text{for} \quad I > 0,$$

then, $(\mu+\gamma+\alpha)I/f(I) \geq A$ as $R_0 \leq 1$. So the largest invariant set of model (3.2) on the set in which $dV/dt = 0$ is the singleton $\{E_0\}$. It follows from the LaSalle's Invariance Principle [9] that the disease-free equilibrium E_0 is globally stable on the feasible region Ω.

Next, we will prove the global stability of the endemic equilibrium E^*. For simplicity, we introduce a new variable $N = S + I$, (3.2) can become the following system

$$\begin{aligned}\frac{dI}{dt} &= (N-I)f(I) - (\mu+\gamma+\alpha)I,\\ \frac{dN}{dt} &= \mu A - \mu N - (\gamma+\alpha)I.\end{aligned} \tag{3.5}$$

Correspondingly, system (3.5) has a unique positive equilibrium $\bar{E}^*(I^*, N^*)$ as $R_0 > 1$, where $N^* = S^* + I^*$. Since I^* and N^* satisfy the following equations

$$\begin{aligned}(N^* - I^*) &= (\mu+\gamma+\alpha)\frac{I^*}{f(I^*)},\\ \mu A &= \mu N^* + (\gamma+\alpha)I^*,\end{aligned}$$

system (3.5) can be rewritten as

$$\begin{aligned}\frac{dI}{dt} &= f(I)\left\{\left[(N-N^*) - (I-I^*)\right] - (\mu+\gamma+\alpha)\left[\frac{I}{f(I)} - \frac{I^*}{f(I^*)}\right]\right\},\\ \frac{dN}{dt} &= -\mu(N-N^*) - (\gamma+\alpha)(I-I^*).\end{aligned} \tag{3.6}$$

Define a Lyapunov function

$$V_1 = \int_{I^*}^{I} \frac{u-I^*}{f(u)}du + \frac{1}{2(\gamma+\alpha)}(N-N^*)^2, \tag{3.7}$$

then the derivative of function V_1 along solutions of system (3.6) is given by

$$\frac{dV_1}{dt} = -(I - I^*)^2 - (\mu + \gamma + \alpha)(I - I^*)\left[\frac{I}{f(I)} - \frac{I^*}{f(I^*)}\right] - \frac{\mu}{\gamma + \alpha}(N - N^*)^2.$$

According to the assumption (ii) in Section 2, $(I - I^*)\,[I/f(I) - I^*/f(I^*)] \geq 0$ for $I > 0$, then dV_1/dt is negative definite with respect to $I = I^*$ and $N = N^*$. Thus, it follows from the Lyapunov Stability Theorem [19] that the positive equilibrium $\bar{E}^*$ of (3.5) is globally stable in the first quadrant. Therefore, the endemic equilibria E^* of models (3.2) is globally stable in the feasible region Ω if it is feasible.

This completes the proof of Theorem 3.2. □

Theorem 3.2 showed that the endemic equilibrium E^* of model (3.2) is globally stable in the feasible region if it exists. Note that we first changed the original model (3.2) into the system (3.5) by introducing the transformation of variables, and then applied the Lyapunov function (3.7) with the linear combination form of the perfect square expression $(N - N^*)^2$ and the integral one $\int_{I^*}^{I} [(u - I^*)/f(u)]\,du$ to prove the global stability. In the following remarks, we illustrate the advantage of the method for proving the global stability of the endemic equilibrium E^*.

Remark 3.1. Introducing the transformation of variables changes the original model (3.2) into the system (3.5) which includes a linear differential equation. This makes constructing the appropriate Lyapunov function easier, and determining the negative definiteness of the associated derivative much simpler. Theoretically, although these operations are not necessary, since the new variable N can be expressed by the original ones, they are indeed such that the investigated system and calculations become relatively simple.

Remark 3.2. The Lyapunov function used here is the linear combination form of the perfect square expression and the integral one (2.3). According to the common idea of constructing Lyapunov function, we could also expect that one of the following functions would be available,

$$V_2 = \int_{I^*}^{I} \frac{f(u) - f(I^*)}{f(u)}du + \frac{m}{2}(N - N^*)^2,$$
$$V_3 = \int_{I^*}^{I} \frac{u - I^*}{f(u)}du + m\int_{N^*}^{N} \frac{u - N^*}{u}du,$$

and

$$V_4 = \int_{I^*}^{I} \frac{f(u) - f(I^*)}{f(u)}du + m\int_{N^*}^{N} \frac{u - N^*}{u}du.$$

However, we barely obtain conditions such that one of their derivatives is negative or seminegative definite, which means these functions may not be suitable for this system under assumptions in Section 2. Therefore, only the Lyapunov function V_1 is appropriate for model (3.5).

4. Discussion

In the preceding section, we considered an SIR epidemic model and proved the global stability of the endemic equilibrium E^* by changing the original model (3.2)

into the system (3.5) by introducing a transformation of variable and then applying the Lyapunov function (3.7). The process of proving the global stability provides indeed a new technique. It can be easily verified that the proposed technique is suitable for SIS model

$$\begin{aligned}\frac{dS}{dt} &= \mu A - \mu S - Sf(I) + \gamma I,\\ \frac{dI}{dt} &= Sf(I) - (\mu + \gamma + \alpha)I,\end{aligned} \tag{4.1}$$

and SIRS model

$$\begin{aligned}\frac{dS}{dt} &= \mu A - \mu S - Sf(I) + \varepsilon R,\\ \frac{dI}{dt} &= Sf(I) - (\mu + \gamma + \alpha)I,\\ \frac{dR}{dt} &= \gamma I - (\mu + \varepsilon)R,\end{aligned} \tag{4.2}$$

where $f(I)$ satisfies assumptions in Section 2.

It is important to mention that, to some extent, introducing a transformation of variables is necessary in our technique. If we directly consider the original model (3.2), we usually apply one of the following functions as the Lyapunov function

$$\begin{aligned}V_5 &= m\int_{S^*}^{S}\frac{u-S^*}{u}du + \int_{I^*}^{I}\frac{f(u)-f(I^*)}{f(u)}du\\ V_6 &= \frac{m}{2S^*}(S-S^*)^2 + \int_{I^*}^{I}\frac{f(u)-f(I^*)}{f(u)}du,\\ V_7 &= m\int_{S^*}^{S}\frac{u-S^*}{u}du + \int_{I^*}^{I}\frac{u-I^*}{f(u)}du,\end{aligned}$$

and

$$V_8 = \frac{m}{2S^*}(S-S^*)^2 + \int_{I^*}^{I}\frac{u-I^*}{f(u)}du$$

to proving the global stability of the endemic equilibrium E^*, where the constant m is positive and left unspecified.

For functions V_5 and V_6, the condition

$$(I-I^*)\left[f(I)-f(I^*)\right] > 0 \qquad \text{for} \quad I \neq I^*,\ I > 0 \tag{4.3}$$

is necessary to ensure their positive definiteness. When we choose $m = 1$, the derivatives of functions V_5 and V_6 along solutions of (3.2) are

$$\frac{dV_5}{dt} = -(\mu+\alpha)\left[f(I)-f(I^*)\right]\left[\frac{I}{f(I)} - \frac{I^*}{f(I^*)}\right] - \mu A\frac{(S-S^*)^2}{S},$$

and

$$\frac{dV_6}{dt} = -(\mu+\alpha)\left[f(I)-f(I^*)\right]\left[\frac{I}{f(I)} - \frac{I^*}{f(I^*)}\right] - [\mu + f(I)]\frac{(S-S^*)^2}{S^*},$$

respectively. In order to make $dV_5/dt \leq 0$ or $dV_6/dt \leq 0$, both of them need the condition

$$\left[\frac{f(I)}{I} - \frac{f(I^*)}{I^*}\right]\left[\frac{f(I)}{f(I^*)} - 1\right] \leq 0 \qquad \text{for} \quad I > 0. \tag{4.4}$$

This implies that, when applying either V_5 or V_6 as Lyapunov function, both conditions (4.3) and (4.4) are needed except for assumptions in Section 2. But Theorem 3.2 have shown that both conditions (4.3) and (4.4) could be unnecessary, then the conditions obtained by applying functions V_5 and V_6 are sufficient but unnecessary. Additionally, it is difficult to obtain conditions such that the derivative of function V_7 or V_8 is negative or seminegative definite. This implies that they are also not suitable for model (3.2).

In summary, we think that the technique presented here is effective for some epidemic models with nonlinear incidence $Sf(I)$ with $f(I)$ satisfying the assumptions in Section 2. The contribution of this study on constructing the Lyapunov function is to propose a new form (2.3) of the Lyapunov function. Note that the incidence of form $Sf(I)$ is relatively specific compared to the form $g(S)f(I)$ or $f(S, I)$. For example, function $I^p/(1 + aI^q)$ does not satisfy the assumptions in Section 2 for certain values of $p > 0$ and $q > 0$. This means that formulating a suitable Lyapunov function is still a challenge to prove global stability of the more general systems. We could suggest that one may take priority of the linear combination of some of the functions

$$(x - x^*)^2, \int_{x^*}^{x} \frac{u - x^*}{u} du, \int_{x^*}^{x} \frac{u - x^*}{f(u)} du, \int_{x^*}^{x} \frac{f(u) - f(x^*)}{f(u)} du, \int_{x^*}^{x} \frac{f(u) - f(x^*)}{u} du.$$

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