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OPTIMAL TEMPORAL PATH ON SPATIAL DECAYING NETWORKS*

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Abstract We introduce temporal effect to the classical Kleinberg model and study how it affects the spatial structure of optimal transport network. The initial network is built from a regular d-dimensional lattice added by shortcuts with probability $p(r_{ij}) \sim r_{ij}^{-\alpha}$, where r_{ij} is the geometric distance between node i and j. By assigning each shortcut an energy $E = r \cdot \tau$, a link with length r survives within period τ, which leads the network to a decaying dynamics of constantly losing long-range links. We find new optimal transport in the dynamical system for $\alpha = \frac{3}{4}d$, in contrast to any other result in static systems. The conclusion does not depend on the information used for navigation, being based on local or global knowledge of the network, which indicates the possibility of the optimal design for general transport dynamics in the time-varying network.

Keywords Optimal transport, time-varying, small-world.

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1. Introduction

Finding short path is of great importance for understanding the propagation dynamics on complex networks. Much attention has been dedicated to the issue due to its wide application, ranging from small-world effect [33, 35], search and navigation [2–4, 6, 13, 14, 16, 17, 19, 21, 22, 29, 30, 36] to optimal network design [7, 20, 23]. The most famous study was done by Kleinberg, who proved the existence of the unique spatial structure supporting efficient navigation with local information [16, 17]. The framework of the model inspired the subsequent work, including optimal design for cost-limited system [19, 21, 22] and the enhanced Laplacian flow conditions in small-world networks [24]. All of them contributed to our knowledge of transport dynamics on spatial networks with static topology.

On the other hand, many complex systems are inherently dynamic as connections can appear and disappear adaptively. In particular, when the time scale of the transport dynamics is comparable to that of the topology fluctuation, the propagation process goes beyond our traditional knowledge. In this case, a transport path relies on not only the topology but also how the topologies are ordered

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in time. In other words, the transport dynamics relates to the time arrows of the topology fluctuation rather than the aggregated structure as in static network [11, 12, 15, 25–27, 33, 34]. The path in such system is called temporal path and the network is named time-varying network [11,12]. The temporal path generalizes the basic concept of shortest path in static graph, so that a variety of metrics, such as connectivity, clustering, betweeness centrality should be modified, as has been done in previous studies [11,12,15,25,33,34]. Consequently the ongoing transport dynamics based on these metrics can fundamentally change, which urges us to reconsider and generalize the traditional conclusions on the efficient structure for navigation.

Fortunately, the topology fluctuation is not completely random but dominated by intrinsic rules [5,8,28,31]. And various networks share some common properties: i) the network is usually spatially embedded, ii) the communication frequency or duration decays with the geometric distance, iii) the system is energy limited so that the link activity gradually dies out in a period. A typical example for the three properties is the human communication dynamics where the individuals are spatially embedded [1, 9, 32]. To our intuitive, we can easily communicate with people geographically or socially close but less frequently with those of distance. Besides, the biological nature restricts our activity intensity, forming clear period pattern as one day or one week. Many other systems associated with human dynamics such as airport network and Internet of Vehicles exhibit similar properties. But the geometric constraint on link durations, which surely plays a role in real-world networks, has not yet been revealed in most time-varying models [10, 27, 34].

Motivated by these considerations, we modify the Kleinberg model by introducing the temporal metrics and study the common influence of geometric and temporal effect on the navigation characteristics. We aim to determine its general behavior of the optimal transport, including the spatial structure and small-world phenomenon for both local and global information used. The paper is organized as follows. In Section II we propose the temporal network model and briefly present its basic properties. In Section III and IV, we study in detail the optimal path based on global and local knowledge of the network structure, respectively. We find that the introduction of the temporal effect subjected to energy and geometric constraints maintains the optimal behavior of the short path but changes the optimal exponent to $\alpha = \frac{3}{4}d$, in sharp contrast to the results obtained for any static systems. In Section V we draw the conclusion.

2. Model description

The classical Kleinberg model is built on a regular d-dimensional lattice. Each node i creates a shortcut(long-range link) to a distant node j according to the probability

$$p(r_{ij}) = \frac{r_{ij}^{-\alpha}}{\sum_j r_{ij}^{-\alpha}} \ , \tag{2.1}$$

where $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ is the Manhattan distance between i and j, and α is an exponent that controls the average length of shortcuts. When $\alpha = 0$, the network minimizes the average distance of its shortest paths [18]. If each node uses only the position of the target and its neighbor for routing, without global topological knowledge,

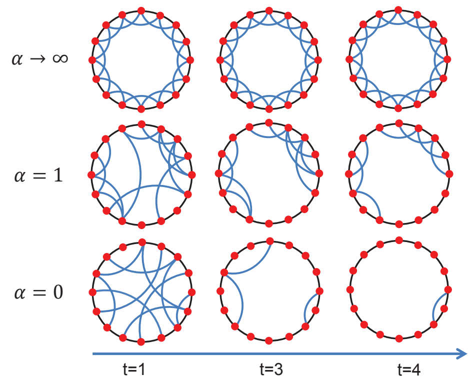

Figure 1. Examples of our model for the one-dimensional case. The evolutionary process of $\alpha = 0$, $\alpha = 1$ and $\alpha \to \infty$ are illustrated. The model starts from the Kleinberg network and degenerates gradually to the static lattice. The shortcut disconnects when time t exceeds its life time τ. For $\alpha = 0$, long-range links disappear too quickly to support later transport. For $\alpha = \infty$, lengths of shortcuts are not long enough to decrease the path distance. The intermediate case of $\alpha = 1$ shows better transport performance, with shortcuts of moderate length existing for a long time.

the actual navigated paths exhibit optimal at $\alpha = d$ [16, 17]. Some other conditions such as cost constraints can change the optimal exponent to $\alpha = d + 1$ [19].

Our model follows the similar construction except for two more new rules:

i) *Temporal rule*: each shortcut is assigned an energy

$$E = \tau \cdot r \ , \tag{2.2}$$

where r is the Manhattan distance and τ is the lifetime of the shortcut.

ii) *Evolution rule*: the network starts with the classical Kleinberg model and evolves according to i).

The temporal rule follows the general condition that communication duration decreases with the spatial distance. It introduces the time-varying effect by lifetime τ. The evolution rule indicates a structure decaying process, as the network loses shortcuts constantly and degenerates to a static lattice. It mimics the dissipative process in the energy-limited systems. The whole evolution of our model is visualized in Figure 1. With the increasing time, the network evolves from the Kleinberg model to the periodic lattice. The temporal path can be calculated from the graph sequence according to its definition. Note that a shortcut contributes to the transport dynamics only within its lifetime.

The equations (2.1) and (2.2) determine the basic properties of our model. The lifetime distribution of shortcuts can be easily deduced to be $p(\tau) \sim \tau^{-(d+1-\alpha)}$, indicating that large α causes more long-life shortcuts. Intuitively, more long-life shortcuts improve the transport performance. Therefore large α can decrease the average temporal path distance. On the other hand, the equation (2.1) indicates that larger α causes more local shortcuts of small r, which increases the path distance. Thus we expect an optimal α, balancing the distribution of τ and r, that leads the system to the most efficient transport. We present an example in Figure 1. For $\alpha = 0$, shortcuts of long-range are abundant at initial time but decrease so quickly that they disappear before messages arrive. On the other hand, large

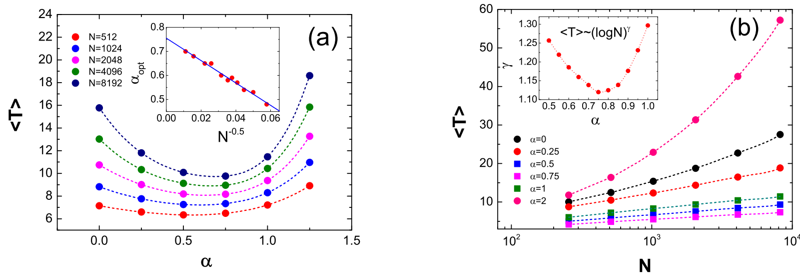

Figure 2. (a) Average temporal shortest path length $\langle T\rangle$ versus α for one-dimensional case. The optimal value α_{opt} is near 0.75. Inset: Finite size analysis presents $\alpha_{opt} = -4.651 \cdot N^{-0.5} + 0.754$(fitted by blue solid line), indicating $\alpha_{opt} \to 0.75$ in the limit of $N \to \infty$. (b) Average temporal shortest path length $\langle T\rangle$ versus the network size N. For $0.5 \leq \alpha \leq 1$, $\langle T\rangle$ grows slowly as $\langle T\rangle \sim log^{\gamma}(N)$, which is a signature of small-world effect. The minimum of γ occurs at $\alpha = 0.75$, consistent with the value of α_{opt}, as shown in the inset. For $\alpha < 0.5$ and $\alpha > 1$, a power-law relation $\langle T\rangle \sim N^{\beta}$ emerges.

$\alpha \to \infty$ causes short-length shortcuts. The network becomes tightly clustered, leading to low transport efficiency as regular lattice even though shortcuts maintain in the whole period. A better case occurs at $\alpha = 1$, where sufficient shortcuts with moderate length exist for a long time.

3. Navigation with global information

To investigate the temporal path quantitatively, we study the relation between exponent α and the average distance of shortest temporal path $\langle T\rangle$ by extensive simulations. We assume that the energy E equals to the diameter of the underlying lattice, i.e. the longest shortest path. This assumption is self-consistent in the sense that for those shortcuts of $\tau = 1$, their length does not exceed the upper bound of the geometric distance constrained by the spatial structure, as indicated by the equation (2.2). In the case of the periodic lattice of one and two dimension, $E = \frac{N}{2}$ and $\sqrt{N}$ respectively.

We firstly investigate the navigation path with global information, i.e. the average shortest temporal path $\langle T\rangle$, for one-dimensional case. As is depicted in Figure 2, it shows clearly the existence of the optimal transport at $\alpha_{opt} \approx 0.75$ for large networks of size N. To get the precise α_{opt} of $N \to \infty$, we apply the finite size scaling analysis. As shown in inset of Figure 2(a) and fitted by the blue dashed line, we find $\alpha_{opt} = -4.651 \cdot N^{-0.5} + 0.754$, which confirms $\alpha_{opt} = 0.75$.

In Figure 2(b), we show the relation between $\langle T\rangle$ and network size N. We observe two distinctive behaviors. For $0.5 \leq \alpha \leq 1$, the average shortest path length $\langle T\rangle$ increases as $\langle T\rangle \sim \log^{\gamma} N$, which is a typical signature of small-world effect. For those α tested out of the range, the small-world property disappears. Instead, a power-law dependence of $\langle T\rangle \sim N^{\beta}$ takes over. Intuitively, smaller γ causes slower increase of the path length, indicating shorter $\langle T\rangle$. Therefore relation between γ and α provides another way to confirm the optimal transport exponent. As shown in the inset, the minimum of γ stays exactly at $\alpha = 0.75$, consistent with the result obtained by finite size scaling approach.

For the case of two dimension, we apply similar analysis. The optimal $\langle T\rangle$ locates at $\alpha_{opt} \approx 1.5$, as shown in Figure 3(a). The finite size scaling analysis gives

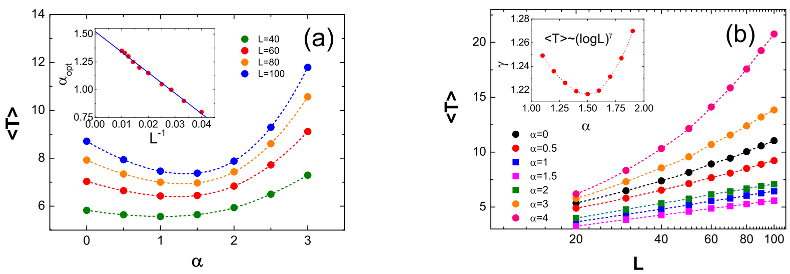

Figure 3. (a) Average temporal shortest path length $\langle T \rangle$ versus α for two-dimensional case. The optimal value α_{opt} is near 1.5. Inset:Finite size analysis presents $\alpha_{opt} = -19.572 \cdot L^{-1} + 1.568$ (fitted by blue solid line), where $L = \sqrt{N}$. It indicates $\alpha_{opt} \to 1.5$ in the limit of $N \to \infty$. (b) Average temporal shortest path length $\langle T \rangle$ versus the network linear size L. For $1 \le \alpha \le 2$, $\langle T \rangle$ grows slowly as $\langle T \rangle \sim log^{\gamma}(L)$, which is a signature of small-world effect. The minimum of γ occurs at $\alpha = 1.5$, consistent with the value of α_{opt}, as shown in the inset. For $\alpha < 1$ and $\alpha > 2$, a power-law relation $\langle T \rangle \sim L^{\beta}$ emerges.

$\alpha_{opt} = -19.572 \cdot L^{-1} + 1.568$, confirming $\alpha_{opt} = 1.5$ for $L \to \infty$, where $L = \sqrt{N}$ is the linear size of the network. The relation between $\langle T \rangle$ and L again shows two different behaviors. For $1 \le \alpha \le 2$, $\langle T \rangle$ grows slowly as $\langle T \rangle \sim \log^{\gamma} N$. For $\alpha < 1$ and $\alpha > 2$, a power-law relation $\langle T \rangle \sim L^{\beta}$ emerges.

The simulation results for both one and two dimension indicate the following conclusion. There exists an optimal state for the transport dynamics in the time-varying network. Based on several different simulations and finite size scaling analysis, we conjecture that the optimal state occurs in a new spatial structure with the exponent

$$\alpha_{opt} = \frac{3}{4} d, \tag{3.1}$$

in sharp contrast to $\alpha_{opt} = 0$ [16, 17]and $\alpha_{opt} = d + 1$ [19] found in static network. The small-world property emerges in the new range of

$$d/2 \le \alpha \le d. \tag{3.2}$$

In Ref. [19], the cost constraint leads to the small world occuring at a single value of α. Under the similar constraint, our model shows a wider α for efficient routing, which indicates a much more relaxed condition for transport design.

4. Navigation with local information

In many practical cases, individuals can use only local information for transport routing. Here local information refers to the coordinate of target and neighbors only, without knowledge of global topological details. The limited information may cause the actual navigation path to deviate from the optimal one and change the whole transport dynamics.

To study the optimal path navigated with local information, we follow the routing strategy of classical Kleinberg model. Specifically, each message holder passes the message to one of its neighbors, who locates the nearest to the target node. The strategy gives optimal navigation exponent of $\alpha_{local} = d$ for Kleinberg model [16,17].

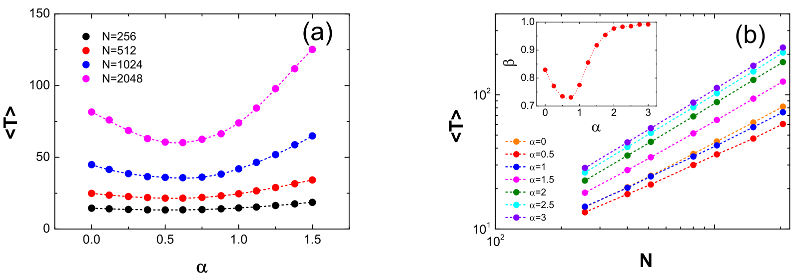

Figure 4. (a) Average length of temporal navigation path $\langle T \rangle$ with local information versus α for one-dimensional case. The optimal α is observed close to 0.75. (b) $\langle T \rangle$ versus the network size N for different α. All the $\langle T \rangle$ follow a power-law behavior, $\langle T \rangle \sim N^{\beta}$. The minimum of β is observed at $\alpha = 0.75$, consistent with the result in (a), as depicted in the Inset. All the results are averaged over 100 independent realizations.

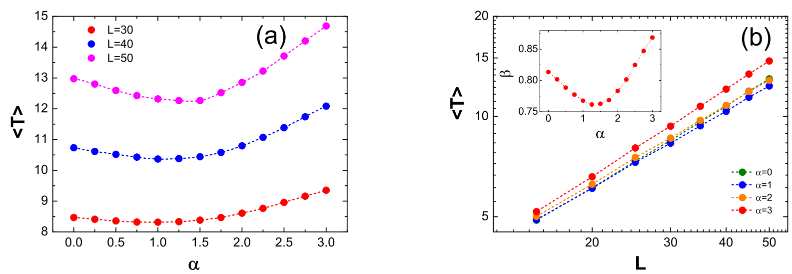

Figure 5. (a) Average length of temporal navigation path $\langle T \rangle$ with local information versus α for two-dimensional case. The optimal α is observed close to 1.5. (b) $\langle T \rangle$ versus the network linear size $L = \sqrt{N}$ for different α. All the $\langle T \rangle$ follow a power-law behavior, $\langle T \rangle \sim L^{\beta}$. The minimum of β is observed at $\alpha = 1.5$, consistent with the result in (a), as depicted in the Inset. All the results are averaged over 100 independent realizations.

In our model, the extensive simulations indicate that the optimal navigation in one-dimensional case occurs near $\alpha_{local} = 0.75$, as shown in Figure 4. However, all the $\langle T \rangle$ scale as $\langle T \rangle \sim N^{\beta}$ regardless of the value that α takes. To eliminate the finite size effect and find the precise α_{local}, we measure the scaling exponent β. The minimum β corresponds to the shortest $\langle T \rangle$. As shown in the inset of Figure 4(b), the minimum β locates at $\alpha_{local} = 0.75$, which is the same value as navigation with global information found in Section III. The similar result is also found in two-dimensional case. As shown in Figure 5, both the simulations and the finite size scaling analysis give $\alpha_{local} = \alpha_{opt} = 1.5$. Therefore in the case of navigation with local information, the optimal transport occurs at $\alpha_{local} = \frac{3}{4}d$, identical with α_{opt} of global shortest temporal path, but the small-world effect vanishes.

5. Conclusion

We propose a time-varying model embedded on a periodic lattice and study how the temporal effect affects the transport process. We find that the optimal transport occurs in a new spatial structure with the optimal exponent $\alpha_{opt} = \frac{3}{4}d$, in sharp contrast to $\alpha = d$ and $\alpha = d + 1$ established in static networks [16, 17, 19]. The conclusion does not depend on the information used for navigation, being based on local or global knowledge of the network. We also determine the area of α for the small-world feature.

Our model presents a natural way to introduce the temporal effect into spatial-constrained networks. The decaying picture mimics the dissipative process in the energy-limited systems, which can be relaxed to other general conditions. We have checked our model in a more general scenario of assigning random birth time to the shortcuts and have obtained similar results. Therefore our conclusion is not strongly model-dependent. All these results indicate the possibility of the optimal design for the general transport dynamics in the time-varying network. The investigations on more practical generalizations such as the limited capacity of each node and heterogeneous energy distribution are left open for further efforts.

References

[1] A.-L. Barabási, *The origin of bursts and heavy tails in human dynamics*, Nature, 435(2005)(7039), 207–211.

[2] M. Boguñá, D. Krioukov and K. C. Claffy, *Navigability of complex networks*, Nat. Phys., 5(2008)(1), 74–80.

[3] S. Carmi, S. Carter, J. Sun and D. Ben-Avraham, *Asymptotic behavior of the kleinberg model*, Phys. Rev. Lett., 102(2009)(23), 238702.

[4] C. C. Cartozo and P. De Los Rios, *Extended navigability of small world networks: exact results and new insights*, Phys. Rev. Lett., 102(2009)(23), 238703.

[5] Q. Chen, J.-H. Qian and D.-D. Han, *Non-Gaussian behavior of the internet topological fluctuations*, Int. J. Mod Phys C, 25(2014)(05), 1440012.

[6] A. P. de Moura, A. E. Motter and C. Grebogi, *Searching in small-world networks*, Phys. Rev. E, 68(2003)(3), 036106.

[7] M. T. Gastner and M. Newman, *Optimal design of spatial distribution networks*, Phys. Rev. E, 74(2006)(1), 016117.

[8] A. Gautreau, A. Barrat and M. Barthelemy, *Microdynamics in stationary complex networks*, Proc. Natl. Acad. Sci. U.S.A., 106(2009)(22), 8847–8852.

[9] M. C. González, C. A. Hidalgo and A.-L. Barabási, *Understanding individual human mobility patterns*, Nature, 453(2008)(7196), 779–782.

[10] S. A. Hill and D. Braha, *Dynamic model of time-dependent complex networks*, Phys. Rev. E, 82(2010)(4), 046105.

[11] P. Holme, *Network reachability of real-world contact sequences*, Phys. Rev. E, 71(2005)(4), 046119.

[12] P. Holme and J. Saramäki, *Temporal networks*, Phys. Rep., 519(2012)(3), 97–125.

[13] Y. Hu, Y. Wang, D. Li et al., *Possible origin of efficient navigation in small worlds*, Phys. Rev. Lett., 106(2011)(10), 108701.

[14] W. Huang, S. Chen and W. Wang, *Navigation in spatial networks: A survey*, Physica A, 393(2014), 132–154.

[15] H. Kim and R. Anderson, *Temporal node centrality in complex networks*, Phys. Rev. E, 85(2012)(2), 026107.

[16] J. M. Kleinberg, *Navigation in a small world*, Nature, 406(2000)(6798), 845–845.

[17] J. M. Kleinberg, *The small-world phenomenon: An algorithmic perspective*, in *Proceedings of the thirty-second annual ACM symposium on Theory of computing*, ACM, New York, 2000, 163–170.

[18] K. Kosmidis, S. Havlin and A. Bunde, *Structural properties of spatially embedded networks*, EPL (Europhysics Letters), 82(2008)(4), 48005.

[19] G. Li, S. Reis, A. Moreira et al., *Towards design principles for optimal transport networks*, Phys. Rev. Lett., 104(2010)(1), 018701.

[20] G. Li, S. Reis, A. Moreira et al., *Optimal transport exponent in spatially embedded networks*, Phys. Rev. E, 87(2013)(4), 042810.

[21] Y. Li, D. Zhou, Y. Hu et al., *Exact solution for optimal navigation with total cost restriction*, EPL (Europhysics Letters), 92(2010)(5), 58002.

[22] W. Liu, A. Zeng and Y. Zhou, *Degree heterogeneity in spatial networks with total cost constraint*, EPL (Europhysics Letters), 98(2012)(2), 28003.

[23] C. F. Moukarzel and M. A. de Menezes, *Shortest paths on systems with power-law distributed long-range connections*, Phys. Rev. E, 65(2002)(5), 056709.

[24] C. L. Oliveira, P. A. Morais, A. A. Moreira and J. S. Andrade Jr, *Enhanced flow in small-world networks*, Phys. Rev. Lett., 112(2014)(14), 148701.

[25] R. K. Pan and J. Saramäki, *Path lengths, correlations, and centrality in temporal networks*, Phys. Rev. E, 84(2011)(1), 016105.

[26] N. Perra, A. Baronchelli, D. Mocanu et al., *Random walks and search in time-varying networks*, Phys. Rev. Lett., 109(2012)(23), 238701.

[27] N. Perra, B. Gonçalves, R. Pastor-Satorras and A. Vespignani, *Activity driven modeling of time varying networks*, Sci. Rep., 2(2012)(469).

[28] J.-H. Qian, Q. Chen, D.-D. Han et al., *Origin of gibrat law in internet: Asymmetric distribution of the correlation*, Phys. Rev. E, 89(2014)(6), 062808.

[29] A. Riascos and J. L. Mateos, *Long-range navigation on complex networks using lévy random walks*, Phys. Rev. E, 86(2012)(5), 056110.

[30] M. R. Roberson and D. Ben-Avraham, *Kleinberg navigation in fractal small-world networks*, Phys. Rev. E, 74(2006)(1), 017101.

[31] D. Rybski, S. V. Buldyrev, S. Havlin et al., *Scaling laws of human interaction activity*, Proc. Natl. Acad. Sci. U.S.A., 106(2009)(31), 12640–12645.

[32] M. Starnini, A. Baronchelli and R. Pastor-Satorras, *Modeling human dynamics of face-to-face interaction networks*, Phys. Rev. Lett., 110(2013)(16), 168701.

[33] J. Tang, S. Scellato, M. Musolesi et al., *Small-world behavior in time-varying graphs*, Phys. Rev. E, 81(2010)(5), 055101.

[34] S. Trajanovski, S. Scellato and I. Leontiadis, *Error and attack vulnerability of temporal networks*, Phys. Rev. E, 85(2012)(6), 066105.

[35] D. J. Watts and S. H. Strogatz, *Collective dynamics of small-world networks*, Nature, 393(1998)(6684), 440–442.

[36] H. Zhu and Z.-X. Huang, *Navigation in a small world with local information*, Phys. Rev. E, 70(2004)(3), 036117.