

## SEVERAL FORMULAS FOR SPECIAL VALUES OF THE BELL POLYNOMIALS OF THE SECOND KIND AND APPLICATIONS

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**Abstract** In the paper, the authors establish several explicit formulas for special values of the Bell polynomials of the second kind, connect these formulas with the Bessel polynomials, and apply these formulas to give new expressions for the Catalan numbers and to compute arbitrary higher order derivatives of elementary functions such as the sine, cosine, exponential, logarithm, arcsine, and arccosine of the square root for the variable.

**Keywords** Bell polynomial of the second kind, formula, Catalan number, connection, Bessel polynomial, derivative, elementary function.

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### 1. Notation and main results

In combinatorial analysis, the Bell polynomials of the second kind, also known as the partial Bell polynomials, denoted by  $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ , can be defined by

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n, \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^n i \ell_i = n \\ \sum_{i=1}^n \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}$$

for  $n \geq k \geq 0$ . The well-known Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind  $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$  by

$$\frac{d^n}{dx^n} f \circ g(x) = \sum_{k=0}^n f^{(k)}(g(x)) B_{n,k}(g'(x), g''(x), \dots, g^{(n-k+1)}(x)), \quad (1.1)$$

see [2, p. 139, Theorem C].

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The first aim of this paper is to discover an explicit formula for special values

$$B_{n,k}((-1)!!, 1!!, 3!!, \dots, [2(n-k) - 1]!!). \quad (1.2)$$

**Theorem 1.1.** For  $n \geq k \geq 0$ , we have

$$B_{n,k}((-1)!!, 1!!, 3!!, \dots, [2(n-k) - 1]!!) = \frac{(-1)^n}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{m=0}^{n-1} (\ell - 2m). \quad (1.3)$$

In recent years, several explicit formulas of special values for the Bell polynomials of the second kind  $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$  were discovered, recovered, and applied in [4, 6, 19, 22, 30, 37] and references cited therein.

The Bessel polynomials were defined in [9] by

$$y_n(x) = \sum_{k=0}^n \frac{(n+k)!}{(n-k)!k!} \left(\frac{x}{2}\right)^k \triangleq \sum_{k=0}^n b_{n,k} x^k. \quad (1.4)$$

The first five Bessel polynomials  $y_n(x)$  for  $0 \leq n \leq 4$  are

$$\begin{aligned} y_0(x) &= 1, & y_1(x) &= x + 1, & y_2(x) &= 3x^2 + 3x + 1, \\ y_3(x) &= 15x^3 + 15x^2 + 6x + 1, & y_4(x) &= 105x^4 + 105x^3 + 45x^2 + 10x + 1. \end{aligned}$$

For more information on the Bessel polynomials  $y_n(x)$ , please refer to the web sites [33, 34, 38].

The second aim is to simplify the right hand side of the identity (1.3) in Theorem 1.1 and, as a consequence, to find a connection between special values of the quantity (1.2) and coefficients  $b_{n,k}$ , defined in (1.4), of the Bessel polynomials  $y_n(x)$ .

**Theorem 1.2.** For  $n \geq k \geq 0$ , we have

$$B_{n,k}((-1)!!, 1!!, 3!!, \dots, [2(n-k) - 1]!!) = \binom{2n-k-1}{2(n-k)} [2(n-k) - 1]!!. \quad (1.5)$$

Consequently, coefficients  $b_{n,k}$  of the Bessel polynomials  $y_n(x)$  and special values of the quantity (1.2) satisfy

$$b_{n,k} = B_{n+1, n-k+1}((-1)!!, 1!!, 3!!, \dots, (2k-1)!), \quad n \geq k \geq 0. \quad (1.6)$$

In combinatorial number theory, the Catalan numbers  $C_n$  for  $n \geq 0$  form a sequence of natural numbers that occur in tree enumeration problems such as ‘‘In how many ways can a regular  $n$ -gon be divided into  $n - 2$  triangles if different orientations are counted separately?’’ The first twelve Catalan numbers  $C_n$  for  $0 \leq n \leq 11$  are

$$1, \quad 1, \quad 2, \quad 5, \quad 14, \quad 42, \quad 132, \quad 429, \quad 1430, \quad 4862, \quad 16796, \quad 58786.$$

Let us recall some conclusions in [3, 8, 35, 36] as follows. Explicit formulas of  $C_n$  for  $n \geq 0$  include

$$\begin{aligned} C_n &= \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{n!(n+1)!} = \frac{2^n (2n-1)!!}{(n+1)!} = (-1)^n 2^{2n+1} \binom{1/2}{n+1} \\ &= \frac{1}{n} \binom{2n}{n-1} = \frac{4^n \Gamma(n+1/2)}{\sqrt{\pi} \Gamma(n+2)} = {}_2F_1(1-n, -n; 2; 1), \end{aligned} \quad (1.7)$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0$$

is the classical Euler gamma function and

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{n=0}^\infty \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}$$

is the generalized hypergeometric series defined for positive integers  $p, q \in \mathbb{N}$ , for complex numbers  $a_i \in \mathbb{C}$  and  $b_i \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ , and in terms of the rising factorials

$$(x)_n = \begin{cases} x(x+1)(x+2)\dots(x+n-1), & n \geq 1, \\ 1, & n = 0. \end{cases}$$

The asymptotic form for the Catalan function

$$C_x \triangleq \frac{4^x \Gamma(x + \frac{1}{2})}{\sqrt{\pi} \Gamma(x + 2)}$$

is

$$C_x \sim \frac{4^x}{\sqrt{\pi}} \left( \frac{1}{x^{3/2}} - \frac{9}{8} \frac{1}{x^{5/2}} + \frac{145}{128} \frac{1}{x^{7/2}} + \dots \right).$$

Motivated by the sixth expression in (1.7) and by virtue of an integral representation of the gamma function  $\ln \Gamma(x)$ , the authors represented in [32, Theorem 1] the Catalan function  $C_x$  as

$$C_x = \frac{e^{3/2} 4^x (x + 1/2)^x}{\sqrt{\pi} (x + 2)^{x+3/2}} \exp \left[ \int_0^\infty \frac{1}{t} \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) (e^{-t/2} - e^{-2t}) e^{-xt} dt \right]$$

for  $x \geq 0$ . Hereafter, the above integral representation was further deeply cultivated in [15, 29]. For more detailed information on the Catalan numbers  $C_n$ , please refer to the monographs and websites [2, 3, 35, 39] and references cited therein.

The third aim of this paper is to apply the formulas (1.3) and (1.5) in Theorems 1.1 and 1.2 to express the Catalan numbers as new forms below.

**Theorem 1.3.** *For  $n \geq 0$ , the Catalan numbers can be expressed by*

$$C_n = (-1)^n \frac{2^n}{n!} \sum_{k=0}^n \frac{1}{2^k} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{m=0}^{n-1} (\ell - 2m) \tag{1.8}$$

$$= \frac{2^n}{n!} \sum_{k=0}^n \frac{k!}{2^k} \binom{2n-k-1}{2(n-k)} [2(n-k)-1]!! \tag{1.9}$$

Finally, the fourth aim of this paper is to apply the formula (1.5) in Theorem 1.2 to compute arbitrary higher order derivatives of several elementary functions of the form  $f(\sqrt{a+bx})$  for  $a, b \in \mathbb{R}$  and  $b \neq 0$ . As examples, we obtain the following results.

**Theorem 1.4.** *Let  $g(x) = \sqrt{a+bx}$  for  $a, b \in \mathbb{R}$  and  $b \neq 0$  and let  $n \in \mathbb{N}$ . Then the Bell polynomials of the second kind  $B_{n,k}$  satisfy*

$$\begin{aligned} & B_{n,k}(g'(x), g''(x), \dots, g^{(n-k+1)}(x)) \\ &= (-1)^{n+k} [2(n-k)-1]!! \left(\frac{b}{2}\right)^n \binom{2n-k-1}{2(n-k)} \frac{1}{(a+bx)^{n-k/2}}. \end{aligned} \tag{1.10}$$

Consequently, for  $n \geq 0$ , we have

$$\begin{aligned}\frac{d^n(\sin \sqrt{x})}{dx^n} &= \frac{(-1)^n}{(2x)^n} \sum_{k=0}^n (-1)^k [2(n-k) - 1]!! \binom{2n-k-1}{2(n-k)} x^{k/2} \sin\left(\sqrt{x} + k\frac{\pi}{2}\right), \\ \frac{d^n(\cos \sqrt{x})}{dx^n} &= \frac{(-1)^n}{(2x)^n} \sum_{k=0}^n (-1)^k [2(n-k) - 1]!! \binom{2n-k-1}{2(n-k)} x^{k/2} \cos\left(\sqrt{x} + k\frac{\pi}{2}\right), \\ \frac{d^n(e^{\sqrt{x}})}{dx^n} &= \frac{(-1)^n}{(2x)^n} e^{\sqrt{x}} \sum_{k=0}^n (-1)^k [2(n-k) - 1]!! \binom{2n-k-1}{2(n-k)} x^{k/2},\end{aligned}$$

and

$$\begin{aligned}\frac{d^n[\ln(1 \pm \sqrt{x})]}{dx^n} &= \frac{(-1)^{n+1}}{(2x)^n} \sum_{k=0}^n (\pm 1)^k (k-1)! [2(n-k) - 1]!! \binom{2n-k-1}{2(n-k)} \left(\frac{\sqrt{x}}{1 \pm \sqrt{x}}\right)^k.\end{aligned}$$

For  $n \geq 1$ , we have

$$\begin{aligned}\frac{d^n(\arcsin x)}{dx^n} &= -\frac{d^n(\arccos x)}{dx^n} = \frac{d^{n-1}}{dx^{n-1}} \left(\frac{1}{\sqrt{1-x^2}}\right) \\ &= \frac{1}{(2x)^n} \sum_{k=0}^{n-1} 2^{k+1} (2k-1)! (n-k-1)! \binom{n-1}{k} \binom{k}{n-k-1} \left(\frac{x^2}{1-x^2}\right)^{k+1/2}.\end{aligned}\tag{1.11}$$

Consequently, for  $n \geq 1$ , we have

$$\begin{aligned}\frac{d^n(\arcsin \sqrt{x})}{dx^n} &= -\frac{d^n(\arccos \sqrt{x})}{dx^n} \\ &= \frac{(-1)^n}{(2x)^n} \sum_{k=1}^n \frac{(-1)^k}{2^k} [2(n-k) - 1]!! \binom{2n-k-1}{2(n-k)} \frac{1}{(\sqrt{x})^k} \\ &\quad \times \sum_{\ell=0}^{k-1} 2^{\ell+1} (2\ell-1)! (k-\ell-1)! \binom{k-1}{\ell} \binom{\ell}{k-\ell-1} \left(\frac{x^2}{1-x^2}\right)^{\ell+1/2}.\end{aligned}$$

## 2. Proofs of Theorems 1.1 to 1.4

Now we are in a position to prove Theorems 1.1 to 1.4 in details.

**Proof.** [First proof of Theorem 1.1] In [2, p. 133], it was collected that

$$\frac{1}{k!} \left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \frac{t^n}{n!}$$

for  $k \geq 0$ . From this, it follows that

$$\begin{aligned}\sum_{n=k}^{\infty} B_{n,k}((-1)!!, 1!!, 3!!, \dots, [2(n-k) - 1]!!) \frac{t^n}{n!} &= \frac{1}{k!} \left(\sum_{m=1}^{\infty} (2m-3)!! \frac{t^m}{m!}\right)^k \\ &= \frac{1}{k!} (1 - \sqrt{1-2t})^k.\end{aligned}$$

Differentiating  $m \geq k$  times on both sides of the above equation reveals that

$$\begin{aligned} & \sum_{n=m}^{\infty} B_{n,k}((-1)!!, 1!!, 3!!, \dots, [2(n-k)-1]!!) \frac{t^{n-m}}{(n-m)!} \\ &= \frac{1}{k!} \frac{d^m}{dt^m} (1 - \sqrt{1-2t})^k = (-1)^k \frac{1}{k!} \frac{d^m}{dt^m} (\sqrt{1-2t} - 1)^k \\ &= (-1)^k \frac{1}{k!} \frac{d^m}{dt^m} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} (1-2t)^{\ell/2} = \frac{1}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \frac{d^m}{dt^m} (1-2t)^{\ell/2} \\ &= \frac{1}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{p=0}^{m-1} \left(\frac{\ell}{2} - p\right) (-2)^m (1-2t)^{\ell/2-m}. \end{aligned}$$

Further letting  $t \rightarrow 0$  yields

$$\begin{aligned} B_{m,k}((-1)!!, 1!!, 3!!, \dots, (2(m-k)-1)!!) &= \frac{1}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{p=0}^{m-1} \left(\frac{\ell}{2} - p\right) (-2)^m \\ &= (-1)^m \frac{1}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{p=0}^{m-1} (\ell - 2p). \end{aligned}$$

The formula (1.3) is thus proved. The first proof of Theorem 1.1 is complete.  $\square$

**Proof.** [Second proof of Theorem 1.1] The  $n$ th derivative of the function  $\frac{2}{1+\sqrt{1-4x}}$  can be computed by the Faà di Bruno formula (1.1) or by the following formula

$$\frac{d^n y}{dx^n} = \sum_{k=0}^n \frac{(-1)^k}{k!} \sum_{\alpha=0}^k (-1)^\alpha \binom{k}{\alpha} u^{k-\alpha} \frac{d^n(u^\alpha)}{dx^n} \frac{d^k y}{du^k}, \quad n \geq 0 \tag{2.1}$$

in [31, p. 12, (83)], where  $y = \phi(u)$  and  $u = f(x)$ .

Taking in (1.1)  $f(u) = \frac{2}{1+u}$  and  $u = g(x) = \sqrt{1-4x}$  yields

$$\begin{aligned} & \left(\frac{2}{1+\sqrt{1-4x}}\right)^{(n)} \\ &= 2 \sum_{k=0}^n (-1)^k \frac{k!}{(1+u)^{k+1}} \\ & \quad \times B_{n,k}\left(-\frac{2}{(1-4x)^{1/2}}, -\frac{2^2}{(1-4x)^{3/2}}, \dots, -\frac{2^{n-k+1}[2(n-k+1)-3]!!}{(1-4x)^{[2(n-k+1)-1]/2}}\right) \\ &= 2 \sum_{k=0}^n (-1)^k \frac{k!}{(1+\sqrt{1-4x})^{k+1}} (-1)^k 2^n \frac{(1-4x)^{k/2}}{(1-4x)^n} \\ & \quad \times B_{n,k}((-1)!!, 1!!, \dots, [2(n-k)-1]!!) \\ &= \frac{2^{n+1}}{(1-4x)^n} \sum_{k=0}^n \frac{k!(\sqrt{1-4x})^k}{(1+\sqrt{1-4x})^{k+1}} B_{n,k}((-1)!!, 1!!, \dots, [2(n-k)-1]!!), \tag{2.2} \end{aligned}$$

where the formula

$$B_{n,k}(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \tag{2.3}$$

for complex numbers  $a$  and  $b$ , which was listed in [2, p. 135], was employed above.

Taking in (2.1)  $y = \phi(u) = \frac{2}{1+u}$  and  $u = f(x) = \sqrt{1-4x}$  gives

$$\begin{aligned}
\left(\frac{2}{1+\sqrt{1-4x}}\right)^{(n)} &= \sum_{k=0}^n \frac{(-1)^k}{k!} \sum_{\alpha=0}^k (-1)^\alpha \binom{k}{\alpha} (\sqrt{1-4x})^{k-\alpha} \\
&\quad \times \frac{d^n[(1-4x)^{\alpha/2}]}{dx^n} \frac{d^k}{du^k} \left(\frac{2}{1+u}\right) \\
&= \sum_{k=0}^n \frac{(-1)^k}{k!} \sum_{\alpha=0}^k (-1)^\alpha \binom{k}{\alpha} (\sqrt{1-4x})^{k-\alpha} \\
&\quad \times \prod_{\ell=0}^{n-1} (\alpha-2\ell)(-2)^n (1-4x)^{\alpha/2-n} \frac{2(-1)^k k!}{(1+\sqrt{1-4x})^{k+1}} \\
&= (-1)^n \frac{2^{n+1}}{(1-4x)^n} \sum_{k=0}^n \frac{k!(\sqrt{1-4x})^k}{(1+\sqrt{1-4x})^{k+1}} \sum_{\alpha=0}^k \frac{(-1)^\alpha}{\alpha!(k-\alpha)!} \prod_{\ell=0}^{n-1} (\alpha-2\ell).
\end{aligned} \tag{2.4}$$

Comparing derivatives (2.2) and (2.4) leads to (1.3). The second proof of Theorem 1.1 is complete.  $\square$

**Proof.** [Proof of Theorem 1.2] In [33], it was mentioned that

$$B_{n,k}(f', f'', f''', \dots, f^{(n-k+1)}) = T(n-1, k-1)(1-2x)^{k/2-n}, \tag{2.5}$$

where  $f(x) = 1 - \sqrt{1-2x}$  and

$$T(n, k) = \binom{2n-k}{2(n-k)} [2(n-k)-1]!!, \quad n \geq k \geq 0. \tag{2.6}$$

On the other hand, as done in (2.2), it follows that

$$\begin{aligned}
&B_{n,k}(f', f'', f''', \dots, f^{(n-k+1)}) \\
&= B_{n,k} \left( \frac{1}{(1-2x)^{1/2}}, \frac{1}{(1-2x)^{3/2}}, \dots, \frac{[2(n-k+1)-3]!!}{(1-2x)^{[2(n-k+1)-1]/2}} \right) \\
&= B_{n,k}((-1)!!, 1!!, \dots, [2(n-k)-1]!!) (1-2x)^{k/2-n}.
\end{aligned}$$

Since the sequence of the functions  $(1-2x)^{k/2-n}$  is linearly independent, we have

$$B_{n,k}((-1)!!, 1!!, \dots, [2(n-k)-1]!!) = T(n-1, k-1) = \binom{2n-k-1}{2(n-k)} [2(n-k)-1]!!,$$

that is, the identity (1.5) follows immediately.

The equality (1.6) comes from directly verifying  $b_{n,k} = T(n, n-k)$ . The proof of Theorem 1.2 is complete.  $\square$

**Proof.** [Proof of Theorem 1.3] The Catalan numbers  $C_n$  can be generated [35, 39] by

$$\frac{2}{1+\sqrt{1-4x}} = \frac{1-\sqrt{1-4x}}{2x} = \sum_{k=0}^{\infty} C_k x^k. \tag{2.7}$$

Hence, making use of the equations (2.2) and (2.4) and employing Theorem 1.1, it

follows that

$$\begin{aligned}
 C_n &= \frac{1}{n!} \lim_{x \rightarrow 0} \left( \frac{2}{1 + \sqrt{1 - 4x}} \right)^{(n)} \\
 &= \frac{2^{n+1}}{n!} \sum_{k=0}^n \frac{k!}{2^{k+1}} B_{n,k}((-1)!!, 1!!, \dots, [2(n-k) - 1]!!) \\
 &= \frac{2^{n+1}}{n!} \sum_{k=0}^n \frac{k!}{2^{k+1}} (-1)^n \frac{1}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{m=0}^{n-1} (\ell - 2m) \\
 &= (-1)^n \frac{2^n}{n!} \sum_{k=0}^n \frac{1}{2^k} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{m=0}^{n-1} (\ell - 2m).
 \end{aligned}$$

Therefore, the equality (1.8) is deduced readily.

Similarly, we can derive the equality (1.9). The proof of Theorem 1.3 is complete.  $\square$

**Proof.** [Proof of Theorem 1.4] By virtue of (2.3) and the formula (1.5), we have

$$\begin{aligned}
 &B_{n,k}(g'(x), g''(x), \dots, g^{(n-k+1)}(x)) \\
 &= B_{n,k} \left( \frac{1}{2} \frac{b}{(a+bx)^{1/2}}, \frac{1}{2} \left( -\frac{1}{2} \right) \frac{b^2}{(a+bx)^{3/2}}, \right. \\
 &\quad \left. \frac{1}{2} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \frac{b^3}{(a+bx)^{5/2}}, \dots, (-1)^{n-k} \frac{[2(n-k) - 1]!!}{2^{n-k+1}} \frac{b^{n-k+1}}{(a+bx)^{(2(n-k)+1)/2}} \right) \\
 &= (-1)^{n+k} \frac{1}{2^n} (a+bx)^{k/2} \frac{b^n}{(a+bx)^n} B_{n,k}((-1)!!, 1!!, 3!!, \dots, [2(n-k) - 1]!!) \\
 &= (-1)^{n+k} \frac{1}{2^n} \binom{2n-k-1}{2(n-k)} [2(n-k) - 1]!! \frac{b^n}{(a+bx)^{n-k/2}}.
 \end{aligned}$$

The formula (1.10) is thus derived.

Further, by the Faà di Bruno formula (1.1) applied to the functions  $f(u) = \sin u$  and  $u = g(x) = \sqrt{x}$  and by the formula (1.10) applied to  $a = 0$  and  $b = 1$ , we easily obtain

$$\begin{aligned}
 \frac{d^n}{dx^n} (\sin \sqrt{x}) &= \sum_{k=0}^n \sin^{(k)} u B_{n,k}(g'(x), g''(x), \dots, g^{(n-k+1)}(x)) \\
 &= \sum_{k=0}^n \sin \left( \sqrt{x} + k \frac{\pi}{2} \right) (-1)^{n+k} \frac{1}{2^n} \binom{2n-k-1}{2(n-k)} [2(n-k) - 1]!! \frac{1}{x^{n-k/2}} \\
 &= \frac{(-1)^n}{(2x)^n} \sum_{k=0}^n (-1)^k \binom{2n-k-1}{2(n-k)} [2(n-k) - 1]!! x^{k/2} \sin \left( \sqrt{x} + k \frac{\pi}{2} \right).
 \end{aligned}$$

As did just now, considering  $\cos^{(n)} x = \cos \left( x + n \frac{\pi}{2} \right)$  leads to the formula for the derivative  $\frac{d^n}{dx^n} (\cos \sqrt{x})$  in Theorem 1.4.

Similarly, we acquire

$$\frac{d^n}{dx^n} (e^{\sqrt{x}}) = e^{\sqrt{x}} \sum_{k=0}^n B_{n,k}(g'(x), g''(x), \dots, g^{(n-k+1)}(x))$$

$$\begin{aligned}
&= e^{\sqrt{x}} \sum_{k=0}^n (-1)^{n+k} \frac{1}{2^n} \binom{2n-k-1}{2(n-k)} [2(n-k)-1]!! \frac{1}{x^{n-k/2}} \\
&= (-1)^n \frac{e^{\sqrt{x}}}{(2x)^n} \sum_{k=0}^n (-1)^k \binom{2n-k-1}{2(n-k)} [2(n-k)-1]!! x^{k/2}.
\end{aligned}$$

Furthermore, we see that

$$\begin{aligned}
&\frac{d^n}{dx^n} [\ln(1 \pm \sqrt{x})] \\
&= \sum_{k=0}^n \frac{(-1)^{k-1} (k-1)!}{(1 \pm \sqrt{x})^k} B_{n,k}(\pm g'(x), \dots, \pm g^{(n-k+1)}(x)) \\
&= \sum_{k=0}^n \frac{(-1)^{k-1} (k-1)!}{(1 \pm \sqrt{x})^k} (\pm 1)^k B_{n,k}(g'(x), \dots, g^{(n-k+1)}(x)) \\
&= \sum_{k=0}^n \frac{(-1)^{k-1} (k-1)!}{(1 \pm \sqrt{x})^k} (\pm 1)^k (-1)^{n+k} \frac{1}{2^n} \binom{2n-k-1}{2(n-k)} [2(n-k)-1]!! \frac{1}{x^{n-k/2}} \\
&= \frac{(-1)^{n+1}}{(2x)^n} \sum_{k=0}^n (\pm 1)^k (k-1)! \binom{2n-k-1}{2(n-k)} [2(n-k)-1]!! \left( \frac{\sqrt{x}}{1 \pm \sqrt{x}} \right)^k.
\end{aligned}$$

In the proof of [30, Theorem 3.1, pp. 601–602], it was derived that

$$\frac{d^n}{dx^n} \left( \frac{1}{\sqrt{1-x^2}} \right) = \sum_{k=0}^n \frac{(2k-1)!!}{(1-x^2)^{k+1/2}} B_{n,k}(x, 1, 0, \dots, 0), \quad n \geq 0.$$

In [6, Theorem 4.1], it was established that the Bell polynomials of the second kind  $B_{n,k}$  for  $0 \leq k \leq n$  satisfy

$$B_{n,k}(x, 1, 0, \dots, 0) = \frac{(n-k)!}{2^{n-k}} \binom{n}{k} \binom{k}{n-k} x^{2k-n}. \quad (2.8)$$

As a result, it follows that

$$\begin{aligned}
&\frac{d^n}{dx^n} (\arcsin x) \\
&= -\frac{d^n}{dx^n} (\arccos x) = \frac{d^{n-1}}{dx^{n-1}} \left( \frac{1}{\sqrt{1-x^2}} \right) \\
&= \sum_{k=0}^{n-1} \frac{(2k-1)!!}{(1-x^2)^{k+1/2}} B_{n-1,k}(x, 1, 0, \dots, 0) \\
&= \sum_{k=0}^{n-1} \frac{(2k-1)!!}{(1-x^2)^{k+1/2}} \frac{(n-k-1)!}{2^{n-k-1}} \binom{n-1}{k} \binom{k}{n-k-1} x^{2k-n+1} \\
&= \frac{1}{(2x)^n} \sum_{k=0}^{n-1} (2k-1)!! (n-k-1)! 2^{k+1} \binom{n-1}{k} \binom{k}{n-k-1} \left( \frac{x^2}{1-x^2} \right)^{k+1/2}
\end{aligned}$$

for  $n \geq 1$ . The formula (1.11) is proved.



By the Faà di Bruno formula (1.1), the formula (1.10) applied to  $a = 0$  and  $b = 1$ , and the formula (1.11), we arrive at

$$\begin{aligned} & \frac{d^n}{dx^n}(\arcsin \sqrt{x}) = -\frac{d^n}{dx^n}(\arccos \sqrt{x}) \\ &= \sum_{k=0}^n (\arcsin^{(k)} u) B_{n,k}(g'(x), g''(x), \dots, g^{(n-k+1)}(x)) \\ &= \sum_{k=1}^n \frac{1}{(2x)^k} \sum_{\ell=0}^{k-1} (2\ell - 1)!!(k - \ell - 1)!2^{\ell+1} \binom{k-1}{\ell} \binom{\ell}{k-\ell-1} \left(\frac{x^2}{1-x^2}\right)^{\ell+1/2} \\ & \quad \times (-1)^{n+k} \frac{1}{2^n} \binom{2n-k-1}{2(n-k)} [2(n-k)-1]!! \frac{1}{x^{n-k/2}} \\ &= \frac{(-1)^n}{(2x)^n} \sum_{k=1}^n \frac{(-1)^k}{(2x)^k} \binom{2n-k-1}{2(n-k)} [2(n-k)-1]!! x^{k/2} \\ & \quad \times \sum_{\ell=0}^{k-1} (2\ell - 1)!!(k - \ell - 1)!2^{\ell+1} \binom{k-1}{\ell} \binom{\ell}{k-\ell-1} \left(\frac{x^2}{1-x^2}\right)^{\ell+1/2} \end{aligned}$$

for  $n \geq 1$ . The proof of Theorem 1.4 is complete. □

### 3. Remarks

Finally, we give several remarks on the recovery of the third formula in (1.7), on the formulas (1.11) and (2.8), and on the derivatives of some elementary functions.

**Remark 3.1.** Let  $u = u(z)$  and  $v = v(z) \neq 0$  be differentiable functions. In [1, p. 40], the formula

$$\frac{d^k}{dz^k} \left( \frac{u}{v} \right) = \frac{(-1)^k}{v^{k+1}} \begin{vmatrix} u & v & 0 & \dots & 0 \\ u' & v' & v & \dots & 0 \\ u'' & v'' & 2v' & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ u^{(k-1)} & v^{(k-1)} & \binom{k-1}{1} v^{(k-2)} & \dots & v \\ u^{(k)} & v^{(k)} & \binom{k}{1} v^{(k-1)} & \dots & \binom{k}{k-1} v' \end{vmatrix} \quad (3.1)$$

for the  $k$ th derivative of the ratio  $\frac{u(z)}{v(z)}$  was listed. In [19, Section 2.2, p. 849], [22, (3.2) and (3.3)], and [37, Lemma 2.1], the formula (3.1) was reformulated as

$$\frac{d^k}{dz^k} \left( \frac{u}{v} \right) = \frac{(-1)^k}{v^{k+1}} \left| A_{(k+1) \times 1} B_{(k+1) \times k} \right|_{(k+1) \times (k+1)}, \quad (3.2)$$

where the matrices

$$A_{(k+1) \times 1} = (a_{\ell,1})_{0 \leq \ell \leq k} \quad \text{and} \quad B_{(k+1) \times k} = (b_{\ell,m})_{0 \leq \ell \leq k, 0 \leq m \leq k-1}$$

satisfy

$$a_{\ell,1} = u^{(\ell)}(z) \quad \text{and} \quad b_{\ell,m} = \binom{\ell}{m} v^{(\ell-m)}(z)$$

under the conventions that  $v^{(0)}(z) = v(z)$  and that  $\binom{p}{q} = 0$  and  $v^{(p-q)}(z) \equiv 0$  for  $p < q$ . Applying the formula (3.1) or (3.2) to the function  $\frac{1-\sqrt{1-4x}}{2x}$  by taking  $u(x) = 1 - \sqrt{1 - 4x}$  and  $v(x) = 2x$  reveals that

$$\frac{d^n}{dx^n} \left( \frac{1 - \sqrt{1 - 4x}}{2x} \right) = \frac{(-1)^n}{2x^{n+1}} \begin{vmatrix} 1 - \sqrt{1 - 4x} & x & 0 & \dots & 0 & 0 \\ \frac{2}{\sqrt{1-4x}} & 1 & x & \dots & 0 & 0 \\ \frac{4}{(1-4x)^{3/2}} & 0 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{2^{n-2}(2n-7)!!}{(1-4x)^{(2n-5)/2}} & 0 & 0 & \dots & x & 0 \\ \frac{2^{n-1}(2n-5)!!}{(1-4x)^{(2n-3)/2}} & 0 & 0 & \dots & n-1 & x \\ \frac{2^n(2n-3)!!}{(1-4x)^{(2n-1)/2}} & 0 & 0 & \dots & 0 & n \end{vmatrix}$$

$$\triangleq \frac{(-1)^n}{2x^{n+1}} D_{(n+1) \times (n+1)},$$

where the determinant  $D_{(n+1) \times (n+1)}$  satisfies

$$D_{(n+1) \times (n+1)} = nD_{n \times n} + (-1)^n \frac{2^n(2n-3)!!}{(1-4x)^{(2n-1)/2}} \begin{vmatrix} x & 0 & \dots & 0 & 0 \\ 1 & x & \dots & 0 & 0 \\ 0 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & x & 0 \\ 0 & 0 & \dots & n-1 & x \end{vmatrix}_{n \times n}$$

$$= nD_{n \times n} + (-1)^n 2^n(2n-3)!! \frac{x^n}{(1-4x)^{n-1/2}}.$$

Consequently, we obtain

$$\begin{aligned} \frac{d^n}{dx^n} \left( \frac{1 - \sqrt{1 - 4x}}{2x} \right) &= \frac{(-1)^n}{2x^{n+1}} \left[ nD_{n \times n} + (-1)^n 2^n(2n-3)!! \frac{x^n}{(1-4x)^{n-1/2}} \right] \\ &= -\frac{1}{x} \left[ n \frac{(-1)^{n-1}}{2x^n} D_{n \times n} - \frac{2^{n-1}(2n-3)!!}{(1-4x)^{n-1/2}} \right] \\ &= -\frac{1}{x} \left[ n \frac{d^{n-1}}{dx^{n-1}} \left( \frac{1 - \sqrt{1 - 4x}}{2x} \right) - \frac{2^{n-1}(2n-3)!!}{(1-4x)^{n-1/2}} \right]. \end{aligned}$$

This implies that

$$\lim_{x \rightarrow 0} \frac{d^{n-1}}{dx^{n-1}} \left( \frac{1 - \sqrt{1 - 4x}}{2x} \right) = \frac{1}{n} \lim_{x \rightarrow 0} \frac{2^{n-1}(2n-3)!!}{(1-4x)^{n-1/2}} = \frac{2^{n-1}(2n-3)!!}{n}.$$

As a result, by virtue of (2.7), we acquire

$$C_{n-1} = \frac{2^{n-1}(2n-3)!!}{n!}$$

which recovers the third formula in (1.7) for the Catalan numbers  $C_n$  for  $n \geq 0$ .

**Remark 3.2.** On 14 September 2015, when the original version of this paper was finalized, we found the papers [13, 14, 17] which are related to the generating function (2.7) of the Catalan numbers  $C_n$  and its derivatives.

**Remark 3.3.** As said in [6, Remark 4.1], the formulas in (1.11) simplify those results obtained in [30, Theorem 2.1 and Corollaries 2.1 and 2.2] while the formula (2.8) simplifies [30, Theorem 3.1].

**Remark 3.4.** Recently, among other things, several explicit formulas and their applications of higher order derivatives for the tangent and cotangent functions were collected and established in [19]. By virtue of conclusions obtained in [5, 7, 41], some nice closed formulas for higher order derivatives of the tangent, cotangent, secant, cosecant, hyperbolic tangent, hyperbolic cotangent, hyperbolic secant, and hyperbolic cosecant functions were found in [40]. Hence, we can regard this paper as a companion of the papers [19, 30, 40] and closely related reference therein.

**Remark 3.5.** This paper is a companion of the articles [15, 16, 23–25, 29, 32] and the preprints [18, 20, 21, 26, 28] and is a slightly revised version of the preprint [27].

### 4. Appendix

For completeness and accuracy, we now directly and alternatively verify the formulas (2.5) and (2.6) as follows.

In [10–12] and closely-related references therein, the composita  $Y^\Delta(n, k, x)$  of the function

$$Y(x, z) = y(x + z) - y(x)$$

was introduced by

$$Y^\Delta(n, k, x) = \sum_{\pi_k \in S_n} \prod_{i=1}^k \frac{y^{(\lambda_i)}(x)}{\lambda_i!},$$

where  $y^{(i)}$  is the  $i$ th derivative of the function  $y(x)$ ,  $S_n$  is the set of compositions of  $n$ , and  $\pi_k$  is the composition of  $n$  with  $k$  parts such that  $\sum_{\ell=1}^k \lambda_\ell = n$ . In those papers, it was proved that the composita  $Y^\Delta(n, k, x)$  can be generated by

$$[Y(x, z)]^k = [y(x + z) - y(x)]^k = \sum_{n=k}^\infty Y^\Delta(n, k, x) z^n \tag{4.1}$$

and that the Bell polynomials  $B_{n,k}$  and the composita  $Y^\Delta(n, k, x)$  have the relation

$$B_{n,k}(y'(x), y''(x), \dots, y^{(n-k+1)}(x)) = \frac{n!}{k!} Y^\Delta(n, k, x). \tag{4.2}$$

It is straightforward that

$$[f(x + y) - f(x)]^k = (1 - 2x)^{k/2} \left( 1 - \sqrt{1 - \frac{2y}{1 - 2x}} \right)^k,$$

where  $f(x) = 1 - \sqrt{1 - 2x}$ . Since

$$(1 - \sqrt{1 - z})^k = \sum_{n=k}^\infty \frac{k 2^{k-2n} \binom{2n-k-1}{n-k}}{n} z^n,$$

by regarding  $\frac{2}{1-2x}$  as a coefficient, we derive

$$\left(1 - \sqrt{1 - \frac{2y}{1-2x}}\right)^k = \sum_{n=k}^{\infty} \left(\frac{2}{1-2x}\right)^n \frac{k 2^{k-2n} \binom{2n-k-1}{n-k}}{n} y^n.$$

Hence, by (4.1), the composita  $F^\Delta(n, k, x)$  of the function  $F(x, z) = f(x+z) - f(x)$  is equal to

$$\begin{aligned} F^\Delta(n, k, x) &= (1-2x)^{k/2} \left(\frac{2}{1-2x}\right)^n \frac{k}{n} 2^{k-2n} \binom{2n-k-1}{n-k} \\ &= 2^{k-n} \frac{k}{n} \binom{2n-k-1}{n-k} (1-2x)^{k/2-n}. \end{aligned}$$

Therefore, by virtue of the relation (4.2), the Bell polynomial of the second kind  $B_{n,k}$  for the function  $f(x) = 1 - \sqrt{1-2x}$  is

$$\begin{aligned} B_{n,k}(f'(x), f''(x), \dots, f^{(n-k+1)}(x)) &= \frac{n!}{k!} F^\Delta(n, k, x) \\ &= 2^{k-n} \frac{(n-1)!}{(k-1)!} \binom{2n-k-1}{n-k} (1-2x)^{k/2-n} \end{aligned}$$

which can be reformulated as (2.5) and (2.6).

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## References

- [1] N. Bourbaki, *Elements of Mathematics: Functions of a Real Variable: Elementary Theory*, Translated from the 1976 French original by Philip Spain. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2004. Available online at <http://dx.doi.org/10.1007/978-3-642-59315-4>.
- [2] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Revised and Enlarged Edition, D. Reidel Publishing Co., 1974.
- [3] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics—A Foundation for Computer Science*, 2nd ed., Addison-Wesley Publishing Company, Reading, MA, 1994.
- [4] B.-N. Guo and F. Qi, *An explicit formula for Bernoulli numbers in terms of Stirling numbers of the second kind*, J. Anal. Number Theory, 2015, 3(1), 27–30. Available online at <http://dx.doi.org/10.12785/jant/030105>.

- [5] B.-N. Guo and F. Qi, *Explicit formulae for computing Euler polynomials in terms of Stirling numbers of the second kind*, J. Comput. Appl. Math., 2014, 272, 251–257. Available online at <http://dx.doi.org/10.1016/j.cam.2014.05.018>.
- [6] B.-N. Guo and F. Qi, *Explicit formulas for special values of the Bell polynomials of the second kind and for the Euler numbers and polynomials*, Mediterr. J. Math., 2017, 14(6), in press. Available online at <http://dx.doi.org/10.13140/2.1.3794.8808>.
- [7] B.-N. Guo and F. Qi, *Some identities and an explicit formula for Bernoulli and Stirling numbers*, J. Comput. Appl. Math., 2014, 255, 568–579. Available online at <http://dx.doi.org/10.1016/j.cam.2013.06.020>.
- [8] T. Koshy, *Catalan Numbers with Applications*, Oxford University Press, Oxford, 2009.
- [9] H. L. Krall and O. Frink, *A new class of orthogonal polynomials: The Bessel polynomials*, Trans. Amer. Math. Soc., 1949, 65, 100–115.
- [10] D. V. Kruchinin and V. V. Kruchinin, *Application of a composition of generating functions for obtaining explicit formulas of polynomials*, J. Math. Anal. Appl., 2013, 404(1), 161–171; Available online at <http://dx.doi.org/10.1016/j.jmaa.2013.03.009>.
- [11] V. V. Kruchinin, *Derivation of Bell polynomials of the second kind*, arXiv preprint, 2011. Available online at <http://arxiv.org/abs/1104.5065>.
- [12] V. V. Kruchinin and D. V. Kruchinin, *Composita and its properties*, J. Anal. Number Theory, 2014, 2(2), 37–44.
- [13] W. Lang, *On polynomials related to derivatives of the generating function of Catalan numbers*, Fibonacci Quart., 2002, 40(4), 299–313.
- [14] W. Lang, *On polynomials related to powers of the generating function of Catalan's numbers*, Fibonacci Quart., 2000, 38(5), 408–419.
- [15] F.-F. Liu, X.-T. Shi, and F. Qi, *A logarithmically completely monotonic function involving the gamma function and originating from the Catalan numbers and function*, Glob. J. Math. Anal., 2015, 3(4), 140–144. Available online at <http://dx.doi.org/10.14419/gjma.v3i4.5187>.
- [16] M. Mahmoud and F. Qi, *Three identities of the Catalan–Qi numbers*, Mathematics, 2016, 4(2), Article 35, 7 pages. Available online at <http://dx.doi.org/10.3390/math4020035>.
- [17] A. Nkwanta and A. Tefera, *Curious relations and identities involving the Catalan generating function and numbers*, J. Integer Seq., 2013, 16(9), Article 13.9.5, 15 pages.
- [18] F. Qi, *Asymptotic expansions, complete monotonicity, and inequalities of the Catalan numbers*, ResearchGate Technical Report, 2015. Available online at <http://dx.doi.org/10.13140/RG.2.1.4371.6321>.
- [19] F. Qi, *Derivatives of tangent function and tangent numbers*, Appl. Math. Comput., 2015, 268, 844–858. Available online at <http://dx.doi.org/10.1016/j.amc.2015.06.123>.

- [20] F. Qi, *Some properties and generalizations of the Catalan, Fuss, and Fuss–Catalan numbers*, Mathematical Analysis and Applications: Selected Topics, 35 pages, edited by H. Dutta, M. Ruzhansky, and R. P. Agarwal, Wiley, October 2017. (ISBN 9781119414346). Available online at <http://dx.doi.org/10.13140/RG.2.1.1778.3128>.
- [21] F. Qi, *Two product representations and several properties of the Fuss–Catalan numbers*, ResearchGate Research, 2015. Available online at <http://dx.doi.org/10.13140/RG.2.1.1655.6004>.
- [22] F. Qi and R. J. Chapman, *Two closed forms for the Bernoulli polynomials*, J. Number Theory, 2016, 159, 89–100. Available online at <http://dx.doi.org/10.1016/j.jnt.2015.07.021>.
- [23] F. Qi and B.-N. Guo, *Logarithmically complete monotonicity of a function related to the Catalan–Qi function*, Acta Univ. Sapientiae Math., 2016, 8(1), 93–102. Available online at <http://dx.doi.org/10.1515/ausm-2016-0006>.
- [24] F. Qi and B.-N. Guo, *Logarithmically complete monotonicity of Catalan–Qi function related to Catalan numbers*, Cogent Math., 2016, 3, 1179379, 6 pages. Available online at <http://dx.doi.org/10.1080/23311835.2016.1179379>.
- [25] F. Qi, M. Mahmoud, X.-T. Shi, and F.-F. Liu, *Some properties of the Catalan–Qi function related to the Catalan numbers*, SpringerPlus, 2016, 5, 1126, 20 pages. Available online at <http://dx.doi.org/10.1186/s40064-016-2793-1>.
- [26] F. Qi, X.-T. Shi, and F.-F. Liu, *An integral representation, complete monotonicity, and inequalities of the Catalan numbers*, ResearchGate Technical Report, 2015. Available online at <http://dx.doi.org/10.13140/RG.2.1.3754.4806>.
- [27] F. Qi, X.-T. Shi, and F.-F. Liu, *Several formulas for special values of the Bell polynomials of the second kind and applications*, ResearchGate Technical Report, 2015. Available online at <http://dx.doi.org/10.13140/RG.2.1.3230.1927>.
- [28] F. Qi, X.-T. Shi, M. Mahmoud, and F.-F. Liu, *Schur-convexity of the Catalan–Qi function related to the Catalan numbers*, Tbilisi Math. J., 2016, 9(2), 141–150. Available online at <http://dx.doi.org/10.1515/tmj-2016-0026>.
- [29] F. Qi, X.-T. Shi, M. Mahmoud, and F.-F. Liu, *The Catalan numbers: a generalization, an exponential representation, and some properties*, J. Comput. Anal. Appl., 2017, 23(5), 937–944.
- [30] F. Qi and M.-M. Zheng, *Explicit expressions for a family of the Bell polynomials and applications*, Appl. Math. Comput., 2015, 258, 597–607. Available online at <http://dx.doi.org/10.1016/j.amc.2015.02.027>.
- [31] I. J. Schwatt, *An Introduction to the Operations with Series*, Chelsea Publishing Co., New York, 1924. Available online at <http://hdl.handle.net/2027/wu.89043168475>.
- [32] X.-T. Shi, F.-F. Liu, and F. Qi, *An integral representation of the Catalan numbers*, Glob. J. Math. Anal., 2015, 3(3), 130–133. Available online at <http://dx.doi.org/10.14419/gjma.v3i3.5055>.

- 
- [33] N. J. A. Sloane, *Triangle of coefficients of Bessel polynomials*, From The On-Line Encyclopedia of Integer Sequences. Available online at <http://oeis.org/A001497>.
- [34] R. Stanley and E. W. Weisstein, *Bessel polynomial*, From MathWorld—A Wolfram Web Resource. Available online at <http://mathworld.wolfram.com/BesselPolynomial.html>.
- [35] R. Stanley and E. W. Weisstein, *Catalan number*, From MathWorld—A Wolfram Web Resource. Available online at <http://mathworld.wolfram.com/CatalanNumber.html>.
- [36] I. Vardi, *Computational Recreations in Mathematica*, Addison-Wesley, Redwood City, CA, 1991.
- [37] C.-F. Wei and F. Qi, *Several closed expressions for the Euler numbers*, J. Inequal. Appl., 2015, 219, 8 pages. Available online at <http://dx.doi.org/10.1186/s13660-015-0738-9>.
- [38] Wikipedia, *Bessel polynomials*, From the Free Encyclopedia. Available online at [https://en.wikipedia.org/wiki/Bessel\\_polynomials](https://en.wikipedia.org/wiki/Bessel_polynomials).
- [39] Wikipedia, *Catalan number*, From the Free Encyclopedia. Available online at [https://en.wikipedia.org/wiki/Catalan\\_number](https://en.wikipedia.org/wiki/Catalan_number).
- [40] A.-M. Xu and G.-D. Cen, *Closed formulas for computing higher-order derivatives of functions involving exponential functions*, Appl. Math. Comput., 2015, 270, 136–141. Available online at <http://dx.doi.org/10.1016/j.amc.2015.08.051>.
- [41] A.-M. Xu and Z.-D. Cen, *Some identities involving exponential functions and Stirling numbers and applications*, J. Comput. Appl. Math., 2014, 260, 201–207. Available online at <http://dx.doi.org/10.1016/j.cam.2013.09.077>.