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FEJÉR TYPE INEQUALITIES FOR HARMONICALLY-CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract In this paper, a new weighted identity involving harmonically symmetric functions and differentiable functions is established. By using the notion of harmonic symmetricity, harmonic convexity and some auxiliary results, some new Fejér type integral inequalities are presented. Applications to special means of positive real numbers are given as well.

Keywords Hermite-Hadamard's inequality, Fejér's inequality, convex function, harmonically-convex function, Hölder's inequality, power mean inequality.

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1. Introduction

Let $\mathbb{R}$ be the set of real numbers and $I \subseteq \mathbb{R}$ be an interval. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex in the classical sense if it satisfies the following inequality

$$f(tx + (1-t)t) \leq tf(x) + (1-t)f(y)$$

for all x, $y \in I$ and $t \in [0,1]$.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ and $a, b \in I$ with $a < b$. Then the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds if and only if f is convex, and is known in the literature as Hermite-Hadamard inequality, after the name of C. Hermite and J. Hadamard (see [23]). The inequalities in (1.1) hold in reversed direction if f is a concave function.

A vast literature related to (1.1) have been produced by a large number of mathematicians [9] since it is considered to be one of the most famous inequality for convex functions due to its usefulness and many applications in various branches of Pure and Applied Mathematics, such as Numerical Analysis [4], Information Theory [2], Operator Theory [7] and others.

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The definition of usual convex functions has been generalized in a variety of ways and as a consequence many researchers have established a number of Hermite-Hadamard type inequalities, see for instance [5–28] and the references mentioned in these papers. These results have also many applications of interest for special means such as *p-logarithmic* and *identric means* that are of use in some problems in Chemistry or Electrical Engineering [9].

One of the generalizations of classical convexity is the harmonic convexity stated in the definition below.

Definition 1.1 ([15]). Let $I\subset\mathbb{R}\backslash\{0\}$ be a real interval. A function $f:I\to\mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)\,y}\right)\leq tf\left(y\right)+(1-t)\,f\left(x\right) \tag{1.2}$$

for all x, $y\in I$ and $t\in[0,1]$. If the inequality in (1.2) is reversed, then f is said to be harmonically concave.

The following result explains how the usual convexity and the harmonic convexity are connected.

Proposition 1.1 ([15]). *Let $I\subset\mathbb{R}\backslash\{0\}$ be a real interval and $f:I\to\mathbb{R}$ is function, then:*

- *if $I\subset(0,\infty)$ and f is convex and nondecreasing function then f is harmonically convex.*
- *if $I\subset(0,\infty)$ and f is harmonically convex and nonincreasing function then f is convex.*
- *if $I\subset(-\infty,0)$ and f is harmonically convex and nondecreasing function then f is convex.*
- *if $I\subset(-\infty,0)$ and f is convex and nonincreasing function then f is harmonically convex.*

Most recently, İşcan [15], has proved the following results for harmonically convex functions.

Theorem 1.1 ([15]). *Let $I\subset\mathbb{R}\backslash\{0\}$ be a harmonically convex function and a, $b\in I$ with $a<b$. If $f\in L([a,b])$ then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\leq\frac{f(a)+f(b)}{2}.$$

The above inequalities are sharp.

Theorem 1.2 ([15]). *Let $f:(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, a, $b\in I^\circ$ with $a<b$, and $f'\in L([a,b])$. If $\left|f'\right|^q$ is harmonically convex $[a,b]$ for $q\geq1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|$$
$$\leq\frac{ab(b-a)}{2}\lambda_1^{1-\frac{1}{q}}\left[\lambda_2\left|f'(a)\right|^q+\lambda_3\left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

where

$$\lambda_1 = \frac{1}{ab} - \frac{2}{(b-a)^2} \ln \left(\frac{(a+b)^2}{4ab} \right),$$
$$\lambda_2 = -\frac{1}{b(b-a)} + \frac{3a+b}{(b-a)^3} \ln \left(\frac{(a+b)^2}{4ab} \right),$$
$$\lambda_2 = \frac{1}{a(b-a)} - \frac{3b+a}{(b-a)^3} \ln \left(\frac{(a+b)^2}{4ab} \right) = \lambda_1 - \lambda_2.$$

Theorem 1.3 ([15]). *Let $f : (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, a, $b \in I^\circ$ with $a < b$, and $f' \in L([a,b])$. If $\left|f'\right|^q$ is harmonically convex $[a,b]$ for $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right|$$
$$\leq \frac{ab(b-a)}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\mu_1 \left| f'(a) \right|^q + \mu_1 \left| f'(b) \right|^q \right]^{\frac{1}{q}},$$

where

$$\mu_1 = \frac{a^{2-2q} + b^{1-2q}\left[(b-a)(1-2q) - a\right]}{2(b-a)^2(1-q)(1-2q)},$$
$$\mu_2 = \frac{b^{2-2q} - a^{1-2q}\left[(b-a)(1-2q) + b\right]}{2(b-a)^2(1-q)(1-2q)}.$$

Some applications of the above results can also be found in [15].

In 1906, Fejér [10] while studying trigonometric polynomials, obtained inequalities which generalize that of Hermite & Hadamard:

Theorem 1.4. *Consider the integral $\int_a^b h(x) w(x) dx$, where h is a convex function in the interval (a,b) and w is a positive function in the same interval such that*

$$w(a+t) = w(b-t), \quad 0 \leq t \leq \frac{1}{2}(a+b),$$

i.e., $y = w(x)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b), 0\right)$ and is normal to the $x-$axis. Under those conditions the following inequalities are valid:

$$h\left(\frac{a+b}{2}\right) \int_a^b w(x) dx \leq \int_a^b h(x) w(x) dx \leq \frac{h(a)+h(b)}{2} \int_a^b w(x) dx. \tag{1.3}$$

If h is concave on (a,b), then the inequalities reverse in (1.3).

Chen and Wu [6] established the following Fejér type inequality for harmonically convex functions which provides a weighted generalization of the result given in Theorem 1.1.

Theorem 1.5 ([6]). *Let $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function and a, $b \in I$ with $a < b$. If $f \in L([a, b])$, them one has be continuous*

$$f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx \leq \int_a^b \frac{f(x)g(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx, \quad (1.4)$$

$g : [a, b] \to \mathbb{R}$ is nonnegative, integrable and satisfies

$$g\left(\frac{ab}{x}\right) = g\left(\frac{ab}{a+b-x}\right).$$

Motivated by the above results, the main purpose of the present paper is to introduce a new notion of harmonically symmetric functions and to establish a useful identity involving a harmonically symmetric function and a differentiable function. This identity will then be used as a main tool to prove some Fejér type inequalities related to the first part of the inequality (1.4). Some applications to special means of positive real numbers will also be provided in Section 3.

2. Main Results

Throughout this section we take $L(t) = \frac{2ab}{(1-t)a+(1+t)b}$ and $U(t) = \frac{2ab}{(1+t)a+(1-t)b}$ for all $t \in [0, 1]$. The Beta function, the Gamma function and the integral from of the hypergeometric function are defined as follows to be used in the sequel of the paper

$$B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)\beta - 1dt, \ \alpha > 0, \ \beta > 0,$$
$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt, \ \alpha > 0$$

and

$${}_2F_1(\alpha, \beta; \gamma; z) = \frac{1}{B(\beta, \gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt$$

for $|z| < 1, \gamma > \beta > 0$.

The notion of harmonically symmetric functions is given in following definition.

Definition 2.1. A function $g : [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonically symmetric with respect to $\frac{2ab}{a+b}$ if

$$g(x) = g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}\right)$$

holds for all $x \in [a, b]$.

Now we prove a weighted integral identity which will help us to prove our main results.

Lemma 2.1. *Let $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a differentiable function on I° and a, $b \in I^\circ$ with $a < b$ and let $g : [a, b] \to [0, \infty)$ be continuous positive mapping and* *harmonically symmetric to* $\frac{2ab}{a+b}$. *If* $f' \in L([a,b])$, *then the following equality holds*

$$\begin{aligned} & f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)\,g(x)}{x^2}dx \\ =& \left(\frac{b-a}{2ab}\right)\int_0^1 \left(\int_a^{L(t)} \frac{g(x)}{x^2}dx\right)\left[(L(t))^2 f'(L(t)) - (U(t))^2 f'(U(t))\right]dt. \end{aligned} \tag{2.1}$$

Proof. Let

$$I_1 = \int_0^1 \left(\int_a^{L(t)} \frac{g(x)}{x^2}dx\right)(U(t))^2 f'(U(t))\,dt$$

and

$$I_2 = \int_0^1 \left(\int_a^{L(t)} \frac{g(x)}{x^2}dx\right)(L(t))^2 f'(L(t))\,dt.$$

Since $g:[a,b]\to[0,\infty)$ is harmonically symmetric to $\frac{2ab}{a+b}$, then $g(U(t)) = g(L(t))$ for all $t\in[0,1]$ and

$$\int_a^{L(t)} \frac{g(x)}{x^2}dx = \int_{U(t)}^b \frac{g(x)}{x^2}dx.$$

Hence, we have

$$\begin{aligned} I_1 &= \int_0^1 \left(\int_a^{L(t)} \frac{g(x)}{x^2}dx\right)(U(t))^2 f'(U(t))\,dt \\ &= \frac{2ab}{b-a}\int_0^1 \left(\int_{U(t)}^b \frac{g(x)}{x^2}dx\right)d\left[f(U(t))\right] \\ &= \frac{2ab}{b-a}\left(\int_{U(t)}^b \frac{g(x)}{x^2}dx\right)f(U(t))\Bigg|_0^1 + \int_0^1 g(U(t))\,f(U(t))\,dt \\ &= -\frac{2ab}{b-a}f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx - \int_0^1 g(U(t))\,f(U(t))\,dt \\ &= -\frac{2ab}{b-a}f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx + \frac{2ab}{b-a}\int_{\frac{2ab}{a+b}}^b \frac{g(x)\,f(x)}{x^2}dx. \end{aligned} \tag{2.2}$$

Analogously, we have

$$I_2 = \frac{2ab}{b-a}f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx - \frac{2ab}{b-a}\int_a^{\frac{2ab}{a+b}} \frac{g(x)\,f(x)}{x^2}dx. \tag{2.3}$$

Adding (2.2) and (2.3) and multiplying the result by $\frac{b-a}{2ab}$, we get the required identity. This completes the proof of the Lemma. □

Lemma 2.2. *For* $v > u > 0$, *we have*

$$\int_0^1 (1-t)\left[\frac{2uv}{(1+t)\,u + (1-t)\,v}\right]^2 dt = \left(\frac{2uv}{v-u}\right)^2 \lambda_1(u,v),$$

$$\int_0^1 (1-t)^2\left[\frac{2uv}{(1+t)\,u + (1-t)\,v}\right]^2 dt = \left(\frac{2uv}{v-u}\right)^2 \lambda_2(u,v),$$

$$\int_0^1 \left(1-t^2\right)\left[\frac{2uv}{(1+t)\,u+(1-t)\,v}\right]^2 dt = \left(\frac{2uv}{v-u}\right)^2 \lambda_3\,(u,v)\,,$$

where

$$\lambda_1\,(u,v) \triangleq \frac{u-v}{u+v} - \ln\left(\frac{2u}{u+v}\right),$$

$$\lambda_2\,(u,v) \triangleq \frac{4u^2-(u+v)^2}{u^2-v^2} - \frac{4u}{u-v}\ln\left(\frac{2u}{u+v}\right)$$

and

$$\lambda_3\,(u,v) \triangleq \frac{4v^2-(u+v)^2}{u^2-v^2} + \frac{2\,(u+v)}{u-v}\ln\left(\frac{2u}{u+v}\right).$$

Proof. The proof follows from a straightforward computation. □

Lemma 2.3. *For $v > u > 0$ and $p > 1$, we have*

$$\int_0^1 (1+t)\left[\frac{2uv}{(1+t)\,u+(1-t)\,v}\right]^{2p} dt = \left(\frac{2uv}{v+u}\right)^{2p} \zeta_1\,(u,v;p)\,,$$
$$\int_0^1 (1-t)\left[\frac{2uv}{(1+t)\,u+(1-t)\,v}\right]^{2p} dt = \left(\frac{2uv}{v+u}\right)^{2p} \zeta_2\,(u,v;p)\,,$$
$$\int_0^1 \left[\frac{2ab}{(1+t)\,u+(1-t)\,v}\right]^{2p} dt = \left(\frac{2uv}{v+u}\right)^{2p} \varsigma\,(u,v;p)$$

where

$$\zeta_1(u,v;p) \triangleq \frac{2^{1-2p}u\left(\frac{u}{u+v}\right)^{-2p}[(1-2p)(u-v)-v]}{(v-u)^2(p-1)(2p-1)} - \frac{(u+v)\left[(1-2p)(u-v)-2v\right]}{2(v-u)^2(p-1)(2p-1)},$$

$$\zeta_2\,(u,v;p) \triangleq \frac{4^{1-p}u^2\left(\frac{u}{u+v}\right)^{-2p} + (u+v)\left[(2p-1)\,(u-v)-2u\right]}{2\,(v-u)^2\,(p-1)\,(2p-1)}$$

and

$$\varsigma\,(u,v;p) \triangleq \frac{(u+b)^{-2p+1}-(2u)^{-2p+1}}{(2p-1)\,(u-v)\,(u+v)^{-2p}}.$$

Proof. The proof follows from a straightforward computation. □

Now we give some new Fejér type inequalities for harmonically-convex functions, which provide weighted generalization of some of the results established in recent literature.

Theorem 2.1. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° and a, $b \in I^\circ$ with $a < b$ and let $g : [a,b] \to [0,\infty)$ be continuous positive mapping and harmonically symmetric to $\frac{2ab}{a+b}$ such that $f' \in L\,([a,b])$. If $\left|f'\right|^q$ is harmonically-*

convex on $[a,b]$ *for* $q \geq 1$*, then the following inequality holds*

$$\begin{aligned}&\left|f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)g(x)}{x^2}dx\right| \\ &\leq \left(\frac{1}{2}\right)^{1/q}\|g\|_\infty\left\{[\lambda_1(a,b)]^{1-1/q}\left[\lambda_2(a,b)\left|f'(a)\right|^q + \lambda_3(a,b)\left|f'(b)\right|^q\right]^{1/q}\right. \\ &\quad\left.+[\lambda_1(b,a)]^{1-1/q}\left[\lambda_3(b,a)\left|f'(a)\right|^q + \lambda_2(b,a)\left|f'(b)\right|^q\right]^{1/q}\right\},\end{aligned} \tag{2.4}$$

where $\|g\|_\infty = \sup_{x\in[a,b]} g(x) < \infty$ *and* $\lambda_1(\cdot,\cdot)$*,* $\lambda_2(\cdot,\cdot)$*,* $\lambda_3(\cdot,\cdot)$ *are defined in Lemma 2.2.*

Proof. From Lemma 2.1, we get

$$\begin{aligned}&\left|f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)g(x)}{x^2}dx\right| \\ &\leq \left(\frac{b-a}{2ab}\right)^2\|g\|_\infty\left\{\left(\int_0^1(1-t)(U(t))^2dt\right)^{1-1/q}\left(\int_0^1(1-t)(U(t))^2\left|f'(U(t))\right|^q dt\right)^{1/q}\right. \\ &\quad\left.+\left(\int_0^1(1-t)(L(t))^2dt\right)^{1-1/q}\left(\int_0^1(1-t)(L(t))^2\left|f'(L(t))\right|^q dt\right)^{1/q}\right\}.\end{aligned} \tag{2.5}$$

By the harmonic-convexity of $\left|f'\right|^q$ on $[a,b]$ for $q \geq 1$ and by using Lemma 2.2, we have

$$\begin{aligned}&\int_0^1(1-t)(U(t))^2\left|f'(U(t))\right|^q dt \\ &=\int_0^1(1-t)\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^2\times\left|f'\left(\frac{2ab}{(1+t)a+(1-t)b}\right)\right|^q dt \\ &\leq\frac{1}{2}\left|f'(a)\right|^q\int_0^1(1-t)^2\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^2 dt \\ &\quad+\frac{1}{2}\left|f'(b)\right|^q\int_0^1\left(1-t^2\right)\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^2 dt \\ &=\frac{1}{2}\left(\frac{2ab}{b-a}\right)^2\left\{\lambda_2(a,b)\left|f'(a)\right|^q+\lambda_3(a,b)\left|f'(b)\right|^q\right\}\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}&\int_0^1(1-t)(L(t))^2\left|f'(L(t))\right|^q dt \\ &=\int_0^1(1-t)\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^2\times\left|f'\left(\frac{2ab}{(1-t)a+(1+t)b}\right)\right|^q dt \\ &\leq\frac{1}{2}\left|f'(a)\right|^q\int_0^1\left(1-t^2\right)\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^2 dt \\ &\quad+\frac{1}{2}\left|f'(b)\right|^q\int_0^1(1-t)^2\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^2 dt\end{aligned}$$

$$=\frac{1}{2}\left(\frac{2ab}{b-a}\right)^2\left\{\lambda_3(b,a)\left|f'(a)\right|^q+\lambda_2(b,a)\left|f'(b)\right|^q\right\}. \tag{2.7}$$

Moreover,

$$\begin{aligned}\int_0^1(1-t)(U(t))^2\,dt&=\int_0^1(1-t)\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^2dt\\&=\left(\frac{2ab}{b-a}\right)^2\lambda_1(a,b)\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\int_0^1(1-t)(L(t))^2\,dt&=\int_0^1(1-t)\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^2dt\\&=\left(\frac{2ab}{b-a}\right)^2\lambda_1(b,a).\end{aligned} \tag{2.9}$$

A combination of (2.5), (2.6), (2.7), (2.8) and (2.9) gives the required result. This completes the proof of the theorem. □

Corollary 2.1. *Suppose the assumptions of Theorem 2.1 are satisfied. If $q=1$, then the following inequality holds*

$$\begin{aligned}&\left|f\left(\frac{2ab}{a+b}\right)\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{f(x)g(x)}{x^2}dx\right|\\&\leq\left(\frac{1}{2}\right)\|g\|_\infty\left\{[\lambda_2(a,b)+\lambda_3(b,a)]\left|f'(a)\right|+[\lambda_3(a,b)+\lambda_2(b,a)]\left|f'(b)\right|\right\},\end{aligned} \tag{2.10}$$

where $\|g\|_\infty=\sup_{x\in[a,b]}g(x)<\infty$ and $\lambda_2(\cdot,\cdot)$, $\lambda_3(\cdot,\cdot)$ are defined in Lemma 2.2.

Corollary 2.2. *If $g(x)=\frac{ab}{b-a}$ for all $x\in[a,b]$ in Theorem 2.1, then*

$$\begin{aligned}&\left|f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\\&\leq\left(\frac{1}{2}\right)^{1/q}\left(\frac{ab}{b-a}\right)\times\left\{[\lambda_1(a,b)]^{1-1/q}\left[\lambda_2(a,b)\left|f'(a)\right|^q+\lambda_3(a,b)\left|f'(b)\right|^q\right]^{1/q}\right.\\&\quad\left.+[\lambda_1(b,a)]^{1-1/q}\left[\lambda_3(b,a)\left|f'(a)\right|^q+\lambda_2(b,a)\left|f'(b)\right|^q\right]^{1/q}\right\},\end{aligned} \tag{2.11}$$

where $\|g\|_\infty=\sup_{x\in[a,b]}g(x)<\infty$ and $\lambda_1(\cdot,\cdot)$, $\lambda_2(\cdot,\cdot)$, $\lambda_3(\cdot,\cdot)$ are defined in Lemma 2.2.

Corollary 2.3. *If $q=1$ in Corollary 2.2, then the following inequality holds*

$$\begin{aligned}&\left|f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\\&\leq\left(\frac{1}{2}\right)\left(\frac{ab}{b-a}\right)\times\left\{[\lambda_2(a,b)+\lambda_3(b,a)]\left|f'(a)\right|+[\lambda_3(a,b)+\lambda_2(b,a)]\left|f'(b)\right|\right\},\end{aligned} \tag{2.12}$$

where $\lambda_2(\cdot,\cdot)$, $\lambda_3(\cdot,\cdot)$ are defined in Lemma 2.2.

Theorem 2.2. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° and a, $b \in I^\circ$ with $a < b$ and let $g : [a,b] \to [0,\infty)$ be continuous positive mapping and harmonically symmetric to $\frac{2ab}{a+b}$ such that $f^{'} \in L([a,b])$. If $\left|f^{'}\right|^q$ is harmonically-convex on $[a,b]$ for $q > 1$, then the following inequality holds*

$$\begin{aligned} &\left| f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)g(x)}{x^2}dx \right| \\ &\leq \|g\|_\infty \left(\frac{b-a}{b+a}\right)^2 \left(\frac{1}{2}\right)^{1/q} \left(\frac{q-1}{2q-1}\right)^{1-1/q} \\ &\quad\times \left\{ \left[\zeta_2(a,b;q)\left|f^{'}(a)\right|^q + \zeta_1(a,b;q)\left|f^{'}(b)\right|^q\right]^{1/q} \right. \\ &\quad \left. + \left[\zeta_1(b,a;q)\left|f^{'}(a)\right|^q + \zeta_2(b,a;q)\left|f^{'}(b)\right|^q\right]^{1/q} \right\}, \end{aligned} \tag{2.13}$$

where $\zeta_1(\cdot,\cdot;\cdot)$ and $\zeta_2(\cdot,\cdot;\cdot)$ are defined in Lemma 2.3.

Proof. From Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned} &\left| f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)g(x)}{x^2}dx \right| \\ &\leq \left(\frac{b-a}{2ab}\right)^2 \|g\|_\infty \left(\int_0^1 (1-t)^{q/(q-1)}\right)^{1-1/q} \left\{ \left(\int_0^1 (U(t))^{2q}\left|f^{'}(U(t))\right|^q\right)^{1/q} \right. \\ &\quad \left. + \left(\int_0^1 (L(t))^{2q}\left|f^{'}(L(t))\right|^q\right)^{1/q} \right\}. \end{aligned} \tag{2.14}$$

Since $\left|f^{'}\right|^q$ is harmonically-convex on $[a,b]$, we obtain

$$\begin{aligned} &\int_0^1 [U(t)]^{2q}\left|f^{'}(U(t))\right|^q \\ &= \int_0^1 \left[\frac{2ab}{(1+t)a+(1-t)b}\right]^{2q} \left|f^{'}\left(\frac{2ab}{(1+t)a+(1-t)b}\right)\right|^q dt \\ &\leq \frac{1}{2}\left|f^{'}(a)\right|^q \int_0^1 (1-t)\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^{2q} dt \\ &\quad + \frac{1}{2}\left|f^{'}(b)\right|^q \int_0^1 (1+t)\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^{2q} dt \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} &\int_0^1 [L(t)]^{2q}\left|f^{'}(L(t))\right|^q \\ &= \int_0^1 \left[\frac{2ab}{(1-t)a+(1+t)b}\right]^{2q} \left|f^{'}\left(\frac{2ab}{(1-t)a+(1+t)b}\right)\right|^q dt \\ &\leq \frac{1}{2}\left|f^{'}(a)\right|^q \int_0^1 (1+t)\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^{2q} dt \end{aligned}$$

$$+\frac{1}{2}\left|f^{'}(b)\right|^{q}\int_{0}^{1}(1-t)\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^{2q}dt. \tag{2.16}$$

By applying Lemma 2.3 in inequalities (2.15) and (2.16) and then using the resulting inequalities in (2.14), we get the required inequality. □

Corollary 2.4. *If the assumptions of Theorem 2.2 are satisfied and if $g(x)=\frac{ab}{b-a}$ for all $x\in[a,b]$, then the following inequality holds*

$$\begin{aligned}&\left|f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_{a}^{b}\frac{f(x)}{x^{2}}dx\right|\\&\leq\left(\frac{ab}{b-a}\right)\left(\frac{b-a}{b+a}\right)^{2}\left(\frac{1}{2}\right)^{1/q}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\\&\quad\times\left\{\left[\zeta_{2}(a,b;q)\left|f^{'}(a)\right|^{q}+\zeta_{1}(a,b;q)\left|f^{'}(b)\right|^{q}\right]^{1/q}\right.\\&\quad\left.+\left[\zeta_{1}(b,a;q)\left|f^{'}(a)\right|^{q}+\zeta_{2}(b,a;q)\left|f^{'}(b)\right|^{q}\right]^{1/q}\right\},\end{aligned} \tag{2.17}$$

where $\zeta_1(\cdot,\cdot;\cdot)$ and $\zeta_2(\cdot,\cdot;\cdot)$ are defined in Lemma 2.3.

Theorem 2.3. *Let $f:I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I° and a, $b\in I^{\circ}$ with $a<b$ and let $g:[a,b]\to[0,\infty)$ be continuous positive mapping and harmonically symmetric to $\frac{2ab}{a+b}$ such that $f^{'}\in L([a,b])$. If $\left|f^{'}\right|^{q}$ is harmonically-convex on $[a,b]$ for $q>1$, then the following inequality holds*

$$\begin{aligned}&\left|f\left(\frac{2ab}{a+b}\right)\int_{a}^{b}\frac{g(x)}{x^{2}}dx-\int_{a}^{b}\frac{f(x)g(x)}{x^{2}}dx\right|\\&\leq\left(\frac{b-a}{b+a}\right)^{2}\left(\frac{1}{2}\right)^{2/q-1}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\|g\|_{\infty}\left\{\left[\zeta_{2}(a,b;q)+\zeta_{1}(b,a;q)\right]\left|f^{'}(a)\right|^{q}\right.\\&\quad\left.+\left[\zeta_{1}(a,b;q)+\zeta_{2}(b,a;q)\right]\left|f^{'}(b)\right|^{q}\right\}^{1/q},\end{aligned} \tag{2.18}$$

where $\zeta_1(\cdot,\cdot;\cdot)$ and $\zeta_2(\cdot,\cdot;\cdot)$ are defined in Lemma 2.3.

Proof. From Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}&\left|f\left(\frac{2ab}{a+b}\right)\int_{a}^{b}\frac{g(x)}{x^{2}}dx-\int_{a}^{b}\frac{f(x)g(x)}{x^{2}}dx\right|\\&\leq\left(\frac{b-a}{2ab}\right)^{2}\|g\|_{\infty}\left(\int_{0}^{1}t^{q/(q-1)}dt\right)^{1-1/q}\left\{\left(\int_{0}^{1}[U(t)]^{2q}\left|f^{'}(U(t))\right|^{q}dt\right)^{1/q}\right.\\&\quad\left.+\left(\int_{0}^{1}[L(t)]^{2q}\left|f^{'}(L(t))\right|^{q}dt\right)^{1/q}\right\}.\end{aligned} \tag{2.19}$$

By the power-mean inequality $a^{r}+b^{r}\leq2^{1-r}(a+b)^{r}$ for $a>0,b>0$ and $r<1$, we have

$$\begin{aligned}&\left(\int_0^1 [U(t)]^{2q}\left|f'(U(t))\right|^{2q}dt\right)^{1/q}+\left(\int_0^1 [L(t)]^{2q}\left|f'(L(t))\right|^{2q}dt\right)^{1/q}\\&\le 2^{1-1/q}\left(\int_0^1 [U(t)]^{2q}\left|f'(U(t))\right|^{q}dt+\int_0^1 [L(t)]^{2q}\left|f'(L(t))\right|^{q}dt\right)^{1/q}.\end{aligned} \tag{2.20}$$

Since $\left|f'\right|^q$ is harmonically-convex on $[a,b]$ for $q>1$, we obtain

$$\begin{aligned}&\int_0^1 [U(t)]^{2q}\left|f'(U(t))\right|^{q}dt+\int_0^1 [L(t)]^{2q}\left|f'(L(t))\right|^{q}dt\\&\le\frac{1}{2}\left|f'(a)\right|^q\int_0^1(1-t)\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^{2q}dt\\&\quad+\frac{1}{2}\left|f'(b)\right|^q\int_0^1(1+t)\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^{2q}dt\\&\quad+\frac{1}{2}\left|f'(a)\right|^q\int_0^1(1+t)\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^{2q}dt\\&\quad+\frac{1}{2}\left|f'(b)\right|^q\int_0^1(1-t)\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^{2q}dt\\&=\frac{1}{2}\left(\frac{2ab}{b+a}\right)^{2q}\Big\{\left[\zeta_2(a,b;q)+\zeta_1(b,a;q)\right]\left|f'(a)\right|^q\\&\quad+\left[\zeta_1(a,b;q)+\zeta_2(b,a;q)\right]\left|f'(b)\right|^q\Big\}.\end{aligned} \tag{2.21}$$

Using (2.20) in (2.21), we get

$$\begin{aligned}&\left(\int_0^1 [U(t)]^{2q}\left|f'(U(t))\right|^{q}dt\right)^{1/q}+\left(\int_0^1 [L(t)]^{2q}\left|f'(L(t))\right|^{q}dt\right)^{1/q}\\&\le 2^{1-2/q}\left(\frac{2ab}{b+a}\right)^{2}\Big\{\left[\zeta_1(a,b;q)+\zeta_2(b,a;q)\right]\left|f'(a)\right|^q\\&\quad+\left[\zeta_2(a,b;q)+\zeta_1(b,a;q)\right]\left|f'(b)\right|^q\Big\}^{1/q}.\end{aligned} \tag{2.22}$$

Applying (2.22) in (2.19), we obtain the required inequality (2.18). □

Corollary 2.5. *If the assumptions of Theorem 2.3 are satisfied and if* $g(x)=\frac{ab}{b-a}$ *for all* $x\in[a,b]$, *then the following inequality holds*

$$\begin{aligned}&\left|f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\\&\le\left(\frac{ab}{b-a}\right)\left(\frac{b-a}{b+a}\right)^2\left(\frac{1}{2}\right)^{1-2/q}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\\&\quad\times\Big\{\left[\zeta_2(a,b;q)+\zeta_1(b,a;q)\right]\left|f'(a)\right|^q+\left[\zeta_1(a,b;q)+\zeta_2(b,a;q)\right]\left|f'(b)\right|^q\Big\}^{1/q},\end{aligned} \tag{2.23}$$

where $\zeta_1(\cdot,\cdot;\cdot)$ *and* $\zeta_2(\cdot,\cdot;\cdot)$ *are defined in Lemma 2.3.*

Theorem 2.4. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° and a, $b \in I^{\circ}$ with $a < b$ and let $g : [a,b] \to [0,\infty)$ be continuous positive mapping and harmonically symmetric to $\frac{2ab}{a+b}$ such that $f^{'} \in L([a,b])$. If $\left|f^{'}\right|$ is harmonically-convex on $[a,b]$, then the following inequality holds for $q > 1$*

$$\left|f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)g(x)}{x^2}dx\right|$$
$$\leq \frac{1}{2}\left(\frac{b-a}{a+b}\right)^2 \|g\|_{\infty}\left[\left[\varsigma(a,b;q)\right]^{1-1/q}\left\{\left(\frac{1}{2q+1}\right)^{1/q}\left|f^{'}(a)\right| + \left[\omega(q)\right]^{1/q}\left|f^{'}(a)\right|\right\}\right.$$
$$\left. + \left[\varsigma(b,a;q)\right]^{1-1/q}\left\{\left[\omega(q)\right]^{1/q}\left|f^{'}(a)\right| + \left(\frac{1}{2q+1}\right)^{1/q}\left|f^{'}(b)\right|\right\}\right], \tag{2.24}$$

where $\varsigma(\cdot,\cdot;\cdot)$ is defined in Lemma 2.3 and

$$\omega(q) = \frac{\sqrt{\pi}\Gamma(q+1)}{2\Gamma\left(q+\frac{3}{2}\right)}.$$

Proof. From Lemma 2.1 and by using the harmonic-convexity of $\left|f^{'}\right|$ on $[a,b]$, we have

$$\left|f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)g(x)}{x^2}dx\right|$$
$$\leq \left(\frac{b-a}{2ab}\right)^2 \|g\|_{\infty}\int_0^1\left[(1-t)(U(t))^2\left|f^{'}(U(t))\right| + (1-t)(L(t))^2\left|f^{'}(L(t))\right|\right]dt$$
$$\leq \left(\frac{b-a}{2ab}\right)^2 \|g\|_{\infty}\left\{\int_0^1 (U(t))^2\left[\frac{(1-t)^2}{2}\left|f^{'}(a)\right| + \left(\frac{1-t^2}{2}\right)\left|f^{'}(b)\right|\right]dt\right.$$
$$\left. + \int_0^1 (L(t))^2\left[\left(\frac{1-t^2}{2}\right)\left|f^{'}(a)\right| + \frac{(1-t)^2}{2}\left|f^{'}(b)\right|\right]dt\right\}. \tag{2.25}$$

Since

$$\omega(q) = \int_0^1\left(1-t^2\right)^q dt = \int_0^{\frac{\pi}{2}}\cos^{2q+1}(t)\,dt = \frac{\sqrt{\pi}\Gamma(q+1)}{2\Gamma\left(q+\frac{3}{2}\right)} \tag{2.26}$$

and

$$\int_0^1 (1-t)^{2q}dt = \frac{1}{2q+1}. \tag{2.27}$$

Now by using Hölder integral inequality, (2.26), (2.27) and Lemma 2.3, we have

$$\int_0^1\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^2\left[\frac{(1-t)^2}{2}\left|f^{'}(a)\right| + \left(\frac{1-t^2}{2}\right)\left|f^{'}(b)\right|\right]dt$$
$$\leq \left(\int_0^1\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^{2q/(q-1)}dt\right)^{1-1/q}$$

$$\times\left\{\left[\int_0^1\frac{(1-t)^{2q}}{2^q}dt\right]^{1/q}\left|f'(a)\right|+\left[\int_0^1\frac{\left(1-t^2\right)^q}{2^q}dt\right]^{1/q}\left|f'(b)\right|\right\}$$
$$=\frac{1}{2}\left(\frac{2ab}{a+b}\right)^2[\varsigma(a,b;q)]^{1-\frac{1}{q}}\left\{\left(\frac{1}{2q+1}\right)^{\frac{1}{q}}\left|f'(a)\right|+[\omega(q)]^{\frac{1}{q}}\left|f'(b)\right|\right\}. \quad (2.28)$$

Similarly, one has

$$\begin{aligned}&\int_0^1\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^2\left[\left(\frac{1-t^2}{2}\right)\left|f'(a)\right|+\frac{(1-t)^2}{2}\left|f'(b)\right|\right]dt\\&\le\left(\int_0^1\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^{2q/(q-1)}dt\right)^{1-1/q}\\&\quad\times\left\{\left[\int_0^1\frac{\left(1-t^2\right)^q}{2^q}dt\right]^{1/q}\left|f'(a)\right|+\left[\int_0^1\frac{(1-t)^{2q}}{2^q}dt\right]^{1/q}\left|f'(b)\right|\right\}\\&=\frac{1}{2}\left(\frac{2ab}{a+b}\right)^2[\varsigma(b,a;q)]^{1-\frac{1}{q}}\left\{[\omega(q)]^{\frac{1}{q}}\left|f'(a)\right|+\left(\frac{1}{2q+1}\right)^{\frac{1}{q}}\left|f'(b)\right|\right\}.\end{aligned} \quad (2.29)$$

Using (2.28) and (2.29) in (2.25), we obtain the required result (2.24). □

Corollary 2.6. *Under the assumptions of Theorem 2.4, if* $g(x)=\frac{ab}{b-a}$ *for all* $x\in[a,b]$*, then the following inequality holds*

$$\begin{aligned}&\left|f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\\&\le\frac{1}{2}\left(\frac{ab}{b-a}\right)\left(\frac{b-a}{a+b}\right)^2\left[[\varsigma(a,b;q)]^{1-1/q}\left\{\left(\frac{1}{2q+1}\right)^{1/q}\left|f'(a)\right|+[\omega(q)]^{1/q}\left|f'(a)\right|\right\}\right.\\&\quad\left.+[\varsigma(b,a;q)]^{1-1/q}\left\{[\omega(q)]^{1/q}\left|f'(a)\right|+\left(\frac{1}{2q+1}\right)^{1/q}\left|f'(b)\right|\right\}\right],\end{aligned} \quad (2.30)$$

where $\varsigma(\cdot,\cdot;\cdot)$ *and* $\omega(\cdot)$ *are defined in Theorem 2.4.*

Theorem 2.5. *Let* $f:I\subseteq(0,\infty)\to\mathbb{R}$ *be a differentiable function on* I° *and* $a,\ b\in I^\circ$ *with* $0<a<b<1$ *and let* $g:[a,b]\to[0,\infty)$ *be continuous positive mapping and harmonically symmetric to* $\frac{2ab}{a+b}$ *such that* $f'\in L([a,b])$*. If* $\left|f'\right|$ *is harmonically-convex on* $[a,b]$*, then the following inequality holds for* $q>1$

$$\begin{aligned}&\left|f\left(\frac{2ab}{a+b}\right)\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{f(x)g(x)}{x^2}dx\right|\le\frac{1}{2}\left(\frac{b-a}{a+b}\right)^2\|g\|_\infty\\&\times\left\{[\nu(a,b;q)]^{1-1/q}\left[\left(\frac{1}{q+1}\right)^{1/q}\left|f'(a)\right|+\left(\frac{2^{q+1}-1}{q+1}\right)^{1/q}\left|f'(b)\right|\right]\right.\\&\left.+[\nu(b,a;q)]^{1-1/q}\left[\left(\frac{2^{q+1}-1}{q+1}\right)^{1/q}\left|f'(a)\right|+\left(\frac{1}{q+1}\right)^{1/q}\left|f'(b)\right|\right]\right\},\end{aligned} \quad (2.31)$$

where

$$\nu(a,b;q)=\frac{\Gamma\left(\frac{2q-1}{q-1}\right)}{\Gamma\left(\frac{3q-2}{q-1}\right)}\left[a^{\frac{2q-1}{q-1}}{}_2F_1\left(\frac{2q}{q-1},\frac{2q-1}{q-1};\frac{2q-1}{q-1};\frac{a(b-a)}{a+b}\right)\right.$$
$$\left.-b^{\frac{2q-1}{q-1}}{}_2F_1\left(\frac{2q}{q-1},\frac{2q-1}{q-1};\frac{2q-1}{q-1};\frac{b(b-a)}{a+b}\right)\right],$$

$\Gamma(\cdot)$ *is the Gamma function and* ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ *is the hypergeometric function.*

Proof. From Lemma 2.1 and by using the harmonic-convexity of $\left|f^{'}\right|$ on $[a,b]$, we have

$$\begin{aligned}
&\left|f\left(\frac{2ab}{a+b}\right)\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{f(x)g(x)}{x^2}dx\right|\\
&\leq\left(\frac{b-a}{2ab}\right)^2\|g\|_\infty\int_0^1\left[(1-t)(U(t))^2\left|f^{'}(U(t))\right|+(1-t)(L(t))^2\left|f^{'}(L(t))\right|\right]dt\\
&\leq\left(\frac{b-a}{2ab}\right)^2\|g\|_\infty\left\{\int_0^1(1-t)(U(t))^2\left[\left(\frac{1-t}{2}\right)\left|f^{'}(a)\right|+\left(\frac{1+t}{2}\right)\left|f^{'}(b)\right|\right]dt\right.\\
&\quad\left.+\int_0^1(1-t)(L(t))^2\left[\left(\frac{1+t}{2}\right)\left|f^{'}(a)\right|+\left(\frac{1-t}{2}\right)\left|f^{'}(b)\right|\right]dt\right\}.
\end{aligned}\tag{2.32}$$

Application of Hölder integral inequality yields

$$\begin{aligned}
&\int_0^1(1-t)\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^2\left[\left(\frac{1-t}{2}\right)\left|f^{'}(a)\right|+\left(\frac{1+t}{2}\right)\left|f^{'}(b)\right|\right]dt\\
&\leq\left(\int_0^1(1-t)^{q/(q-1)}\left[\frac{2ab}{(1+t)a+(1-t)b}\right]^{2q/(q-1)}dt\right)^{1-1/q}\\
&\quad\times\left\{\left[\int_0^1\left(\frac{1-t}{2}\right)^q dt\right]^{1/q}\left|f^{'}(a)\right|+\left[\int_0^1\left(\frac{1+t}{2}\right)^q dt\right]^{1/q}\left|f^{'}(b)\right|\right\}\\
&=\frac{1}{2}\left(\frac{2ab}{a+b}\right)^2[\nu(a,b;q)]^{1-1/q}\left[\left(\frac{1}{q+1}\right)^{1/q}\left|f^{'}(a)\right|+\left(\frac{2^{q+1}-1}{q+1}\right)^{1/q}\left|f^{'}(b)\right|\right].
\end{aligned}\tag{2.33}$$

Similarly, one has

$$\begin{aligned}
&\int_0^1(1-t)\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^2\left[\left(\frac{1+t}{2}\right)\left|f^{'}(a)\right|+\left(\frac{1-t}{2}\right)\left|f^{'}(b)\right|\right]dt\\
&\leq\left(\int_0^1(1-t)^{q/(q-1)}\left[\frac{2ab}{(1-t)a+(1+t)b}\right]^{2q/(q-1)}dt\right)^{1-1/q}\\
&\quad\times\left\{\left[\int_0^1\left(\frac{1+t}{2}\right)^q dt\right]^{1/q}\left|f^{'}(a)\right|+\left[\int_0^1\left(\frac{1-t}{2}\right)^q dt\right]^{1/q}\left|f^{'}(b)\right|\right\}\\
&=\frac{1}{2}\left(\frac{2ab}{a+b}\right)^2[\nu(b,a;q)]^{1-\frac{1}{q}}\left[\left(\frac{2^{q+1}-1}{q+1}\right)^{\frac{1}{q}}\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\right].
\end{aligned}\tag{2.34}$$

Using (2.33) and (2.34) in (2.32), we obtain the required inequality (2.31). □

Corollary 2.7. *Suppose the assumptions of Theorem 2.4 are satisfied and if* $g(x) = \frac{ab}{b-a}$ *for all* $x \in [a,b]$*, then the following inequality holds*

$$\left| f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \right| \le \frac{1}{2}\left(\frac{ab}{b-a}\right)\left(\frac{b-a}{a+b}\right)^2$$
$$\times \left\{ [\nu(a,b;q)]^{1-1/q} \left[\left(\frac{1}{q+1}\right)^{1/q} \left|f'(a)\right| + \left(\frac{2^{q+1}-1}{q+1}\right)^{1/q} \left|f'(b)\right| \right] \right. \tag{2.35}$$
$$\left. + [\nu(b,a;q)]^{1-\frac{1}{q}} \left[\left(\frac{2^{q+1}-1}{q+1}\right)^{\frac{1}{q}} \left|f'(a)\right| + \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left|f'(b)\right| \right] \right\},$$

where $\nu(\cdot,\cdot;\cdot)$ *is defined in Theorem 2.5.*

Remark 2.1. Some further results can be obtained from (2.25) but we omit the details for the interested readers.

3. Applications to Special Means

In this section we apply some of the above established inequalities of Hermite-Hadamard type involving the product of a harmonically convex function and a harmonically symmetric function to construct inequalities for special means.

For positive numbers $a > 0$ and $b > 0$ with $a \neq b$

$$A(a,b) = \frac{a+b}{2},\ L(a,b) = \frac{b-a}{\ln b - \ln a},\ G(a,b) = \sqrt{ab},\ H(a,b) = \frac{2ab}{a+b}$$

and

$$L_p(a,b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p \neq -1, 0, \\ L(a,b), & p = -1, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & p = 0, \end{cases}$$

are the arithmetic mean, the logarithmic mean, geometric mean, harmonic mean and the generalized logarithmic mean of order $p \in \mathbb{R}$, respectively.

For further information on means, we refer the readers to [3] and the references therein.

Let $g : [a,b] \to \mathbb{R}_0$ be defined as

$$g(x) = \left(\frac{a+b}{2ab} - \frac{1}{x}\right)^2, x \in [a,b].$$

It is obvious that

$$g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}\right) = g(x)$$

for all $x \in [a,b]$. Hence $g(x) = \left(\frac{a+b}{2ab} - \frac{1}{x}\right)^2$, $x \in [a,b]$ is harmonically symmetric with respect to $x = \frac{2ab}{a+b}$.

Now applications of our results are given in the following theorems to come.

Theorem 3.1. *Let $0 < a < b$. Then the following inequality holds*

$$\begin{aligned}&\left|\frac{A^2(a,b)-G^2(a,b)}{3A^2(a,b)}-\frac{A^2(a,b)+G^2(a,b)}{G^2(a,b)}+\frac{2A(a,b)}{L(a,b)}\right|\\&\le\frac{a}{b}\ln\left(\frac{a}{A(a,b)}\right)+\frac{b}{a}\ln\left(\frac{A(a,b)}{b}\right)+\frac{a\ln\left(\frac{b}{A(a,b)}\right)+b\ln\left(\frac{A(a,b)}{a}\right)}{H(a,b)}.\end{aligned}\tag{3.1}$$

Proof. Applying Corollary 2.1 to the functions

$$f(x)=x^2 \text{ for } x>0$$

and

$$g(x)=\left(\frac{a+b}{2ab}-\frac{1}{x}\right)^2, x\in[a,b]$$

we get the desired result. □

Theorem 3.2. *Let $0 < a < b$ and $r \in (-1,\infty)\setminus\{0\}$. Then the following inequality holds*

$$\begin{aligned}&\left|H^{r+2}(a,b)-G^2(a,b)L_r^r(a,b)\right|\\&\le(r+2)\left(\frac{1}{2}\right)\left(\frac{2ab}{b-a}\right)^2\left\{\left(\frac{a^{r+1}}{H(a,b)}-\frac{b^{r+1}}{a}\right)\ln\left(\frac{b}{A(a,b)}\right)\right.\\&\left.+\left(\frac{a^{r+1}}{b}-\frac{b^{r+1}}{H(a,b)}\right)\ln\left(\frac{a}{A(a,b)}\right)\right\}.\end{aligned}\tag{3.2}$$

Proof. The assertion follows from the inequality proved in Corollary 2.3 for $f(x) = x^{r+2}$, $x > 0$, $r \in (-1,\infty)\setminus\{0\}$. □

Theorem 3.3. *Let $0 < a < b$ and $q > 1$. Then*

$$\begin{aligned}&\left|H^2(a,b)-G^2(a,b)\right|\\&\le\left(\frac{1}{2}\right)^{1/q-1}\left(\frac{ab}{b-a}\right)\left(\frac{b-a}{b+a}\right)^2\left(\frac{q-1}{2q-1}\right)^{1-1/q}\\&\times\left\{\left(\left[\frac{2ab^{2q}((1-2q)(a-b)-b)H^{-2q}(a,b)-((1-2q)(a-b)-2b)A(a,b)}{(b-a)^2(q-1)(2q-1)}\right]a^q\right.\right.\\&\left.+\left[\frac{2a^2b^{2q}H^{-2q}(a,b)+((2q-1)(a-b)-2a)A(a,b)}{(b-a)^2(q-1)(2q-1)}\right]b^q\right)^{1/q}\\&+\left(\left[\frac{2a^{2q}b((1-2q)(b-a)-a)H^{-2q}(a,b)-((1-2q)(b-a)-2a)A(a,b)}{(b-a)^2(q-1)(2q-1)}\right]b^q\right.\\&\left.\left.+\left[\frac{2a^{2q}b^2H^{-2q}(a,b)+((2q-1)(b-a)-2b)A(a,b)}{(b-a)^2(q-1)(2q-1)}\right]a^q\right)^{1/q}\right\}.\end{aligned}\tag{3.3}$$

Proof. Applying Corollary 2.4 to the function

$$f(x)=x^2 \text{ for } x>0,$$

we get the desired result. □

Theorem 3.4. *Let* $0 < a < b$ *and* $q > 1$. *Then*

$$\begin{aligned}
&\left|\frac{A^2(a,b) - G^2(a,b)}{3A^2(a,b)} - \frac{A^2(a,b) + G^2(a,b)}{G^2(a,b)} + \frac{2A(a,b)}{L(a,b)}\right| \\
&\leq \left(\frac{b-a}{b+a}\right)^2 \left(\frac{1}{2}\right)^{1/q-2} \left(\frac{q-1}{2q-1}\right)^{1-1/q} \left(\frac{b-a}{2ab}\right)^2 \\
&\times \left\{\left[\frac{\left(a^2b^{2q} + a^{2q}b\left((2q-1)(a-b) - a\right)\right) H^{-2q}(a,b)}{(a-b)^2(q-1)(2q-1)}\right] a^q \right. \\
&\left. + \left[\frac{\left(a^{2q}b^2 + ab^{2q}\left((2q-1)(b-a) - b\right)\right) H^{-2q}(a,b)}{(a-b)^2(q-1)(2q-1)}\right] b^q\right\}^{1/q}.
\end{aligned} \tag{3.4}$$

Proof. Applying Theorem 2.3 to the functions

$$f(x) = x^2 \text{ for } x > 0$$

and

$$g(x) = \left(\frac{a+b}{2ab} - \frac{1}{x}\right)^2, \ x \in [a,b],$$

we get the desired result. □

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