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ON L_P-SOLUTION OF FRACTIONAL HEAT EQUATION DRIVEN BY FRACTIONAL BROWNIAN MOTION*

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Abstract In this paper, we study the fractional stochastic heat equation driven by fractional Brownian motions of the form

$$du(t,x) = \Big(-(-\Delta)^{\alpha/2}u(t,x) + f(t,x)\Big)\,dt + \sum_{k=1}^{\infty} g^k(t,x)\delta\beta_t^k$$

with $u(0,x) = u_0$, $t \in [0,T]$ and $x \in \mathbb{R}^d$, where $\beta^k = \{\beta_t^k, t \in [0,T]\}, k \geq 1$ is a sequence of i.i.d. fractional Brownian motions with the same Hurst index $H > 1/2$ and the integral with respect to fractional Brownian motion is Skorohod integral. By adopting the framework given by Krylov, we prove the existence and uniqueness of L_p-solution to such equation.

Keywords Fractional Brownian motion, fractional heat equation, the Littlewood-Paley inequality.

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1. Introduction

It is known that stochastic partial differential equations (SPDEs) play an important role in describing a real world with random perturbations. Its theory has been developed very fast, and it is widely used in many scientific fields such as nonlinear filtering, the dynamics of population, describing a free field in relativistic and diffraction in random-heterogeneous media in statistical physics. To study the SPDEs, one can choose either Walsh's method [32] using martingale measures or the abstract Hilbert space approach of Da Prato and Zabczyk [8]. Both Walsh's theory and Da Prato and Zabczyk's approach are rather complete and satisfactory in solving of the Cauchy problems for SPDEs.

Recently, Krylov in [19, 22] established a comprehensive L_p theory of second order quasi-linear parabolic SPDEs of the form

$$\begin{aligned} du(t,x) = {} & (\mathcal{L}u(t,x) + cu(t,x) + f(t,x))\,dt \\ & + \sum_k \left(\sum_i \sigma^{ik} \frac{\partial}{\partial x^i} u(t,x) + \nu^k u + g^k(t,x) \right) dW_t^k, \end{aligned} \tag{1.1}$$

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with $u(0,x) = u_0(x)$, where $\mathcal{L} = \sum a_{i,j} \frac{\partial^2}{\partial x^i \partial x^j} + \sum b_i \frac{\partial}{\partial x^i}$, $\{W^k, k \geq 1\}$ is a sequence of independent Brownian motions and the integral with respect to Brownian motion is Itô's integral. This theory is sharp and cannot be improved under his assumptions, and which may be applied to a large class of important equations, including equations of nonlinear filtering, stochastic heat equation with nonlinear noise term, etc. This theory has attracted attentions of many authors and various L^p-results of SPDEs around Krylov's L^p-theory were developed rapidly. Mikulevicius and Rozovskii [24] extended Krylov's L_p-solvability theory to the Cauchy problem for systems of parabolic SPDEs and established some additional integrability and regularity properties. In [17], Kim studied the L_p-theory of stochastic partial differential equations on weighted Sobolev spaces within C^1 and Lipschitz domain, respectively. Zhang [33] introduced a more general L_p-theory of semi-linear SPDEs on general measure spaces with an unbounded linear negative operator on $L^p(E, B, \mu)$. Mikulevicius and Pragarauskas [25] gave some estimates of fractional Sobolev and Besov norms of singular integrals arising in the model problem for the Zakai equation with discontinuous observation. Chen and Kim [7] presented the L_p-theory of non-divergence form SPDEs driven by Lévy processes. K. Kim and P. Kim [16] extended the above results on equation (1.1) to a class of stochastic equations with the random fractional Laplacian driven by Lévy processes. Some more works for the L_p-theory of SPDEs (1.1) and related questions can be fund in I. Kim and K. Kim [15], K. Kim [18], Krylov [19] and the references therein.

On the other hand, in recent years there has been considerable interest in studying fractional Brownian motion due to its compact properties and applications in various scientific areas including telecommunications, turbulence, image processing and finance. Moreover, there also has been some recent interest in studying SPDEs driven by fractional Brownian motion. Grecksch and Ann [12] studied the semilinear stochastic parabolic equation with an infinite dimensional fractional Brownian motion input. In Nualart and Ouknine [29], the authors proved the existence and uniqueness of a solution for a quasilinear parabolic equation in one dimension driven by fractional white noise. Balan and Tudor [3] considered the stochastic heat equation with multiplicative fractional-colored noise. Balan [2] studied a class of stochastic wave equation driven by multiplicative fractional noise by a Malliavin calculus approach. Duncan *et al.* [9] investigated the solutions of a family of semi-linear stochastic equations with a fractional Brownian motion in a Hilbert space. For more material, we refer to Duncan *et al.* [10], Garrido-Atienza *et al.* [11], Hu-Nualart [14], Maslowski-Nualart [23], Tindel *et al.* [31] and references therein.

Motivated by the above results, in this paper, we consider an L_p-theory of the fractional stochastic heat equation driven by fractional Brownian motions of the form

$$\begin{cases} du(t,x) = \big(-(-\Delta)^{\alpha/2} u(t,x) + f(t,x)\big)\,dt + \sum\limits_{k=1}^{\infty} g^k(t,x)\delta\beta_t^k, \\ u(0,x) = u_0(x), \end{cases} \tag{1.2}$$

with $t \in [0,T]$, $x \in \mathbb{R}^d$ and $\alpha \in (0,2]$, where $\beta^k = \{\beta_t^k, t \in [0,T]\}, k \geq 1$ is a sequence of i.i.d. fractional Brownian motions with the same Hurst index $H > 1/2$ and the integral with respect to fractional Brownian motion is Skorohod integral. When $H = \frac{1}{2}$ and the stochastic integral is Itô integral, the equation is considered in Chang-Lee [6], and moreover, when $\alpha = 2$ Balan [1] considered the L_p-theory of equation (1.2). This paper can be viewed as an extension of the results in Balan [1] and its structure is organized as follows. Section 2 contains some necessary preliminaries on the Skorohod integral with respect to fractional Brownian motion and definitions of stochastic function spaces, and the fractional Laplacian operator is also introduced in this section. In Section 3, by using the method introduced by Krylov in [19], we obtain a version of Littlewood-Paley inequality. Finally, in Section 4, we prove the uniqueness and existence of equation (1.2) in the framework of L_p-theory developed by Krylov.

2. Preliminaries

In this section, we firstly recall some basic results of fractional Brownian motion (in short, fBm). For more aspects on the material we refer to Biagini *et al.* [4], Hu [13], Mishura [26], Nourdin [27], Nualart [28] and references therein. In order to state the main results, we explain also the stochastic function spaces and introduce some properties of fractional Laplacian operator in this section. We make the convention that the positive constant C, unless special explanation, depend only on the subscripts and its value may be different in different appearance. When C has subscripts, it will indicate the only dependence on parameters.

2.1. Fractional Brownian motion

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$ be a complete probability space with the filtration $\{\mathscr{F}_t\}$ satisfying the usual condition. The Gaussian process $\beta = \{\beta(t), 0 \le t \le T\}$ defined on $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$, with continuous sample paths, is called a fBm with Hurst index $H \in (0,1)$ if $\beta(0) = 0$, $E\beta(t) = 0$ and

$$E\left[\beta(t)\beta(s)\right] = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right]$$

for all $t, s \ge 0$. FBm β admits the Wiener integral representation of the form

$$\beta(t) = \int_0^t K_H(t,s)dW(s), \qquad 0 \le t \le T,$$

where $\{W(t), 0 \le t \le T\}$ is a standard Brownian motion and the kernel $K_H(t,s)$ satisfies

$$\frac{\partial K_H}{\partial t}(t,s) = \kappa_H\left(H - \frac{1}{2}\right)\left(\frac{s}{t}\right)^{\frac{1}{2}-H}(t-s)^{H-\frac{3}{2}}$$

with a normalizing constant $\kappa_H > 0$ such that $E(\beta_1^2) = 1$. Let $\mathcal{H}$ be the completion of the linear space $\mathcal{E}$ generated by the indicator functions $1_{[0,t]}, t \in [0,T]$ with respect to the inner product

$$\langle 1_{[0,s]}, 1_{[0,t]}\rangle_{\mathcal{H}} = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right].$$

The application $\varphi \in \mathcal{E} \to \beta(\varphi)$ is an isometry from $\mathcal{E}$ to the Gaussian space generated by β and it can be extended to $\mathcal{H}$. Denote by $\mathcal{S}_\beta$ the set of smooth functionals of the form

$$F = f(\beta(\varphi_1), \beta(\varphi_2), \ldots, \beta(\varphi_n)),$$

where $f \in C_b^\infty(\mathbb{R}^n)$ (f and all its derivatives are bounded) and $\varphi_i \in \mathcal{H}$. The *derivative operator* D^β (the Malliavin derivative) of a functional F of the form above is defined as

$$D^\beta F = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\beta(\varphi_1), \beta(\varphi_2), \dots, \beta(\varphi_n))\varphi_j.$$

The derivative operator D^β is then a closable operator from $L^2(\Omega)$ into $L^2(\Omega; \mathcal{H})$. We denote by $\mathbb{D}_\beta^{1,2}$ the closure of $\mathcal{S}_\beta$ with respect to the norm

$$\|F\|_{\mathbb{D}_\beta^{1,2}} := \sqrt{E|F|^2 + E\|D^\beta F\|_{\mathcal{H}}^2}.$$

The *divergence integral* δ^β is the adjoint of derivative operator D^β. That is, we say that a random variable u in $L^2(\Omega; \mathcal{H})$ belongs to the domain of the divergence operator δ^β, denoted by $\mathrm{Dom}(\delta^\beta)$, if

$$E\left|\langle D^\beta F, u\rangle_{\mathcal{H}}\right| \le C\|F\|_{L^2(\Omega)}, \qquad \forall F \in \mathbb{D}_\beta^{1,2}.$$

In this case $\delta^\beta(u)$ is defined by the duality relationship

$$E\left[F\delta^\beta(u)\right] = E\langle D^\beta F, u\rangle_{\mathcal{H}} \tag{2.1}$$

for any $u \in \mathbb{D}_\beta^{1,2}$. We have $\mathbb{D}_\beta^{1,2} \subset \mathrm{Dom}(\delta^\beta)$. We will use the notation

$$\delta^\beta(u) = \int_0^T u_s \delta\beta_s$$

to express the Skorohod integral of a process u, and the indefinite Skorohod integral is defined as $\int_0^t u_s\delta\beta_s = \delta^\beta(u1_{[0,t]})$.

Let K be a Hilbert space with norm $\|\cdot\|_K$. Consider the family $\mathcal{S}_\beta(K)$ of K-valued smooth random variables of the form

$$F = \sum_{j=1}^{n} F_j v_j, \qquad F_j \in \mathcal{S}_\beta, v_j \in K.$$

Clearly, $\mathcal{S}_\beta(K) \subset L_p(\Omega; K)$. Similarly, the Malliavin calculus can be defined as

$$D^\beta F := \sum_{j=1}^{n} (D^\beta F_j) \otimes v_j$$

and we have $D^\beta F \in L_p(\Omega; \mathcal{H} \otimes K)$ for $p \ge 1$. We endow $\mathcal{S}_\beta(K)$ with the norm

$$\|F\|_{\mathbb{D}_\beta^{1,p}(K)}^p := E\|F\|_K^p + E\|D^\beta F\|_{\mathcal{H}\otimes K}^p,$$

and let $\mathbb{D}_\beta^{1,p}(K)$ be the completion of $\mathcal{S}_\beta(K)$ to this norm.

We now introduce some Banach spaces and Hilbert spaces associated with the Malliavin calculus. Let V be an arbitrary Banach space and let $\mathcal{E}_V$ be the class of all elementary processes taking values in V. Define the spaces of strongly measurable functions by

$$|\mathcal{H}_V| := \{f : [0,T] \to V \mid \|f\|_{|\mathcal{H}_V|} < \infty\}$$

and

$$|\mathcal{H}| \otimes |\mathcal{H}_V| := \{f : [0,T]^2 \to V \mid \|f\|_{|\mathcal{H}|\otimes|\mathcal{H}_V|} < \infty\},$$

where

$$\|f\|^2_{|\mathcal{H}_V|} = \alpha_H \int_0^T \int_0^T \|f(t)\|_V \|f(s)\|_V |t-s|^{2H-2} dsdt, \qquad \alpha_H = H(2H-1)$$

and

$$\|f\|^2_{|\mathcal{H}|\otimes|\mathcal{H}_V|} = \alpha_H^2 \int_{[0,T]^4} |f(t,\theta)|_V |f(s,\eta)|_V |t-s|^{2H-2} |\theta-\eta|^{2H-2} d\theta d\eta dsdt.$$

Note that the space $\mathcal{E}_V$ is dense in $|\mathcal{H}_V|$ with respect to the norm $\|\cdot\|_{|\mathcal{H}_V|}$. It is well-known that there exists a constant b_H such that (see Nualart [28])

$$\|f\|_{|\mathcal{H}_V|} \le b_H \|f\|_{L_{1/H}([0,T];V)}.$$

If V is a Hilbert space, we let $\mathcal{H}_V$ be the completion of $\mathcal{E}_V$ with respect to the inner product

$$\langle \phi, \varphi \rangle_{\mathcal{H}_V} = \alpha_H \int_0^T \int_0^T \langle \phi(t), \varphi(s) \rangle_V |t-s|^{2H-2} dsdt$$

and $\mathcal{H}_V \otimes \mathcal{H}_V$ be the completion of $\mathcal{E}_V \otimes \mathcal{E}_V$ with respect to the inner product

$$\langle \phi, \varphi \rangle_{\mathcal{H}_V\otimes\mathcal{H}_V} = \alpha_H^2 \int_{[0,T]^4} \langle \phi(t,\theta), \varphi(s,\eta) \rangle_{V\otimes V} |t-s|^{2H-2} |\theta-\eta|^{2H-2} dsdt.$$

Moreover, we define the space $|\mathcal{H}_V| \otimes |\mathcal{H}_V|$ of strongly measurable functions by

$$|\mathcal{H}_V| \otimes |\mathcal{H}_V| := \{ f : [0,T]^2 \to V \otimes V \mid \|f\|_{|\mathcal{H}_V|\otimes|\mathcal{H}_V|} < \infty \},$$

where

$$\|f\|^2_{|\mathcal{H}_V|\otimes|\mathcal{H}_V|} = \alpha_H^2 \int_{[0,T]^4} |f(t,\theta)|_{V\otimes V} |f(s,\eta)|_{V\otimes V} |t-s|^{2H-2} |\theta-\eta|^{2H-2} d\theta d\eta dsdt.$$

Lemma 2.1 (Balan [1])**.** *We have that $\mathcal{H}_V$ is isomorphic with $\mathcal{H} \otimes V$ and*

$$\|f\|_{\mathcal{H}_V} \le \|f\|_{|\mathcal{H}_V|} \le b_H \|f\|_{L_{1/H}([0,T];V)} \le b_H \|f\|_{L_2([0,T];V)},$$

and

$$\|f\|_{\mathcal{H}_V\otimes\mathcal{H}_V} \le \|f\|_{|\mathcal{H}_V|\otimes|\mathcal{H}_V|} \le b_H \|f\|_{L_{1/H}([0,T]^2;V\otimes V)} \le b_H \|f\|_{L_2([0,T]^2;V\otimes V)}.$$

In particular, $\mathcal{E}_V = \mathcal{E}$, $|\mathcal{H}_V| = |\mathcal{H}|$ and $\mathcal{H}_V = \mathcal{H}$, provided $V = \mathbb{R}$.

2.2. Stochastic function spaces

Let $p \ge 1$ and $n \in \mathbb{R}$. We first recall some basic facts about the space of Bessel potentials. Let $C_0^\infty = C_0^\infty(\mathbb{R}^d)$ be the space of infinitely differentiable functions on $\mathbb{R}^d$ with compact support and let $\mathcal{D} = \mathcal{D}(\mathbb{R}^d)$ be the space of real-valued Schwartz distributions on C_0^∞. Define the spaces

$$\begin{aligned}
&L_p = L_p(\mathbb{R}^d) := \{ f : \mathbb{R}^d \to \mathbb{R} \mid \|f\|^p_{L_p} = \int_{\mathbb{R}^d} |f(x)|^p dx < \infty \},\\
&H_p^n = H_p^n(\mathbb{R}^d) := \{ f \in \mathcal{D} \mid (I-\Delta)^{n/2} f \in L_p \},\\
&H_p^n(\ell_2) := \{ g = (g^1, g^2, \ldots) : \ell_2 - \text{valued functions such that}\\
&\qquad\qquad g^k \in H_p^n \text{ for each k and } |(I-\Delta)^{n/2} g|_{\ell_2} \in L_p \},
\end{aligned}$$

and their norms are given by

$$\|f\|_{H_p^n} := \|(I-\Delta)^{n/2} f\|_{L_p}, \qquad \|g\|_{H_p^n(\ell_2)} := \||(I-\Delta)^{n/2} g|_{\ell_2}\|_{L_p}$$

for $f \in H_p^n$ and $g \in H_p^n(\ell_2)$, respectively. Then we have, for $u \in H_p^n$ and $\phi \in C_0^\infty$

$$(u,\phi) = ((I-\Delta)^{n/2}u, (I-\Delta)^{-n/2}\phi),$$

where the right hand side is a usual Lebesgue integral. Using Hölder's inequality, one can easily obtain

$$|(u,\phi)|^2 \le C\|u\|_{H_p^n}^2,$$

where $C = \|(I-\Delta)^{-n/2}\phi\|_{L_{p/(p-1)}}^2$.

Using the spaces mentioned above, we define the following stochastic function spaces

$$\begin{aligned}
&\mathbb{H}_p^n := L_p(\Omega \times [0,T], \mathscr{F} \times \mathscr{B}([0,T]); H_p^n),\\
&\mathbb{H}_p^n(\ell_2) := L_p(\Omega \times [0,T], \mathscr{F} \times \mathscr{B}([0,T]); H_p^n(\ell_2)),\\
&\mathbb{H}_{p,H}^n := L_p(\Omega \times [0,T], \mathscr{F} \times \mathscr{B}([0,T]); L_{1/H}([0,T], H_p^n)),\\
&\mathbb{H}_{p,H}^n(\ell_2) := L_p(\Omega \times [0,T], \mathscr{F} \times \mathscr{B}([0,T]); L_{1/H}([0,T], H_p^n(\ell_2))),\\
&\mathbb{D}_\beta^{1,p}(|\mathcal{H}_{H_p^n}|) := \{f \in \mathbb{D}_\beta^{1,p}(\mathcal{H}_{H_p^n}) \mid f \in |\mathcal{H}_{H_p^n}|,\\
&\qquad\qquad D^\beta f \in |\mathcal{H}| \otimes |\mathcal{H}_{H_p^n}|, \|f\|_{\mathbb{D}_\beta^{1,p}(|\mathcal{H}_{H_p^n}|)} < \infty\},\\
&\mathbb{L}_{H,\beta}^{1,p}(H_p^n) := \{g \in \mathbb{D}_\beta^{1,p}(|\mathcal{H}_{H_p^n}|) \mid \|g\|_{\mathbb{L}_{H,\beta}^{1,p}(H_p^n)} < \infty\},
\end{aligned}$$

and the norms are given by

$$\begin{aligned}
\|f\|_{\mathbb{D}_\beta^{1,p}(|\mathcal{H}_{H_p^n}|)}^p :=& E\|f\|_{|\mathcal{H}_{H_p^n}|}^p + E\|D^\beta f\|_{|\mathcal{H}|\otimes|\mathcal{H}_{H_p^n}|}^p,\\
\|g\|_{\mathbb{L}_{H,\beta}^{1,p}(H_p^n)}^p :=& E\int_0^T \|g(s,\cdot)\|_{H_p^n}^p ds + E\int_0^T \left(\int_0^T \|D_t^\beta g(s,\cdot)\|_{H_p^n}^{1/H} dt\right)^{pH} ds\\
=& \|g\|_{\mathbb{H}_p^n}^p + \|D^\beta g\|_{\mathbb{H}_{p,H}^n}^p
\end{aligned}$$

for $f \in \mathbb{D}_\beta^{1,p}(|\mathcal{H}_{H_p^n}|)$ and $g \in \mathbb{L}_{H,\beta}^{1,p}(H_p^n)$. Note that C_0^∞ is dense in H_p^n. Then we introduce the set $\mathcal{S}_\beta(\mathcal{E}_{C_0^\infty})$ of smooth elementary processes of the form

$$g(t,\cdot) = \sum_{i=1}^m F_i 1_{(t_{i-1},t_i]}(t)\phi(\cdot), \quad t \in [0,T],$$

where $F_i \in \mathcal{S}_\beta$, $0 \le t_0 < t_1 < \ldots < t_m \le T$ and $\phi \in C_0^\infty$. The set $\mathcal{S}_\beta(\mathcal{E}_{C_0^\infty})$ is dense in $\mathbb{D}_\beta^{1,p}(|\mathcal{H}_{H_p^n}|)$ with respect to the norm $\|\cdot\|_{\mathbb{D}_\beta^{1,p}(|\mathcal{H}_{H_p^n}|)}$. We let $\tilde{\mathbb{L}}_{H,\beta}^{1,p}(H_p^n)$ be the completion of $\mathcal{S}_\beta(\mathcal{E}_{C_0^\infty})$ with respect to the norm $\|\cdot\|_{\mathbb{L}_{H,\beta}^{1,p}(H_p^n)}$. It is easy to know that $\tilde{\mathbb{L}}_{H,\beta}^{1,p}(H_p^n) \subset \mathbb{L}_{H,\beta}^{1,p}(H_p^n)$.

Let $\beta^k = \{\beta_t^k, t \in [0,T]\}, k \ge 1$ be a sequence of independent fractional Brownian motions with the same Hurst index $H > 1/2$, defined on the same probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$. Then we define

$$\begin{aligned}
\mathbb{L}_H^{1,p}(H_p^n, \ell_2) := \{g = (g^1, g^2, \ldots) \mid\ & g^k \in \mathbb{D}_{\beta^k}^{1,p}(|\mathcal{H}_{H_p^n}|)\\
&\text{for each } k \text{ and } \|g\|_{\mathbb{L}_H^{1,p}(H_p^n,\ell_2)} < \infty\}
\end{aligned}$$

with the norm $\|\cdot\|^p_{\mathbb{L}^{1,p}_H(H^n_p,\ell_2)}$ given by

$$\|g\|^p_{\mathbb{L}^{1,p}_H(H^n_p,\ell_2)} := E\int_0^T |g(s,\cdot)|^p_{H^n_p(\ell_2)}ds + E\int_0^T \left(\int_0^T |D_t g(s,\cdot)|^{1/H}_{H^n_p(\ell_2)}dt\right)^{pH} ds$$
$$= \|g\|^p_{\mathbb{H}^n_p(\ell_2)} + \|Dg\|^p_{\mathbb{H}^n_{p,H}(\ell_2)},$$

where $Dg(\cdot,\cdot) := (D^{\beta^k} g^k(\cdot,\cdot))_k$. At last, we let $\tilde{\mathbb{L}}^{1,p}_H(H^n_p,\ell_2)$ be the set of all elements $g \in \mathbb{L}^{1,p}_H(H^n_p,\ell_2)$ for which there exists a sequence $(g_j)_j \subset \mathbb{L}^{1,p}_H(H^n_p,\ell_2)$ such that $\|g_j - g\|_{\mathbb{L}^{1,p}_H(H^n_p,\ell_2)} \to 0$ as $j\to\infty$, $g^k_j = 0$ for $k > C_j$ and $g^k_j \in \mathcal{S}_\beta(\mathcal{E}_{C_0^\infty})$ for $k < C_j$.

Remark 2.1. By Proposition 4.3 and Lemma 4.7 in Balan [1], it is easy to check that for any $n\in\mathbb{R}, \phi\in C_0^\infty$ and $g\in\tilde{\mathbb{L}}^{1,p}_H(H^n_p,\ell_2)$, the series of Skorohod integrals

$$\sum_{k=1}^\infty \int_0^t (g^k(s,\cdot),\phi)\delta\beta^k_s$$

converges in probability uniformly in $t\in[0,T]$.

2.3. Fractional Laplace

Fractional Laplacian operator has been studied by many authors (see, for example Caffarelli-Silvestre [5] and Stein [30]). As we know, the infinitesimal generator of a symmetric α-stable process X^α taking values in $\mathbb{R}^d$ with $\alpha\in(0,2)$ is the fractional Laplacian $-(-\Delta)^{\alpha/2}$. The connection between X^α and $-(-\Delta)^{\alpha/2}$ can be seen as follows. The fundamental solution of

$$\frac{\partial u}{\partial t}(t,x) = -(-\Delta)^{\alpha/2}u(t,x) \tag{2.2}$$

is the transition density of X^α_t. Following the identity

$$Ee^{i\theta X^\alpha_t} = e^{-t|\theta|^\alpha}$$

and Fourier inversion formula, we can give the fundamental solution $G_\alpha(t,x)$ of (2.2)

$$G_\alpha(t,x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\theta x}e^{-t|\theta|^\alpha}d\theta = \mathcal{F}^{-1}(e^{-t|\theta|^\alpha})(x),$$

where $\mathcal{F}(f)(x) = \hat{f}(x) := \int_{\mathbb{R}^d} e^{-i\theta x}f(\theta)d\theta$ is the Fourier transform of f and $\mathcal{F}^{-1}(f)(x) = \check{f}(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\theta x}f(\theta)d\theta$ is the inverse Fourier transform of f. For suitable f, define

$$S^\alpha_t f(x) = (G_\alpha(t,\cdot)*f)(x) := \int_{\mathbb{R}^d} G_\alpha(t,x-y)f(y)dy,$$

where the symbol "$*$" denotes the convolution operation and

$$\partial^\beta_x f(x) := (-\Delta)^{\beta/2}f(x) := \mathcal{F}^{-1}(|\theta|^\beta\mathcal{F}(f)(\theta))(x).$$

For $\beta > 0$, one can obtain by Fourier transform

$$\begin{aligned}\partial_x^\beta S_t^\alpha f(x) &= \mathcal{F}^{-1}(|\theta|^\beta e^{-t|\theta|^\alpha}\mathcal{F}(f)(\theta))(x)\\ &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\theta x}|\theta|^\beta e^{-t|\theta|^\alpha}d\theta * f(x)\\ &= t^{-\frac{\beta+d}{\alpha}}\int_{\mathbb{R}^d} e^{i\theta(t^{-1/\alpha}x)}|\theta|^\beta e^{-|\theta|^\alpha}d\theta * f(x)\\ &= t^{-\frac{\beta+d}{\alpha}}(\partial_x^\beta G_\alpha(1, x/t^{1/\alpha})) * f(x).\end{aligned}$$

Define

$$\Psi_t f(x) := t^{-d/\alpha}\phi(\cdot/t^{1/\alpha}) * f(\cdot)(x),$$

where $\phi(x) = \partial_x^{\alpha/2}G_\alpha(1, x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\theta x}|\theta|^{\alpha/2}e^{-|\theta|^\alpha}d\theta$. The next inequality can be found in I. Kim and K. Kim [15] and Mikulevicius-Pragarauskas [25].

$$\int_{\mathbb{R}^d}\int_0^t\left(\int_0^s |\partial_x^{\alpha/2}S_{t-s}^\alpha g(s,\cdot)(x)|_{\ell_2}^2 dr\right)^{p/2} dsdx \le C\int_{\mathbb{R}^d}\int_0^t |g(s,x)|_{\ell_2}^p dsdx \tag{2.3}$$

with $p \ge 2$.

3. a version of Littlewood-Paley inequality

In this section, we shall prove the following theorem which will be used in Section 4.

Theorem 3.1. *Let $p \ge 2$, $\frac{1}{2} < H < 1$ and let K be a Hilbert space with norm $|\cdot|_K$. Then, the inequality*

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_a^b\left(\int_a^t\left(\int_\mu^\nu |\partial_x^{\alpha/2}S_{t-s}^\alpha f(s,x,r)|_K^{1/H}dr\right)^{2H}ds\right)^{p/2}dtdx\\ \le& C\int_a^b\left(\int_\mu^\nu\left(\int_{\mathbb{R}^d}|f(t,x,r)|_K^p dx\right)^{1/(pH)}dr\right)^{pH}dt\end{aligned}$$

holds for all $f \in C_0^\infty((a,b)\times\mathbb{R}^d, L_{1/H}((\mu,\nu),K))$ with $-\infty \le a < b \le \infty$ and $-\infty \le \mu < \nu \le \infty$.

This result can viewed as a generalization of Littlewood-Paley's inequality, precisely say

$$\int_{\mathbb{R}^d}\int_{-\infty}^\infty\left(|\nabla T_{t-s}g(s,\cdot)|_K^2 ds\right)^{p/2}dtdx \le C_{p,d}\int_{\mathbb{R}^d}\int_{-\infty}^\infty |g(t,x)|_K^p dtdx,$$

where $g \in L^p((-\infty,\infty)\times\mathbb{R}^d, K)$, ∇ is the usual gradient operator, T_t is the Gaussian heat semigroup. In order to prove the theorem above, we introduce some concepts and establish some useful lemmas.

Recall that the maximal function of a real-valued function g with the domain $\mathbb{R}^d$ is defined by

$$\mathbb{M}_x g(x) := \sup_{r>0}\frac{1}{|B_r(x)|}\int_{B_r(x)}|g(y)|dy,$$

where $B_r(x) = \{y; |x-y| < r\}$ with $B_r = B_r(0)$, and $|B_r(x)|$ denotes the Lebesgue measure of $B_r(x)$. Similarly, for a function $g(t) : \mathbb{R} \to \mathbb{R}$, we define the maximal function of g as follows

$$\mathbb{M}_t g(t) := \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} |g(t+s)| ds.$$

For a function $g(t,x)$ of two variables, we set

$$\mathbb{M}_x g(t,x) := \mathbb{M}_x(g(t,\cdot))(x), \qquad \mathbb{M}_t g(t,x) := \mathbb{M}_t(g(\cdot,x))(t).$$

As in Krylov [19,21], we can introduce the filtration $\mathbb{Q}_n = \{Q_n(i_0, i_1, \dots, i_d);$ $i_0, i_1, \dots, i_d \in \mathbb{Z}\}, n \in \mathbb{Z}$ of partitions of $\mathbb{R}^{d+1}$. We denote by $Q_n(t,x)$ the unique $Q \in \mathbb{Q}_n$ containing (t,x). For a measurable function $g(t,x)$, define the sharp function

$$g^{\#}(t,x) := \sup_{n\in\mathbb{Z}} \frac{1}{|Q_n(t,x)|} \int_{Q_n(t,x)} |g(s,y) - g_{|n}(t,x)| dsdy,$$

where $g_{|n}(t,x) = \frac{1}{|Q_n(t,x)|} \int_{Q_n(t,x)} g(s,y) dsdy$. Then by the Fefferman-Stein theorem, we have

$$\|g\|_{L_p} \le C \|g^{\#}\|_{L_p}$$

for any $0 < p < \infty$ and $g \in L_p(\mathbb{R}^{d+1})$.

Let $Q_0 := [-2^\alpha, 0] \times [-1,1]^d$ and

$$\mathcal{G}_\alpha f(t,x) = \left[\int_{-\infty}^{t} \left(\int_\mu^\nu |\Psi_{t-s} f(s,x,r)|_K^{1/H} dr \right)^{2H} \frac{1}{t-s} ds \right]^{1/2},$$

where $f \in C_0^\infty((a,b) \times \mathbb{R}^d, L_{1/H}((\mu,\nu),K))$.

Lemma 3.1. *Assume that $f(t,x,r) = 0$ outside of $[-10,10] \times B_{3d}$. Then for any $(t,x) \in Q_0$*

$$\int_{Q_0} |\mathcal{G}_\alpha f(s,y)|^2 dsdy < C \mathbb{M}_t \left(\int_\mu^\nu (\mathbb{M}_x |f(t,x,r)|_K^2)^{1/(2H)} dr \right)^{2H}. \tag{3.1}$$

Proof. Using Minkowski's inequality, Fourier transform and the fact

$$\int_0^\infty |\hat{\phi}(\xi t^{1/\alpha})|^2 \frac{dt}{t} \le C,$$

we have

$$\begin{aligned}
&\int_{Q_0} |\mathcal{G}_\alpha f(s,y)|^2 dsdy \\
&\le \int_{-\infty}^0 \int_{\mathbb{R}^d} \int_{-\infty}^s \left(\int_\mu^\nu |\Psi_{s-s'} f(s',y,r)|_K^{1/H} dr \right)^{2H} \frac{1}{s-s'} ds' dyds \\
&\le \int_{-\infty}^0 \left[\int_\mu^\nu \left(\int_{\mathbb{R}^d} \int_{s'}^0 |\Psi_{s-s'} f(s',y,r)|_K^2 \frac{1}{s-s'} dsdy \right)^{1/(2H)} dr \right]^{2H} ds' \\
&= \int_{-\infty}^0 \left[\int_\mu^\nu \left((2\pi)^d \int_{\mathbb{R}^d} \int_0^{-s'} |\hat{\phi}(\xi s^{1/\alpha})|^2 |\hat{f}(s',\xi,r)|_K^2 \frac{1}{s} dsd\xi \right)^{1/(2H)} dr \right]^{2H} ds'
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_{-\infty}^{0}\left[\int_{\mu}^{\nu}\left(\int_{\mathbb{R}^d}|f(s,y,r)|_K^2dy\right)^{1/(2H)}dr\right]^{2H}ds\\
&\le C\int_{-10}^{0}\left[\int_{\mu}^{\nu}\left(\mathbb{M}_x|f(s,x,r)|_K^2\right)^{1/(2H)}dr\right]^{2H}ds\\
&\le C\mathbb{M}_t\left(\int_{\mu}^{\nu}\left(\mathbb{M}_x|f(t,x,r)|_K^2\right)^{1/(2H)}dr\right)^{2H}.
\end{aligned}
$$

This completes the proof. □

Lemma 3.2. *Assume that $f(t,x,r)=0$ for $t\notin[-10,10]$. Then (3.1) holds for any $(t,x)\in Q_0$.*

Proof. We take $\zeta\in C_0^\infty(\mathbb{R}^d)$ such that $\zeta=1$ in B_{2d}, $\zeta=0$ outside of B_{3d}. Set $\mathcal{A}=\zeta f$ and $\mathcal{B}=(1-\zeta)f$. Combining Lemma 3.1 and the fact that $\mathcal{G}_\alpha f\le\mathcal{G}_\alpha\mathcal{A}+\mathcal{G}_\alpha\mathcal{B}$, we just need to concentrate on $\mathcal{G}_\alpha\mathcal{B}$. Then one can assume that $f(\cdot,x,\cdot)=0$ for $x\in B_{2d}$ in the following proof. Also notice that if $(s,y)\in Q_0$ and $|z|\le\rho$ with $\rho>1$, then

$$
|x-y|\le 2d,\qquad B_\rho(y)\subset B_{2d+\rho}(x)\subset B_{(2d+1)\rho}(x),
$$

whereas if $|z|\le 1$, then $|y-z|\le 2d$ and $f(\cdot,y-z,\cdot)=0$. Then by using (16.16) in Krylov [19], we obtain, for $0>s>s'>-10$ and $(s,y)\in Q_0$

$$
\begin{aligned}
\left(\int_{\mu}^{\nu}|\Psi_{s-s'}f(s',y,r)|_K^{1/H}dr\right)^{H}\le&(s-s')^{-(d+1)/\alpha}\left(\int_{\mu}^{\nu}\left(\int_{1}^{\infty}|\bar{\phi}'_{\alpha/2}(\rho/(s-s')^{1/\alpha})|\right.\right.\\
&\left.\left.\times\left(\int_{|z|\le\rho}|f(s',y-z,r)|_Kdz\right)d\rho\right)^{1/H}dr\right)^{H}\\
\le&(s-s')^{-(d+1)/\alpha}\left(\int_{\mu}^{\nu}\left(\int_{1}^{\infty}|\bar{\phi}'_{\alpha/2}(\rho/(s-s')^{1/\alpha})|\right.\right.\\
&\left.\left.\times\left(\int_{B_{(2d+1)\rho}(x)}|f(s',z,r)|_Kdz\right)d\rho\right)^{1/H}dr\right)^{H}\\
\le&C(s-s')^{-(d+1)/\alpha}\int_{1}^{\infty}|\bar{\phi}'_{\alpha/2}(\rho/(s-s')^{1/\alpha})|\rho^d d\rho\\
&\times\left(\int_{\mu}^{\nu}\left(\mathbb{M}_x|f(s',x,r)|_K\right)^{1/H}dr\right)^{H},
\end{aligned}
$$

where $\bar{\phi}_{\alpha/2}(\cdot)$ is defined as follows (see Lemma 2.2 of I. Kim and K. Kim [15])

$$
\bar{\phi}_{\alpha/2}(\rho)=\begin{cases}\frac{C}{\rho^{d+\alpha/2}}, & \rho\ge(10)^{-1/\alpha},\\ Ce^{-(d+\alpha/2)(10^{1/\alpha}\rho-1)}, & \rho<(10)^{-1/\alpha}.\end{cases}
$$

Thus, an elementary calculation shows

$$
\begin{aligned}
|\mathcal{G}_\alpha f(s,y)|^2 &= \int_{-\infty}^{s} \left(\int_\mu^\nu |\Psi_{s-s'} f(s',y,r)|_K^{1/H} dr \right)^{2H} \frac{1}{s-s'} ds' \\
&\leq C \int_{-10}^{0} \left(\int_\mu^\nu \left(\mathbb{M}_x |f(s',x,r)|_K \right)^{1/H} dr \right)^{2H} ds' \\
&\leq C \int_{-10}^{0} \left(\int_\mu^\nu \left(\mathbb{M}_x |f(s',x,r)|_K^2 \right)^{1/(2H)} dr \right)^{2H} ds' \\
&\leq C \mathbb{M}_t \left(\int_\mu^\nu \left(\mathbb{M}_x |f(t,x,r)|_K^2 \right)^{1/(2H)} dr \right)^{2H},
\end{aligned}
$$

where the second last inequality above follows from Hölder's inequality. This completes the proof. □

Lemma 3.3. *Assume that $f(t,x,r)=0$ for $t \geq -8$. Then for any $(t,x) \in Q_0$*

$$
\int_{Q_0} |\mathcal{G}_\alpha f(s,y) - \mathcal{G}_\alpha f(t,x)|^2 ds dy < C \mathbb{M}_t \left(\int_\mu^\nu (\mathbb{M}_x |f(t,x,r)|_K^2)^{1/(2H)} dr \right)^{2H}.
$$

Proof. The argument is similar to the proof of Lemma 4.4 of I. Kim and K. Kim [15] and we omit it. □

Lemma 3.4. *Assume that $f \in C_0^\infty(\mathbb{R}^{d+1}, L_{1/H}((\mu,\nu),K))$. Then for any $(t,x) \in \mathbb{R}^{d+1}$*

$$
(\mathcal{G}_\alpha f)^\#(t,x) < C \left(\mathbb{M}_t \left(\int_\mu^\nu (\mathbb{M}_x |f(t,x,r)|_K^2)^{1/(2H)} dr \right)^{2H} \right)^{1/2}.
$$

Proof. By Hölder's inequality, we have

$$
(\mathcal{G}_\alpha f)^\#(t,x) \leq \sup_{n \in \mathbb{Z}} \left(\frac{1}{|Q_n(t,x)|} \int_{Q_n(t,x)} |\mathcal{G}_\alpha f(s,y) - \mathcal{G}_\alpha f_{|n}(t,x)|^2 ds dy \right)^{1/2}.
$$

Therefore it suffices to prove that

$$
\begin{aligned}
&\frac{1}{|Q_n(t,x)|} \int_{Q_n(t,x)} |\mathcal{G}_\alpha f(s,y) - \mathcal{G}_\alpha f_{|n}(t,x)|^2 ds dy \\
\leq& C \mathbb{M}_t \left(\int_\mu^\nu (\mathbb{M}_x |f(t,x,r)|_K^2)^{1/(2H)} dr \right)^{2H}
\end{aligned}
$$

for each integer n. Based on Lemma 3.2, 3.3, the following argument is similar to the proof of relation (5.2) given in I. Kim and K. Kim [15]. □

Proof of Theorem 3.1. It is enough to assume that $a=-\infty$, $b=\infty$ and to prove

$$
\int_{\mathbb{R}^d} \int_{-\infty}^{\infty} |\mathcal{G}_\alpha f(t,x)|^p dt dx \leq C \int_{-\infty}^{\infty} \left[\int_\mu^\nu \left(\int_{\mathbb{R}^d} |f(t,x,r)|_K^p dx \right)^{1/(pH)} dr \right]^{pH} dt
$$

for any $p \in [2,\infty)$.

We first consider the case $p=2$. As the computation in the proof of Lemma 3.1, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{-\infty}^{\infty}|\mathcal{G}_\alpha f(t,x)|^2 dtdx\\ \le&\int_{-\infty}^{\infty}\int_{\mathbb{R}^d}\int_{-\infty}^{t}\left(\int_\mu^\nu|\Psi_{t-s}f(s,y,r)|_K^{1/H}dr\right)^{2H}\frac{dsdydt}{t-s}\\ \le&\int_{-\infty}^{\infty}\left(\int_\mu^\nu\left((2\pi)^d\int_{\mathbb{R}^d}\int_s^{\infty}|\hat{\phi}(\xi t^{1/\alpha})|^2|\hat{f}(s,\xi,r)|_K^2\frac{1}{t}dtd\xi\right)^{1/(2H)}dr\right)^{2H}ds\\ \le&C\int_{-\infty}^{\infty}\left(\int_\mu^\nu\left(\int_{\mathbb{R}^d}|f(t,y,r)|_K^2dy\right)^{1/(2H)}dr\right)^{2H}dt.\end{aligned}$$

We now assume that $p>2$. Due to the Fefferman-Stein theorem, Lemma 3.4 and Hardy-Littlewood maximal theorem, we get

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{-\infty}^{\infty}|\mathcal{G}_\alpha f(t,x)|^p dtdx\le C\int_{\mathbb{R}^d}\int_{-\infty}^{\infty}|(\mathcal{G}_\alpha f)^{\#}(t,x)|^p dtdx\\ \le&C\int_{\mathbb{R}^d}\int_{-\infty}^{\infty}\left(\mathbb{M}_t\left(\int_\mu^\nu(\mathbb{M}_x|f(t,x,r)|_K^2)^{1/(2H)}dr\right)^{2H}\right)^{p/2}dtdx\\ \le&C\int_{\mathbb{R}^d}\int_{-\infty}^{\infty}\left(\int_\mu^\nu(\mathbb{M}_x|f(t,x,r)|_K^2)^{1/(2H)}dr\right)^{pH}dtdx\\ \le&C\int_{-\infty}^{\infty}\left(\int_\mu^\nu\left(\int_{\mathbb{R}^d}(\mathbb{M}_x|f(t,x,r)|_K^2)^{p/2}dx\right)^{1/(pH)}dr\right)^{pH}dt\\ \le&C\int_{-\infty}^{\infty}\left(\int_\mu^\nu\left(\int_{\mathbb{R}^d}|f(t,x,r)|_K^p dx\right)^{1/(pH)}dr\right)^{pH}dt,\end{aligned}$$

where the second last inequality above follows from Minkowski's inequality for integrals, and the Littlewood-Paley inequality follows. □

4. The main results

In this section, we mainly state the L_p-theory of fractional heat equation (1.2).

Definition 4.1. For a $\mathcal{D}$-valued function $u\in\mathbb{H}_p^{n+\alpha}$, we write $u\in\mathcal{H}_{p,H}^{n+\alpha}$ if $u(0,\cdot)\in L_p(\Omega,\mathscr{F}_0,H_p^{n+\alpha-\alpha/p})$ and

$$du=fdt+\sum_{k=0}^{\infty}g^k\delta\beta_t^k,\quad t\in[0,T]$$

in the weak sense for some $f\in\mathbb{H}_p^n$ and $g\in\tilde{\mathbb{L}}_H^{1,p}(H_p^{n+\alpha/2},\ell_2)$. That is, for any $\phi\in C_0^\infty$,

$$(u(t,\cdot),\phi)=(u(0,\cdot),\phi)+\int_0^t(f(s,\cdot),\phi)ds+\sum_{k=1}^{\infty}\int_0^t(g^k(s,\cdot),\phi)\delta\beta_t^k \tag{4.1}$$

holds for any $t \in [0,T]$ *a.s.* In this case, we write

$$\mathbf{D}u := f, \qquad \mathbf{S}^k u := g^k, \qquad \mathbf{S}u = (\mathbf{S}^1 u, \mathbf{S}^2 u, \dots)$$

and define the norm

$$\|u\|_{\mathcal{H}^{n+\alpha}_{p,H}} := \|u\|_{\mathbb{H}^{n+\alpha}_p} + \|\mathbf{D}u\|_{\mathbb{H}^n_p} + \|\mathbf{S}u\|_{\tilde{\mathbb{L}}^{1,p}_H(H^{n+\alpha/2}_p,\ell_2)} + \left(E\|u(0)\|^p_{H^{n+\alpha-\alpha/p}_p}\right)^{1/p}.$$

Theorem 4.1. *The space $\mathcal{H}^{n+\alpha}_{p,H}$ is a Banach space with the norm $\|\cdot\|_{\mathcal{H}^{n+\alpha}_{p,H}}$, and for $u \in \mathcal{H}^{n+\alpha}_{p,H}$,*

$$E\sup_{t\le T}\|u(t,\cdot)\|^p_{H^n_p} \le C_{\alpha,p,d,T}\left(\|\mathbf{D}u\|_{\mathbb{H}^n_p} + \|\mathbf{S}u\|_{\tilde{\mathbb{L}}^{1,p}_H(H^{n+\alpha/2}_p,\ell_2)} + E\|u(0)\|^p_{H^{n+\alpha-\alpha/p}_p}\right)$$

and

$$\|u\|_{\mathbb{H}^n_p} \le C_{\alpha,p,d,T}\|u\|_{\mathcal{H}^{n+\alpha}_{p,H}}.$$

Proof. The proof is almost identical to the proof of Theorem 5.5 in Balan [1], based on the method of proving Theorem 3.7 in Krylov [20], and we omit it. □

Remark 4.1. We know that for any $n,m \in \mathbb{R}$, the operator $(I-\Delta)^{m/2}$ maps isometrically H^n_p onto H^{n-m}_p. Indeed,

$$\|(I-\Delta)^{m/2}u\|_{H^{n-m}_p} = \|(I-\Delta)^{m/2}(I-\Delta)^{(n-m)/2}u\|_{L_p} = \|u\|_{H^n_p}.$$

On the other hand, Proposition 5.4 in Balan [1] shows that the operator $(I-\Delta)^{m/2}$: $\tilde{\mathbb{L}}^{1,p}_H(H^n_p,\ell_2) \to \tilde{\mathbb{L}}^{1,p}_H(H^{n-m}_p,\ell_2)$ is an isometry. Replacing ϕ with $(I-\Delta)^{m/2}\phi$ in (4.1), then one can easily obtain that the operator $(I-\Delta)^{m/2}$ maps isometrically $\mathcal{H}^n_{p,H}$ onto $\mathcal{H}^{n-m}_{p,H}$.

The following results are also known (see, for example K. Kim and P. Kim [16]).

Theorem 4.2. *(i) For any deterministic functions $f = f(t,x)$ and $u_0 = u_0(x)$ satisfying*

$$\int_0^T \|f(t,\cdot)\|^p_{H^n_p}dt < \infty, \quad \|u_0\|_{H^{n+\alpha-\alpha/p}_p} < \infty,$$

the equation

$$du(t,x) = -(-\Delta)^{\alpha/2}u(t,x)dt + f(t,x)dt, \quad u(0,x) = u_0$$

has a unique solution u with $\int_0^T \|u(t,\cdot)\|^p_{H^{n+\alpha}_p}dt < \infty$ and

$$\int_0^T \|u(t,\cdot)\|^p_{H^{n+\alpha}_p}dt < C_{p,d,T}\left(\int_0^T \|f(t,\cdot)\|^p_{H^n_p}dt + \|u_0\|^p_{H^{n+\alpha-\alpha/p}_p}\right).$$

(ii) For any $f \in \mathbb{H}^n_p$ and $u_0 \in L_p(\Omega,\mathscr{F}_0,H^{n+\alpha-\alpha/p}_p)$, the equation

$$du(t,x) = -(-\Delta)^{\alpha/2}u(t,x)dt + f(t,x)dt, \quad u(0,x) = u_0$$

has a unique solution $u \in \mathbb{H}^{n+\alpha}_p$ with

$$\|u\|_{\mathcal{H}^{n+\alpha}_{p,H}} \le C_{p,d,T}\left(\|f\|_{\mathbb{H}^n_p} + \left(E\|u_0\|^p_{H^{n+\alpha-\alpha/p}_p}\right)^{1/p}\right).$$

The next theorem states the main result of this paper.

Theorem 4.3. *Let $p \geq 2$ and $n \in \mathbb{R}$. For any $f \in \mathbb{H}_p^n$, $g \in \tilde{\mathbb{L}}_H^{1,p}(H_p^{n+\alpha/2}, \ell_2)$ and $u_0 \in L_p(\Omega, \mathscr{F}_0, H_p^{n+\alpha-\alpha/p})$, the equation (1.2) admits a unique solution in $\mathcal{H}_{p,H}^{n+\alpha}$, in the weak sense, and for this solution*

$$\|u\|_{\mathcal{H}_{p,H}^{n+\alpha}} \leq C\left(\|f\|_{\mathbb{H}_p^n} + \|g\|_{\tilde{\mathbb{L}}_H^{1,p}(H_p^{n+\alpha/2},\ell_2)} + \left(E\|u_0\|_{H_p^{n+\alpha-\alpha/p}}^p\right)^{1/p}\right), \tag{4.2}$$

where C is a positive constant depending on p, d, T and H.

Proof. By Definition 4.1, the unique solution $u \in \mathcal{H}_{p,H}^{n+\alpha}$ is understood in the weak sense, that is, for any $\phi \in C_0^\infty$,

$$\begin{aligned}(u(t,\cdot),\phi) =&(u(0,\cdot),\phi) + \int_0^t ((u(s,\cdot), -(-\Delta)^{\alpha/2}\phi) + (f(s,\cdot),\phi))ds \\ &+ \sum_{k=1}^\infty \int_0^t (g^k(s,\cdot),\phi)\delta\beta_t^k\end{aligned}$$

holds for any $t \in [0,T]$ *a.s.* Due to Remark 4.1, it suffices to prove that the theorem holds for a particular $n = n_0$. So we let $n = -\alpha/2$ in the following proof and we split the proof in two steps.

Step I. We prove the theorem with assumptions $f = 0$ and $u_0 = 0$. Suppose that $g^k = 0$ for $k > \kappa_0$, and

$$g^k(t,\cdot) = \sum_{i=0}^{m_k} F_i^k 1_{(t_{i-1}^k, t_i^k]}(t) g_i^k(\cdot) \quad t \in [0,T], k \leq \kappa_0,$$

where κ_0 is a fixed positive integer, $F_i^k \in \mathcal{S}_{\beta^k}$, $0 \leq t_0^k < t_1^k < \ldots < t_m^k \leq T$ and $g_i^k(\cdot) \in C_0^\infty$. Let $v(t,x) := \sum_{k=1}^{\kappa_0} \int_0^t g^k(s,x)\delta\beta_s^k$ and

$$u(t,x) := v(t,x) - \int_0^t (-\Delta)^{\alpha/2} S_{t-s}^\alpha v(s,x)ds = v(t,x) - \int_0^t S_{t-s}^\alpha (-\Delta)^{\alpha/2} v(s,x)ds.$$

Then, Fourier transform implies that

$$d(u-v) = (-(-\Delta)^{\alpha/2}(u-v) - (-\Delta)^{\alpha/2}v)dt = -(-\Delta)^{\alpha/2}udt$$

and

$$du = -(-\Delta)^{\alpha/2}udt + dv = -(-\Delta)^{\alpha/2}udt + \sum_{k=1}^{\kappa_0} g^k(t,x)\delta\beta_t^k.$$

Using the stochastic Fubini theorem, we obtain

$$\begin{aligned}u(t,x) &= v(t,x) - \sum_{k=1}^{\kappa_0}\int_0^t \int_0^s (-\Delta)^{\alpha/2} S_{t-s}^\alpha g^k(r,x)\delta\beta_r^k ds \\ &= v(t,x) - \sum_{k=1}^{\kappa_0}\int_0^t \int_r^t (-\Delta)^{\alpha/2} S_{t-s}^\alpha g^k(r,x) ds\delta\beta_r^k \\ &= v(t,x) - \sum_{k=1}^{\kappa_0}\int_0^t (g^k(r,x) - S_{t-r}^\alpha g^k(r,x))\delta\beta_r^k \\ &= \sum_{k=1}^{\kappa_0}\int_0^t S_{t-s}^\alpha g^k(s,x)\delta\beta_s^k.\end{aligned}$$

Therefore we can write

$$\partial_x^{\alpha/2} u(t,x) = \sum_{k=1}^{\kappa_0} \int_0^t \partial_x^{\alpha/2} S_{t-s}^{\alpha} g^k(s,\cdot)(x) \delta\beta_s^k = \sum_{k=1}^{\infty} \int_0^t \partial_x^{\alpha/2} S_{t-s}^{\alpha} g^k(s,\cdot)(x) \delta\beta_s^k.$$

Combining this with Theorem 3.6 in Balan [1], we have

$$\begin{aligned}
&E|\partial_x^{\alpha/2} u(t,x)|^p \\
=&E\left|\sum_{k=1}^{\infty} \int_0^t \partial_x^{\alpha/2} S_{t-s}^{\alpha} g^k(s,\cdot)(x) \delta\beta_s^k\right|^p \\
\le& CE\left(\int_0^t \sum_{k=1}^{\infty} |\partial_x^{\alpha/2} S_{t-s}^{\alpha} g^k(s,\cdot)(x)|^2 ds\right)^{p/2} \\
&+ CE\left(\int_0^t \left(\int_0^T \left(\sum_{k=1}^{\infty} |D_r^{\beta^k}(\partial_x^{\alpha/2} S_{t-s}^{\alpha} g^k(s,\cdot)(x))|^2\right)^{1/(2H)} dr\right)^{2H} ds\right)^{p/2} \\
=&CE\left(\int_0^t |\partial_x^{\alpha/2} S_{t-s}^{\alpha} g(s,\cdot)(x)|_{\ell_2}^2 ds\right)^{p/2} \\
&+ CE\left(\int_0^t \left(\int_0^T |\partial_x^{\alpha/2} S_{t-s}^{\alpha}(D_r g(s,\cdot))(x)|_{\ell_2}^{1/H} dr\right)^{2H} ds\right)^{p/2},
\end{aligned}$$

where we have used the fact that

$$D_r^{\beta^k}(\partial_x^{\alpha/2} S_{t-s}^{\alpha} g^k(s,\cdot)(x)) = \partial_x^{\alpha/2} S_{t-s}^{\alpha}(D_r^{\beta^k} g^k(s,\cdot))(x).$$

According to Theorem 3.1 and the inequality (2.3), we obtain

$$\begin{aligned}
&E\int_0^T \|\partial_x^{\alpha/2} u(t,\cdot)\|_{L_p}^p dt \\
\le& CE\int_{\mathbb{R}^d} \int_0^T \left(\int_0^t |\partial_x^{\alpha/2} S_{t-s}^{\alpha} g(s,\cdot)(x)|_{\ell_2}^2 ds\right)^{p/2} dt dx \\
&+ C\int_{\mathbb{R}^d} \int_0^T \left(\int_0^t \left(\int_0^T |\partial_x^{\alpha/2} S_{t-s}^{\alpha}(D_r g(s,\cdot))(x)|_{\ell_2}^{1/H} dr\right)^{2H} ds\right)^{p/2} dt dx \\
\le& CE\int_{\mathbb{R}^d} \int_0^T |g(s,x)|_{\ell_2}^p ds dx \\
&+ CE\int_0^T \left(\int_0^T \left(\int_{\mathbb{R}^d} |D_r g(s,x)|_{\ell_2}^p dx\right)^{1/(pH)} dr\right)^{pH} ds \\
=& C\|g\|_{\mathbb{H}_p^0(\ell_2)}^p + C\|Dg\|_{\mathbb{H}_{p,H}^0(\ell_2)}^p = C\|g\|_{\mathbb{L}_H^{1,p}(H_p^0,\ell_2)}.
\end{aligned} \tag{4.3}$$

Similarly, we obtain

$$
\begin{aligned}
& E\int_0^T \|u(t,\cdot)\|_{L_p}^p dt \\
\le & CE\int_{\mathbb{R}^d}\int_0^T\left(\int_0^t |S_{t-s}^\alpha g(s,\cdot)(x)|_{\ell_2}^2 ds\right)^{p/2} dtdx \\
& + CE\int_{\mathbb{R}^d}\int_0^T\left(\int_0^t\left(\int_0^T |S_{t-s}^\alpha(D_r g(s,\cdot))(x)|_{\ell_2}^{1/H} dr\right)^{2H} ds\right)^{p/2} dtdx \\
\le & CE\int_{\mathbb{R}^d}\int_0^T\int_0^t |S_{t-s}^\alpha g(s,\cdot)(x)|_{\ell_2}^p dsdtdx \\
& + C\int_{\mathbb{R}^d}\int_0^T\int_0^t\left(\int_0^T |S_{t-s}^\alpha(D_r g(s,\cdot))(x)|_{\ell_2}^{1/H} dr\right)^{pH} dsdtdx \\
\le & CE\int_{\mathbb{R}^d}\int_0^T\int_0^t |g(s,x)|_{\ell_2}^p dsdtdx \\
& + CE\int_0^T\int_0^t\left(\int_0^T\left(\int_{\mathbb{R}^d} |D_r g(s,x)|_{\ell_2}^p dx\right)^{1/(pH)} dr\right)^{pH} dsdtdx \\
\le & C\|g\|_{\mathbb{H}_p^0(\ell_2)}^p + C\|Dg\|_{\mathbb{H}_{p,H}^0(\ell_2)}^p \\
= & C\|g\|_{\mathbb{L}_H^{1,p}(H_p^0,\ell_2)},
\end{aligned} \tag{4.4}
$$

where we have used Hölder's inequality and Minkowski's inequality, and the fact that the operator S_t^α is bounded for any $t\in[0,T]$. It follows from the definition and the well-known inequality

$$\|u\|_{H_p^n} \le C\|u\|_{H_p^{-\epsilon}} + C\|\partial_x^{n/2}u\|_{L_p}$$

with $\epsilon \ge 0$ that

$$
\begin{aligned}
\|u\|_{\mathcal{H}_{p,H}^{\alpha/2}}^p &\le C\left(\|u\|_{\mathbb{H}_p^{\alpha/2}}^p + \|-(-\Delta)^{\alpha/2}u\|_{\mathbb{H}_p^{-\alpha/2}}^p + \|g\|_{\mathbb{L}_H^{1,p}(H_p^0,\ell_2)}^p\right) \\
&\le C\left(\|u\|_{\mathbb{H}_p^0}^p + \|\partial_x^{\alpha/2}u\|_{\mathbb{H}_p^0}^p + \|-(-\Delta)^{\alpha/2}u\|_{\mathbb{H}_p^{-\alpha/2}}^p + \|g\|_{\mathbb{L}_H^{1,p}(H_p^0,\ell_2)}^p\right),
\end{aligned}
$$

which gives

$$
\begin{aligned}
\|u\|_{\mathcal{H}_{p,H}^{\alpha/2}}^p &\le C\left(\|u\|_{\mathbb{H}_p^0}^p + \|\partial_x^{\alpha/2}u\|_{\mathbb{H}_p^0}^p + \|g\|_{\mathbb{L}_H^{1,p}(H_p^0,\ell_2)}^p\right) \\
&\le C\|g\|_{\mathbb{L}_H^{1,p}(H_p^0,\ell_2)}^p
\end{aligned}
$$

by (4.3), (4.4) and $\|-(-\Delta)^{\alpha/2}u\|_{H_p^{-\alpha/2}} \le C\|\partial_x^{\alpha/2}u\|_{H_p^0}^p$. This leads to (4.2) and $u\in\mathcal{H}_{p,H}^{\alpha/2}$, and the uniqueness of the solution follows from Theorem 4.2.

By using standard approximation argument, we can drop the additional assumption on g and the conclusions above still hold for any $g\in\tilde{\mathbb{L}}_H^{1,p}(H_p^0,\ell_2)$, due to the completeness of the spaces $H_p^{-\alpha/2}, \mathbb{L}_H^{1,p}(H_p^0,\ell_2)$ and $\mathcal{H}_{p,H}^{\alpha/2}$.

Step II. We prove the theorem without assumptions on f and u_0 in Step I. Here we still assume $n = -\alpha/2$. Owing to Theorem 4.2, it suffices to prove that there exists a solution u and it satisfies (4.2). Meanwhile, Theorem 4.2 gives that equation

$$dv(t,x) = (-(-\Delta)^{\alpha/2}v(t,x) + f(t,x))dt, \quad v(0,x) = u_0$$

has a solution $v \in \mathcal{H}_{p,H}^{\alpha/2}$ and

$$\|v\|_{\mathcal{H}_{p,H}^{\alpha/2}} \le C\left(\|f\|_{\mathbb{H}_p^{-\alpha/2}} + (E\|u_0\|^p_{H_p^{\alpha/2-\alpha/p}})^{1/p}\right).$$

On the other hand, the assertion proved in Step I implies that the equation

$$dw(t,x) = -(-\Delta)^{\alpha/2}w(t,x)dt + \sum_{k=1}^{\infty} g^k(t,x)\delta\beta_t^k, \quad w(0,x) = 0$$

has a solution w such that $\|w\|_{\mathcal{H}_{p,H}^{\alpha/2}} \le C\|g\|^p_{\mathbb{L}_H^{1,p}(H_p^0,\ell_2)}$. Thus, u is the solution of equation (1.2), provided we set $u = v + w$, and moreover the solution u admits the following estimate:

$$\|u\|_{\mathcal{H}_{p,H}^{\alpha/2}} \le C\left(\|f\|_{\mathbb{H}_p^{-\alpha/2}} + \|g\|^p_{\mathbb{L}_H^{1,p}(H_p^0,\ell_2)} + (E\|u_0\|^p_{H_p^{\alpha/2-\alpha/p}})^{1/p}\right).$$

This completes the proof. $\square$

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