

RESEARCH ON TRAVELING WAVE SOLUTIONS FOR A CLASS OF (3+1)-DIMENSIONAL NONLINEAR EQUATION*

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Abstract Nonlinear wave phenomena are of great importance in the nature, and have become for a long time a challenging research topic for both pure and applied mathematicians. In this paper the solitary wave, kink (anti-kink) wave and periodic wave solutions for a class of (3+1)-dimensional nonlinear equation were obtained by some effective methods from the dynamical systems theory.

Keywords Traveling wave system, bifurcation, solitary wave, kink (anti-kink) wave, periodic wave.

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1. Introduction

In the last few decades, the research on traveling wave solutions for soliton equations is one of most prominent events in the field of nonlinear sciences. Researching the exact traveling wave solutions can help both mathematicians and physicists to understand the mechanism of phenomena in nature which have been described by these soliton equations.

In recent years, some of powerful methods have been proposed to get solutions of different equations. Abourabia and Morad [1] applied two different exact methods to obtain exact traveling wave solutions of the van der Waals normal form for fluidized granular matter. The results show that the exact solutions of the model introduce solitary waves with different types. Applying the G'/G expansion method, Alquran and Qawasmeh [2, 12] investigated the traveling wave

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solutions determined by the generalized shallow water wave equation, and also investigated the Whitham-Broer-Kaup model for dispersive long waves in the shallow water small-amplitude regime. Rehman et.al [13] investigated the possible classes of traveling wave solutions of some members of a recently-derived integrable family of generalized Camassa-Holm equations, and got smooth and non-smooth traveling wave solutions of some generalized Camassa-Holm equations. Zhao and Ruan [19] researched the existence, uniqueness, and asymptotic stability of time periodic traveling wave solutions for a class of periodic advection-reaction-diffusion systems under certain conditions. Lin [11] applied Schauder fixed point theorem to proof the existence of traveling wave solutions for integro-difference systems of higher order. Then the asymptotic behavior of traveling wave solutions was studied by using the idea of contracting rectangles. Li et.al [10] applied the extended Riccati equation method to the Zakharov-Kuznetsov equation and then obtained more general exact traveling wave solutions under specific parametric conditions. Using the bifurcation theory of dynamical systems to the (2+1)-dimensional generalized asymmetric Nizhnik-Novikov-Veselov equation, Zhang and Han [17] obtained the existence of solitary wave solutions and uncountably infinite many smooth and non-smooth periodic wave solutions in different regions of parametric spaces.

In [3, 4, 14], four (2+1)-dimensional nonlinear models generated by the Jaulent-Miodek hierarchy were developed in the form

$$w_t = -(w_{xx} - 2w^3)_x - \frac{3}{2}(w_x \partial_x^{-1} w_y + w w_y), \quad (1.1a)$$

$$w_t = \frac{1}{2}(w_{xx} - 2w^3)_x - \frac{3}{2}\left(-\frac{1}{4}\partial_x^{-1} w_{yy} + w w_y\right), \quad (1.1b)$$

$$w_t = \frac{1}{4}(w_{xx} - 2w^3)_x - \frac{3}{4}\left(\frac{1}{4}\partial_x^{-1} w_{yy} + w_x \partial_x^{-1} w_y\right), \quad (1.1c)$$

$$w_t = 2(w_{xx} - 2w^3)_x - \frac{3}{4}(\partial_x^{-1} w_{yy} - 2w_x \partial_x^{-1} w_y - 6w w_y), \quad (1.1d)$$

where ∂_x^{-1} is the inverse of ∂_x with $\partial_x \partial_x^{-1} = \partial_x^{-1} \partial_x = 1$, and

$$\partial_x^{-1} = \int_{-\infty}^x f(t), \quad (1.2)$$

under the decaying condition at infinity. The model of (1.1a) above was studied by using a perturbation technique in Wu [16]. Wazwaz [14] studied the model (1.1c) to derive multiple kink solutions and multiple singular kink solutions by use of Hirota's bilinear method [6], and then the author gave soliton solutions in terms of exponential polynomials. The other models can be researched in a similar manner.

Wazwaz [15] extended the works in [3, 4, 13], and have further researched four (3+1)-dimensional nonlinear models generated by the Jaulent-Miodek hierarchy. The (3+1)-dimensional nonlinear models were developed in the form

$$w_t = -(w_{xx} - 2w^3)_x - \frac{3}{2}(w_x \partial_x^{-1} w_y + w w_y) + \alpha \partial_x^{-1} w_{zz}, \quad (1.3a)$$

$$w_t = \frac{1}{2}(w_{xx} - 2w^3)_x - \frac{3}{2}\left(-\frac{1}{4}\partial_x^{-1} w_{yy} + w w_y\right) + \alpha \partial_x^{-1} w_{zz}, \quad (1.3b)$$

$$w_t = \frac{1}{4}(w_{xx} - 2w^3)_x - \frac{3}{4}\left(\frac{1}{4}\partial_x^{-1} w_{yy} + w_x \partial_x^{-1} w_y\right) + \alpha \partial_x^{-1} w_{zz}, \quad (1.3c)$$

$$w_t = 2(w_{xx} - 2w^3)_x - \frac{3}{4}(\partial_x^{-1} w_{yy} - 2w_x \partial_x^{-1} w_y - 6w w_y) - \frac{3}{4}\partial_x^{-1} w_{zz} - \frac{1}{4}w_z - \frac{1}{2}w_y, \quad (1.3d)$$

where α is a parameter. It is obvious that these (3+1)-dimensional nonlinear models are developed by adding $\alpha\partial_x^{-1}w_{zz}$ to the first three models in (1.1), and the terms $-\frac{3}{4}\partial_x^{-1}w_{zz} - \frac{1}{4}w_z - \frac{1}{2}w_y$ to the fourth model in (1.1).

Wazwaz [15] applied the method of Hereman-Nuseir [5] to model (1.3a) and (1.3d) to derive multiple soliton solutions. The single soliton solution, two-soliton solutions and three-soliton solutions for model (1.3a) were obtained. In addition, the single soliton solution, two-soliton solutions and three-soliton solutions for model (1.3d) were also obtained respectively.

In order to obtain traveling wave solutions for a class of nonlinear integrable evolution equations, it is always significant to decompose a nonlinear partial differential equation into a pair of systems of ordinary differential equations both in theoretical point and practical point of view. This approach aims to decompose integrable soliton equations into finite-dimensional Hamiltonian systems, and it also makes it very natural to compute solutions of soliton equations. Li [7–9] applied these effective methods from the dynamical systems theory to prove the existence of solitary wave, kink wave and periodic wave solutions of different singular nonlinear traveling wave equations.

The present paper will keep the focus on the nonlinear model (1.3a) by use of the method introduced in [7–9]. We take a traveling wave transformation, such that the partial differential equation becomes a associated “traveling wave system”. We also discuss the dynamical behaviors of the “traveling wave system”. Using the known dynamical behaviors of the “traveling wave system”, we obtain the nonlinear wave profiles for the partial differential equation.

The outline of the paper is as follows. The rest of the present paper is divided into four sections. In section 2, we obtain an ordinary differential system (traveling wave system) in phase plane $(\phi, \eta = d\phi/d\xi)$ from the nonlinear model by using the potential introduced in [15] and the associated traveling wave transformation introduced in [7–9]. In section 3, we investigate the equilibrium points bifurcation and obtain the bifurcation curves(sets) of the ordinary differential system. In section 4, we compute the solitary wave solution, the periodic wave solution and the kink(anti-kink) wave solution of the nonlinear model (1.3a).

2. Traveling Wave System in Phase Plane $(\phi, \eta = d\phi/d\xi)$ from (1.3a)

In this section, we obtain an ordinary differential system in phase plane $(\phi, \eta = d\phi/d\xi)$ from equation (1.3a) by using the potential introduced in [15] and the traveling wave transformation introduced in [7–9].

We apply the potential

$$w(x, y, z, t) = u_x(x, y, z, t), \quad (2.1)$$

to remove the integral term in (1.3a) and then equation (1.3a) becomes as follows

$$u_{xt} + u_{xxxx} - 6u_x^2u_{xx} + \frac{3}{2}u_{xx}u_y + \frac{3}{2}u_xu_{xy} - \alpha u_{zz} = 0. \quad (2.2)$$

By letting $u(x, y, z, t) = u(\xi) = u(ax + by + cz - dt)$, where d is the propagating wave velocity and $a \neq 0$. We have

$$-(ad + \alpha c^2)u'' + a^4u^{(4)} - 6a^4(u')^2u'' + 3a^2bu'u'' = 0, \quad (2.3)$$

where “ ’ ” stands for the derivative with respect to ξ .

Integrating equation (2.3) with respect to ξ once and setting $\phi = u_\xi$, we have

$$-2(ad + \alpha c^2)\phi + 2a^4\phi'' - 4a^4\phi^3 + 3a^2b\phi^2 = 0. \quad (2.4)$$

By letting $\phi' = \eta$, we have the following planar system (traveling wave system) with Hamiltonian function $H = H(\phi, \eta)$,

$$\frac{d\phi}{d\xi} = \eta = \frac{\partial H}{\partial \eta}, \quad \frac{d\eta}{d\xi} = 2\phi^3 + f_1\phi^2 + f_2\phi = -\frac{\partial H}{\partial \phi}, \quad (2.5)$$

where $f_1 = -\frac{3a^2b}{2a^4}$, $f_2 = \frac{ad+\alpha c^2}{a^4}$.

3. Bifurcation of the Phase Portraits and the Wave Profiles Determined by the Orbits of (2.5)

In this section, we investigate the dynamical behaviors of traveling wave system (2.5). Based on the bifurcation theory of dynamical systems [18], we study the bifurcation of equilibrium points and obtain the bifurcation curves(sets) of system (2.5). According to these curves, we consider the bifurcation of the phase portraits of system (2.5) in different regions of parametric spaces.

3.1. Bifurcation of Equilibrium Points of System (2.5)

Through system (2.5), we have theorem as follows.

Theorem 3.1. *We denote that $\Delta_1 = f_1^2 - 8f_2$, so*

- (i) *If $\Delta_1 > 0$, System (2.5) has three different equilibrium points, $(\phi_1, \eta_1) = (0, 0)$, $(\phi_2, \eta_2) = (\frac{-f_1 + \sqrt{f_1^2 - 8f_2}}{4}, 0)$, $(\phi_3, \eta_3) = (\frac{-f_1 - \sqrt{f_1^2 - 8f_2}}{4}, 0)$;*
- (ii) *If $\Delta_1 = 0$, System (2.5) has two equilibrium points, $(\phi_1, \eta_1) = (0, 0)$, $(\phi_2, \eta_2) = (\frac{-f_1}{4}, 0)$;*
- (iii) *If $\Delta_1 < 0$, System (2.5) has only one real equilibrium points, $(\phi_1, \eta_1) = (0, 0)$.*

We notice that the Jacobian of the linearized system of (2.5) at equilibrium point (ϕ_i, η_i) is given by $J(\phi_i, \eta_i)$. Let $M(\phi_i, \eta_i)$ be the coefficient matrix of the linearized system of (2.5) at an equilibrium point (ϕ_i, η_i) . We have

$$\begin{aligned} J(0, 0) &= \det M(0, 0) = -f_2, \\ J\left(\frac{-f_1 + \sqrt{f_1^2 - 8f_2}}{4}, 0\right) &= -\frac{f_1^2 - 8f_2}{4}(\sqrt{f_1^2 - 8f_2} - f_1), \\ J\left(\frac{-f_1 - \sqrt{f_1^2 - 8f_2}}{4}, 0\right) &= -\frac{f_1^2 - 8f_2}{4}(\sqrt{f_1^2 - 8f_2} + f_1), \end{aligned}$$

and

$$\begin{aligned} \text{tr}M(0, 0) &= \text{tr}M\left(\frac{-f_1 + \sqrt{f_1^2 - 8f_2}}{4}, 0\right) \\ &= \text{tr}M\left(\frac{-f_1 - \sqrt{f_1^2 - 8f_2}}{4}, 0\right) \\ &= 0. \end{aligned}$$

Remark 3.1. For a system,

$$\frac{dx}{dt} = \frac{1}{f^2(x)} \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{1}{f^2(x)} \frac{\partial H}{\partial x},$$

where H is the first integral of the system above and $f^2(x)$ is an integral factor. There exist a equilibrium point (x_0, y_0) , according to [17], we have

- (i) If $J(x_0, y_0) < 0$, then the equilibrium point is a saddle point;
- (ii) If $J(x_0, y_0) > 0$ and $(\text{trice}M(x_0, y_0))^2 - 4J(x_0, y_0) < 0 (> 0)$, then the equilibrium point is a center point (a node point);
- (iii) If $J(x_0, y_0) = 0$ and the Poincare index of the equilibrium point is 0, then the equilibrium point is a cusp.

Remark 3.2. System (2.5) is a completely integrable system, so any nondegenerate equilibrium point of system (2.5) is either a saddle point or a center. Furthermore,

- (i) The saddle point of (2.5) corresponds to a strict maximum of the Hamiltonian function;
- (ii) The center of (2.5) corresponds to a strict minimum of the Hamiltonian function.

3.2. Bifurcation of the Phase Portraits of System (2.5)

Based on Hamiltonian function of system (2.5), we denote that $F(\phi) = \frac{1}{2}\phi^2 + \frac{1}{3}f_1\phi + \frac{1}{2}f_2$, and $\Delta_2 = f_1^2 - 9f_2$. There are four bifurcation curves in the (f_1, f_2) parameter plane (see Figure 1) as follows

$$\begin{aligned} L1 : f_1^2 - 8f_2 &= 0, \\ L2 : f_1^2 - 9f_2 &= 0, \\ L3 : f_1 &= 0 \ (f_2 < 0), \\ L4 : f_2 &= 0, \end{aligned}$$

these curves divide (f_1, f_2) parameter plane into several sections. Let

$$h_i = H(\phi_{(i)}, \eta_{(i)}), \quad i = 1, 2, 3, \tag{3.1}$$

for a fixed h , the level curve $H(\phi, \eta) = h$ determines a set of solution curves of system (2.5), which includes different branches of curves. We compute the type of equilibrium point and compare the value of Hamiltonian function at each equilibrium point of system (2.5) under different regions of parametric spaces in Figure 1. Based on the basic information above, we can describe the orbits of system (2.5) approximately.

(1) Section (I). Parameters condition is $f_1^2 = 8f_2$, and $f_1 > 0$, we have $h_2 < h_1$ and the system (2.5) have a saddle point at $(\phi_1, 0)$ and two cusps coincide at $(\phi_2, 0)$.

(2) Section (II). Parameters condition is $8f_2 < f_1^2 < 9f_2$, and $f_1 > 0$, we have $h_2 < h_3 < h_1$ and the system (2.5) have two saddle points at $(\phi_{1,3}, 0)$ and a center at $(\phi_2, 0)$. The curve on the right hand side of saddle point $(\phi_3, 0)$ defined by $H(\phi, \eta) = h_3$ give rise to a homoclinic orbit. The homoclinic orbit is also the limit curve of the family of periodic orbits of (2.5) defined by $H(\phi, \eta) = h, h \in (h_2, h_3)$.

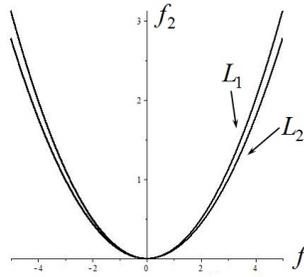


Figure 1. The bifurcation curves(sets) of system (2.5).

(3) Section (III). Parameters condition is $f_1^2 = 9f_2$, and $f_1 > 0$, we have $h_2 < h_1 = h_3$ and the system (2.5) have two saddle points at $(\phi_{1,3}, 0)$ and a center at $(\phi_2, 0)$. The curve connect $(\phi_{1,3}, 0)$ defined by $H(\phi, \eta) = h_1$ give rise to two heteroclinic orbits (or so called connecting orbit). The heteroclinic orbits is also the limit curve of the family of periodic orbits of (2.5) defined by $H(\phi, \eta) = h, h \in (h_2, h_1)$.

(4) Section (IV). Parameters condition is $f_1^2 > 9f_2$, and $f_1 > 0$, we have $h_2 < h_1 < h_3$ and the system (2.5) have two saddle points at $(\phi_{1,3}, 0)$ and a center at $(\phi_2, 0)$. The curve on the left hand side of saddle point $(\phi_1, 0)$ defined by $H(\phi, \eta) = h_1$ give rise to a homoclinic orbit. The homoclinic orbit is also the limit curve of the family of periodic orbits of (2.5) defined by $H(\phi, \eta) = h, h \in (h_2, h_1)$.

(5) Section (V). Parameters condition is $f_2 = 0$, and $f_1 > 0$, we have $h_1 = h_2 < h_3$ and the system (2.5) have a saddle point at $(\phi_3, 0)$ and two cusps coincide at origin.

(6) Section (VI). Parameters condition is $f_2 < 0$, and $f_1 > 0$, we have $h_1 < h_2 < h_3$ and the system (2.5) have two saddle points at $(\phi_{2,3}, 0)$ and a center at $(\phi_1, 0)$. The curve on the left hand side of saddle point $(\phi_2, 0)$ defined by $H(\phi, \eta) = h_2$ give rise to a homoclinic orbit. The homoclinic orbit is also the limit curve of the family of periodic orbits of (2.5) defined by $H(\phi, \eta) = h, h \in (h_1, h_2)$.

(7) Section (VII). Parameters condition is $f_1 = 0$, and $f_2 < 0$, we have $h_1 < h_2 = h_3$ and the system (2.5) have two saddle points at $(\phi_{2,3}, 0)$ and a center at $(\phi_1, 0)$. The curve connect $(\phi_{2,3}, 0)$ defined by $H(\phi, \eta) = h_2$ give rise to two heteroclinic orbits. These heteroclinic orbits are also the limit curve of the family of periodic orbits of (2.5) defined by $H(\phi, \eta) = h, h \in (h_1, h_2)$.

(8) Section (VIII). Parameters condition is $f_1 < 0$, and $f_2 < 0$, we have $h_1 < h_3 < h_2$ and the system (2.5) have two saddle points at $(\phi_{2,3}, 0)$ and a center at $(\phi_1, 0)$. The curve on the right hand side of saddle point $(\phi_3, 0)$ defined by $H(\phi, \eta) = h_3$ give rise to a homoclinic orbit. The homoclinic orbit is also the limit curve of the family of periodic orbits of (2.5) defined by $H(\phi, \eta) = h, h \in (h_1, h_3)$.

(9) Section (IX). Parameters condition is $f_2 = 0$, and $f_1 < 0$, we have $h_1 = h_3 < h_2$ and the system (2.5) have a saddle point at $(\phi_2, 0)$ and two cusps coincide at origin.

(10) Section (X). Parameters condition is $f_1^2 > 9f_2$, and $f_1 < 0$, we have $h_3 < h_1 < h_2$ and the system (2.5) have two saddle points at $(\phi_{1,2}, 0)$ and a center at $(\phi_3, 0)$. The curve on the right hand side of saddle point $(\phi_1, 0)$ defined by $H(\phi, \eta) = h_1$ give rise to a homoclinic orbit. The homoclinic orbit is also the limit curve of the family of periodic orbits of (2.5) defined by $H(\phi, \eta) = h, h \in (h_3, h_1)$.

(11) Section (XI). Parameters condition is $f_1^2 = 9f_2$, and $f_1 < 0$, we have $h_3 < h_1 = h_2$ and the system (2.5) have two saddle points at $(\phi_{1,2}, 0)$ and a center at $(\phi_3, 0)$. The curve connect $(\phi_{1,2}, 0)$ defined by $H(\phi, \eta) = h_1$ give rise to two heteroclinic orbits. The heteroclinic orbits is also the limit curve of the family of periodic orbits of (2.5) defined by $H(\phi, \eta) = h, h \in (h_3, h_1)$.

(12) Section (XII). Parameters condition is $8f_2 < f_1^2 < 9f_2$, and $f_1 < 0$, we have $h_3 < h_2 < h_1$ and the system (2.5) have two saddle points at $(\phi_{1,2}, 0)$ and a center at $(\phi_3, 0)$. The curve on the left hand side of saddle point $(\phi_2, 0)$ defined by $H(\phi, \eta) = h_2$ give rise to a homoclinic orbit. The homoclinic orbit is also the limit curve of the family of periodic orbits of (2.5) defined by $H(\phi, \eta) = h, h \in (h_3, h_2)$.

(13) Section (XIII). Parameters condition is $f_1^2 = 8f_2$, and $f_1 < 0$, we have $h_2 < h_1$ and the system (2.5) have a saddle point at $(\phi_1, 0)$ and two cusps coincide at $(\phi_2, 0)$.

Remark 3.3. According to the qualitative theory of differential equations [18], System (2.5) has only one equilibrium point which is degenerate saddle point at the origin in parametric space under the parameters condition is $f_1 = f_2 = 0$.

We obtain the bifurcation of the phase portraits of system (2.5) in different regions of parametric spaces with maple (see Figure 2).

4. Relationship Between Special Bounded Orbits of System (2.5) and Exact Nonlinear Wave Solutions of System (1.3a)

In this section, we obtain several important wave profiles based on some special phase orbits, such as solitary wave, periodic wave and kink(anti-kink) wave.

4.1. Special bounded orbits of system (2.5) in Figure 2

According to qualitative theory of dynamical system, we suppose that $\phi(\xi)$ is a smooth solution of a system with smoothness for $\xi \in (-\infty, \infty)$ and $\lim_{\xi \rightarrow \infty} \phi(\xi) = \alpha, \lim_{\xi \rightarrow -\infty} \phi(\xi) = \beta$. It is well known that

- (i) $\phi(\xi)$ is called a smooth solitary wave solution if $\alpha = \beta$;
- (ii) $\phi(\xi)$ is called a smooth kink or anti-kink wave solution if $\alpha \neq \beta$.

Usually a smooth solitary wave solution of partial differential system corresponds to a smooth homoclinic orbit of a traveling wave equation. A smooth kink(or antikink) wave solution corresponds to a smooth heteroclinic orbit of a traveling wave equation. In some references, a kink wave solution is called a wavefront. Similarly, a periodic wave solution corresponds to a smooth periodic orbit of traveling wave equation.

Through the analysis above, we can obtain some special and important bounded orbits of system (2.5) from Figure 2.

4.1.1. Smooth homoclinic orbit of system (2.5)

Theorem 4.1. *From Figure 2, we have that*

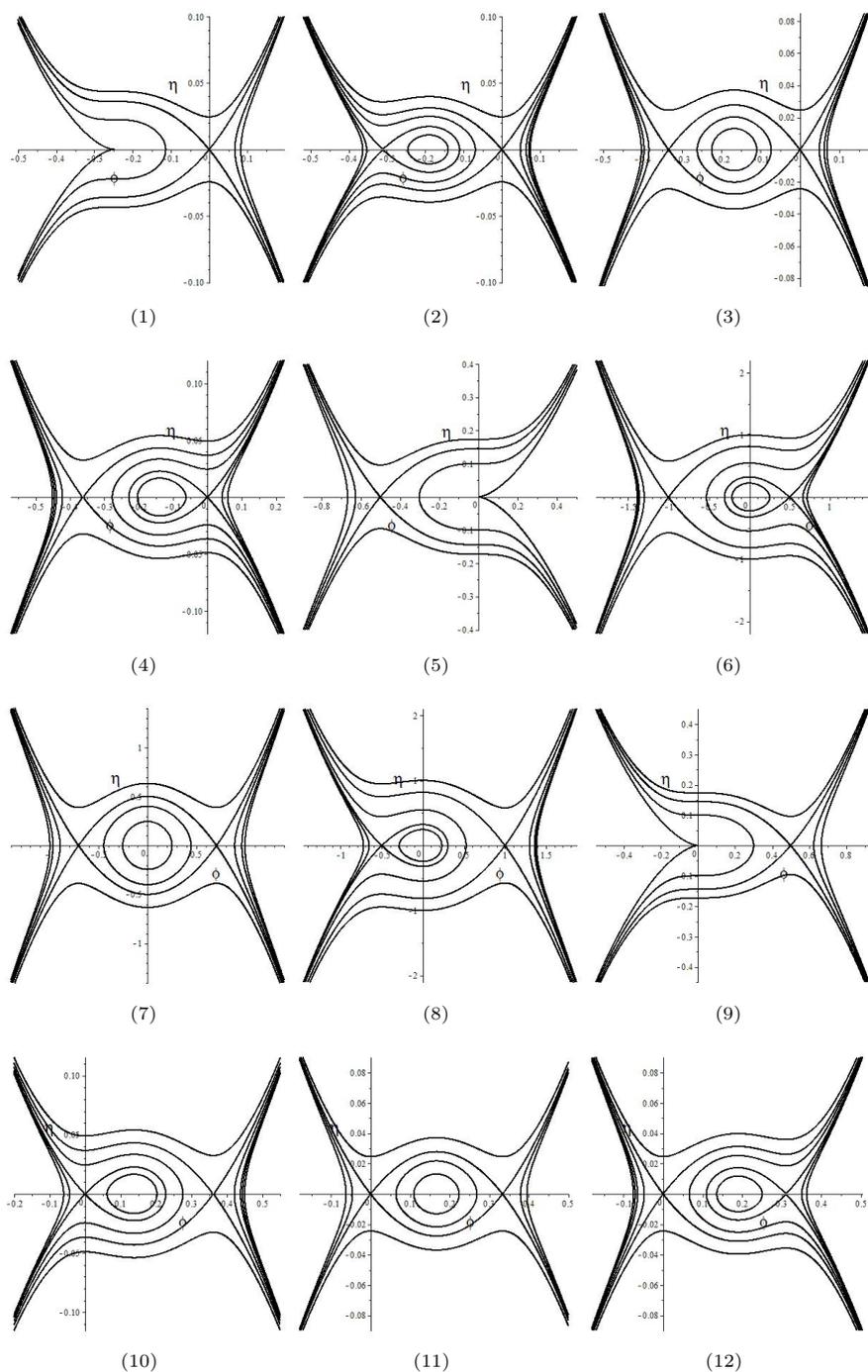
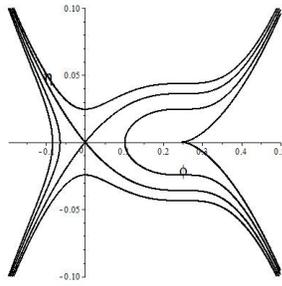
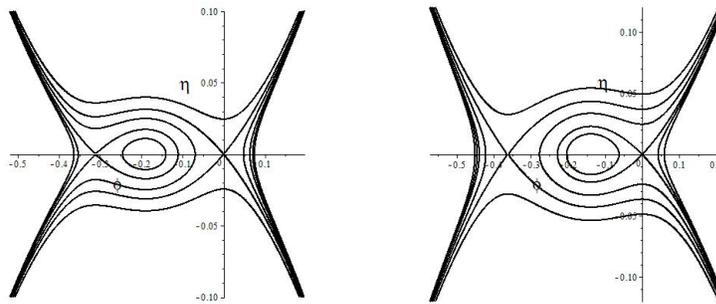


Figure 2. The Bifurcation phase portraits of the system (2.5). (1)-(4) $f_1 > 0, f_2 > 0$. (1) $f_1^2 = 8f_2$. (2) $8f_2 < f_1^2 < 9f_2$. (3) $f_1^2 = 9f_2$. (4) $f_1^2 > 9f_2$. (5) $f_1 > 0, f_2 = 0$. (6) $f_1 > 0, f_2 < 0$. (7) $f_1 = 0, f_2 < 0$. (8) $f_1 < 0, f_2 < 0$. (9) $f_1 < 0, f_2 = 0$. (10)-(13) $f_1 < 0, f_2 > 0$. (10) $f_1^2 > 9f_2$. (11) $f_1^2 = 9f_2$. (12) $8f_2 < f_1^2 < 9f_2$. (13) $f_1^2 = 8f_2$.



(13)

Figure 2. Continued



(1)

(2)

Figure 3. The smooth homoclinic orbit of system (2.5). (1) $f_1, f_2 \in$ section (II). (2) $f_1, f_2 \in$ section (IV).

- (i) $f_1, f_2 \in$ section (II), section (VIII) and section (X), system (2.5) has a smooth homoclinic orbit and the curve is on the right hand side of saddle point;
- (ii) $f_1, f_2 \in$ section (IV), section (VI) and section (XII), system (2.5) has a smooth homoclinic orbit and the curve is on the left hand side of saddle point.

The smooth homoclinic orbit of system (2.5) under parameters conditions that $f_1, f_2 \in$ section (II) and $f_1, f_2 \in$ section (IV) shown in Figure 3 respectively. In section 4.2, we determine the profiles and solutions of nonlinear solitary waves from the known portraits of Figure 3.

4.1.2. Smooth heteroclinic orbit of system (2.5)

Theorem 4.2. From figure 2, we have that $f_1, f_2 \in$ section (III), section (VII) and section (XI), system (2.6) has a smooth heteroclinic orbit.

The smooth heteroclinic orbit of system (2.5) under parameters conditions that $f_1, f_2 \in$ section (III) shown in Figure 4. In section 4.2, we determine the profiles and solutions of nonlinear kink(or anti-kink) waves from the known portraits of Figure 4.

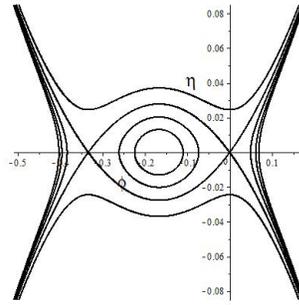


Figure 4. The smooth heteroclinic orbit and the smooth periodic orbit of system (2.6) with $f_1, f_2 \in$ section (III).

4.1.3. Smooth periodic orbit of system (2.5)

Theorem 4.3. *From Figure 2, we have that parameters f_1 and f_2 in the sections in which system (2.5) has a center, system (2.5) has a family of smooth periodic orbit.*

The smooth periodic orbit of system (2.5) under parameters conditions that $f_1, f_2 \in$ section (III) shown in Figure 4. In section 4.2, we determine the profiles and solutions of nonlinear periodic waves from the known portraits of Figure 4.

4.2. Exact Nonlinear Wave Solutions of System (1.3a)

In this section, we compute and determine the profiles and explicit expressions of nonlinear waves from the known portraits of Figure 3 and Figure 4.

4.2.1. Smooth Solitary Wave of System (1.3a)

From Figure 3(1), the intersection points of curve defined by $H(\phi, \eta) = h_3$ with $\eta = 0$ is ϕ_{a1} , ϕ_{b1} and ϕ_{c1} , with

$$\phi_{a1} < \phi_{b1} < \phi_{c1}. \quad (4.1)$$

Based on the Hamiltonian function of system (2.5), we can get that

$$\eta_1 = \pm \sqrt{(\phi - \phi_{a1})^2 (\phi_{b1} - \phi) (\phi_{c1} - \phi)}, \quad (4.2)$$

by using (4.2) and the first equation of system (2.5), we have

$$\pm \int_{\tilde{\phi}_1}^{\phi} \frac{1}{(s - \phi_{a1}) \sqrt{(\phi_{b1} - s)(\phi_{c1} - s)}} ds = \int_0^{\xi} dt, \quad (4.3)$$

where $\phi_{a1} < \tilde{\phi}_1 \ll \phi_{b1}$. From (4.3), we obtain,

$$\phi_1 = \phi_{a1} - 2 \frac{(\phi_{a1} - \phi_{b1})(\phi_{a1} - \phi_{c1})}{(\phi_{b1} - \phi_{c1}) \cosh \left(\sqrt{(\phi_{a1} - \phi_{b1})(\phi_{a1} - \phi_{c1})} \xi \right) + 2\phi_{a1} - \phi_b - \phi_{c1}}. \quad (4.4)$$

By integrating equation (4.4) with respect to ξ once, according to equation (2.1), we have the parametric representation of solitary wave solution of (1.3a) as follows (see Figure 5(1)),

$$w_1 = a(\phi_{a1} - 2 \frac{(\phi_{a1} - \phi_{b1})(\phi_{a1} - \phi_{c1})}{(\phi_{b1} - \phi_{c1}) \cosh(\sqrt{(\phi_{a1} - \phi_{b1})(\phi_{a1} - \phi_{c1})}\xi) + 2\phi_{a1} - \phi_{b1} - \phi_{c1}}).$$

(4.5)

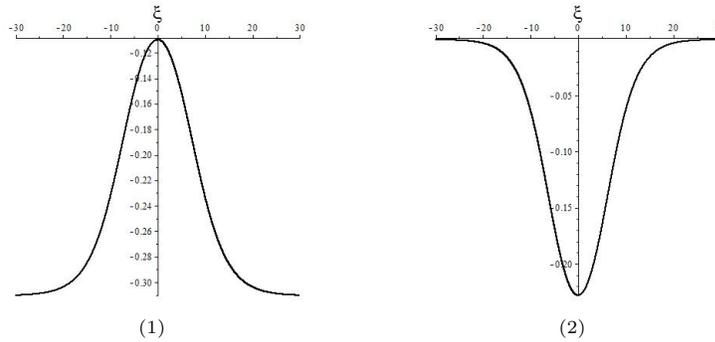


Figure 5. (1) Smooth solitary wave of peak type with $a=1$. (2) Smooth solitary wave of valley type with $a=1$.

Similarly, from Figure 3(2), we obtain parametric representation of the smooth solitary wave of (1.3a) as follows (see Figure 5(2)),

$$w_2 = a(\phi_{a2} - 2 \frac{(\phi_{a2} - \phi_{b2})(\phi_{a2} - \phi_{c2})}{(\phi_{b2} - \phi_{c2}) \cosh(\sqrt{(\phi_{a2} - \phi_{b2})(\phi_{a2} - \phi_{c2})}\xi) + 2\phi_{a2} - \phi_b - \phi_{c2}}),$$

(4.6)

where ϕ_{a2} , ϕ_{b2} and ϕ_{c2} are intersection points of curve defined by $H(\phi, \eta) = h_1$ with $\eta = 0$ under parameter $f_1, f_2 \in \text{section (IV)}$, and

$$\phi_{c2} < \phi_{b2} < \phi_{a2}.$$

(4.7)

4.2.2. Smooth Kink(or Anti-Kink) Wave of System (1.3a)

From Figure 4, the intersection points of curve defined by $H(\phi, \eta) = h_1$ with $\eta = 0$ is ϕ_{a1} , ϕ_{a3} and ϕ_{b3} , with

$$\phi_{b3} < \phi_{a3}.$$

(4.8)

Based on the Hamiltonian function of system (2.5), we can get that

$$\eta_3 = \pm \sqrt{(\phi_{a3} - \phi)^2(\phi - \phi_{b3})^2}.$$

(4.9)

by using (4.9) and the first equation of system (2.5), we have

$$\pm \int_{\tilde{\phi}_3}^{\phi} \frac{1}{\sqrt{(\phi_{a3} - \phi)^2(\phi - \phi_{b3})^2}} ds = \int_0^{\xi} dt,$$

(4.10)

where $\phi_{b3} < \tilde{\phi}_3 \ll \phi_{a3}$.

From (4.10), we obtain,

$$\phi_3 = \frac{\phi_{a3} + \phi_{b3}}{2} + \frac{\phi_{a3} - \phi_{b3}}{2} \tanh\left(\frac{\phi_{a3} - \phi_{b3}}{4}\xi\right), \tag{4.11}$$

and

$$\phi_{3'} = \frac{\phi_{a3} + \phi_{b3}}{2} + \frac{\phi_{a3} - \phi_{b3}}{2} \tanh\left(-\frac{\phi_{a3} - \phi_{b3}}{4}\xi\right). \tag{4.12}$$

By integrating equation (4.11) and (4.12) with respect to ξ once, according to equation (2.1), we have the parametric representation of kink wave and anti-kink wave solution of (1.3a) as follows (see Figure 6 respectively),

$$w_3 = a\left(\frac{\phi_{a3} + \phi_{b3}}{2} + \frac{\phi_{a3} - \phi_{b3}}{2} \tanh\left(\frac{\phi_{a3} - \phi_{b3}}{4}\xi\right)\right), \tag{4.13}$$

and

$$w_{3'} = a\left(\frac{\phi_{a3} + \phi_{b3}}{2} + \frac{\phi_{a3} - \phi_{b3}}{2} \tanh\left(-\frac{\phi_{a3} - \phi_{b3}}{4}\xi\right)\right). \tag{4.14}$$

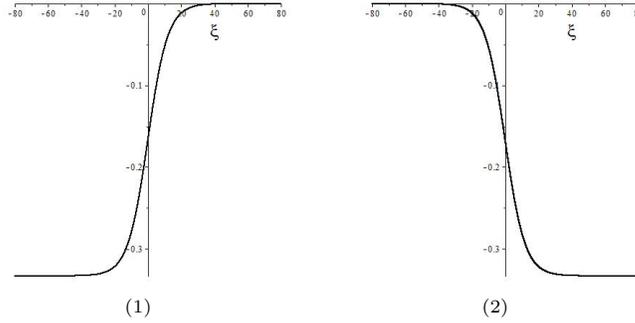


Figure 6. (1) Smooth kink wave with a=1. (2) Smooth anti-kink wave with a=1.

4.2.3. Smooth Periodic Wave of System (1.3a)

From Figure 4, the intersection points of curve defined by defined by $H(\phi, \eta) = h, h \in (h_2, h_1)$ with $\eta = 0$ is $\phi_{a4}, \phi_{b4}, \phi_{c4}$ and ϕ_{d4} with

$$\phi_{d4} < \phi_{c4} < \phi_{b4} < \phi_{a4}. \tag{4.15}$$

Based on the Hamiltonian function of system (2.5), we can get that

$$\eta_4 = \pm \sqrt{(\phi_{a4} - \phi)(\phi_{b4} - \phi)(\phi - \phi_{c4})(\phi - \phi_{d4})}, \tag{4.16}$$

by using (4.16) and the first equation of system (2.5), we have

$$\pm \int_{\tilde{\phi}_4}^{\phi} \frac{1}{\sqrt{(\phi_{a4} - \phi)(\phi_{b4} - \phi)(\phi - \phi_{c4})(\phi - \phi_{d4})}} ds = \int_0^{\xi} dt, \tag{4.17}$$

where $\phi_{c4} < \tilde{\phi}_4 \ll \phi_{b4}$.

From (4.17), we obtain,

$$\phi_4 = \frac{(\phi_{b4} - \phi_{d4})\phi_{c4} - (\phi_{b4} - \phi_{c4})\phi_{d4}sn^2(\frac{\xi}{g}, k)}{(\phi_{b4} - \phi_{d4}) - (\phi_{b4} - \phi_{c4})sn^2(\frac{\xi}{g}, k)}, \tag{4.18}$$

where $g = \frac{2}{\sqrt{(\phi_{a4} - \phi_{c4})(\phi_{b4} - \phi_{d4})}}$ and $k = \sqrt{\frac{(\phi_{b4} - \phi_{c4})(\phi_{a4} - \phi_{d4})}{(\phi_{a4} - \phi_{c4})(\phi_{b4} - \phi_{d4})}}$.

By integrating equation (4.18) with respect to ξ once, according to equation (2.1), we have the parametric representation of periodic wave solution of (1.3a) as follows (see Figure 7),

$$w_4 = a \left(\frac{(\phi_{b4} - \phi_{d4})\phi_{c4} - (\phi_{b4} - \phi_{c4})\phi_{d4}sn^2(\frac{\xi}{g}, k)}{(\phi_{b4} - \phi_{d4}) - (\phi_{b4} - \phi_{c4})sn^2(\frac{\xi}{g}, k)} \right). \tag{4.19}$$

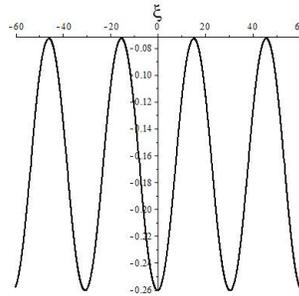


Figure 7. The smooth periodic wave with $h=-0.00018$ and $a=1$.

Thus, to summarize, we have the following main results.

Theorem 4.4. *Through the analysis and computation above, we have*

- (i) *Under parameters $f_1, f_2 \in$ section(II), (VIII) and (X), partial differential equation (1.3a) have a solitary wave of peak type. Under parameters $f_1, f_2 \in$ section(IV), (VI) and (XII), partial differential equation (1.3a) have a solitary wave of valley type.*
- (ii) *Under parameters $f_1, f_2 \in$ section(III), (VII) and (XI), partial differential equation (1.3a) have a Kink (anti-kink) wave.*
- (iii) *Under parameters f_1, f_2 in the sections mentioned in theorem above, partial differential equation (1.3a) have a family of periodic wave.*

Remark 4.1. To sum up, for ease of understanding, we show the wave profiles determined by different phase portraits of system (2.5)(see Figure 8).

5. Conclusion

In this paper, we apply the bifurcation theory method of dynamical systems to find exact traveling wave solutions and their dynamics. We obtain the profiles and solutions of nonlinear waves for a class of (3+1)-dimensional nonlinear equation

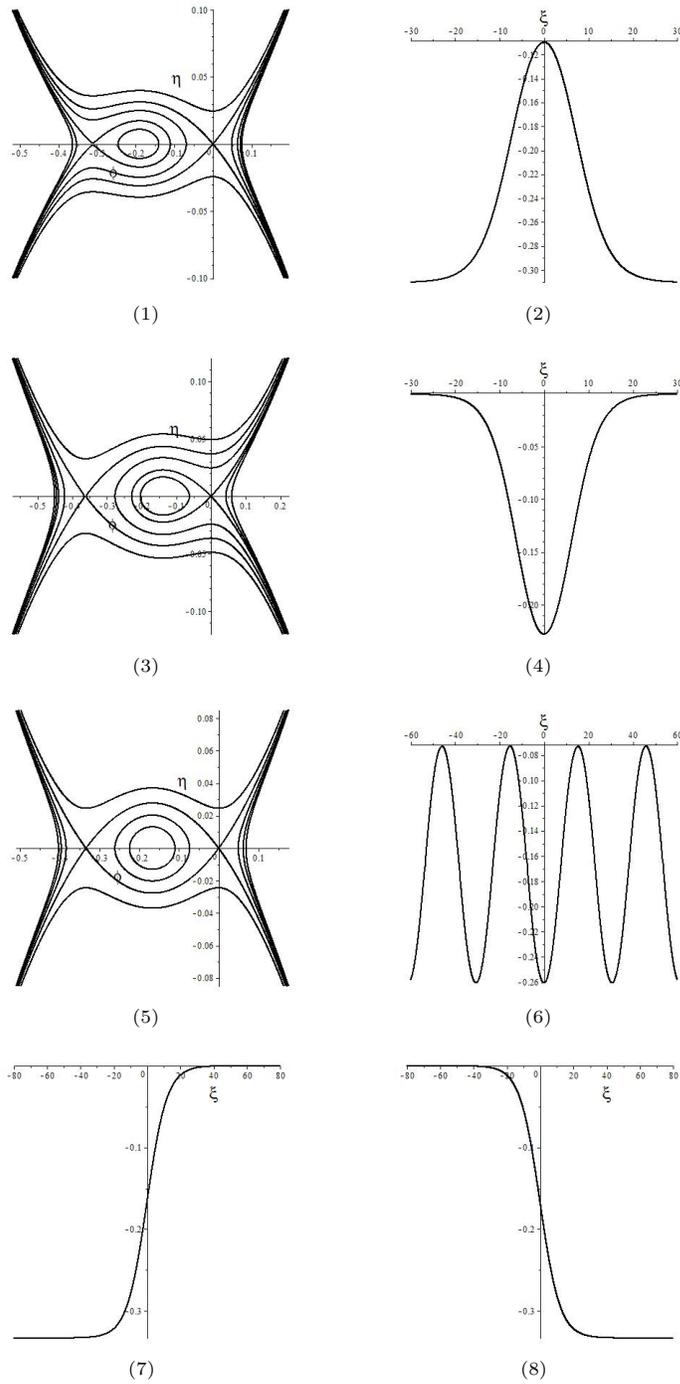


Figure 8. The profiles of wave determined by phase portraits of the system (2.5). (1) Homoclinic orbit to left equilibrium. (2) Solitary wave of peak type. (3) Homoclinic orbit to right equilibrium. (4) Solitary wave of valley type. (5) Smooth periodic orbit and two smooth heteroclinic orbits. (6) Smooth periodic wave. (7) Smooth kink wave. (8) Smooth anti-kink wave.

from known phase portraits of traveling wave equations. We can find that system (2.5) has not singular property and new waves for singular nonlinear traveling wave equations were not arised in this paper.

In fact, nonlinear wave phenomena are of great importance in the physical world and have been for a long time a challenging topic of research for both pure and applied mathematicians. There are numerous nonlinear evolution equations for which we need to analyze the properties of the solutions for time evolution of the systems. The investigation of the traveling wave solutions to nonlinear evolution equations plays an important role in the mathematical physics.

To find exact traveling wave solutions for a given nonlinear wave system, since 1970's, a lot of methods have been developed such as the inverse scattering method, Darboux transformation method, Hirota bilinear method, algebraic geometric method, et al. Usually, the mathematical modeling of important phenomena arising in physics and biology often leads to integrable nonlinear wave equations. Generally, their traveling systems are ordinary differential equations. The studies of solitons and complete integrability of nonlinear wave equations and bifurcations, chaos of dynamical systems are two very active fields in nonlinear science [7–9]. A homoclinic orbit of a traveling wave system corresponds to a solitary wave solution of a nonlinear wave equation, while a heteroclinic orbit of a traveling wave system corresponds to a kink wave solution of a nonlinear wave equation. These relationships provide intersection points for the above two study fields. To consider traveling wave solutions of a partial differential equation, the essential work is to investigate the dynamical behavior of the corresponding ordinary differential equations (traveling wave systems) [7–9]. Therefore, the theory and method of dynamical systems play the pivotal role in the qualitative study of traveling wave solutions.

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