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# ANALYSIS OF A STOCHASTIC TWO-PREDATORS ONE-PREY SYSTEM WITH MODIFIED LESLIE-GOWER AND HOLLING-TYPE II SCHEMES*

Yao Xu[1], Meng Liu[1,2,†] and Yun Yang[1]

**Abstract** In this paper, we consider a stochastic two-predators one-prey model with modified Leslie-Gower and Holling-type II schemes. Analytically, we completely classify the parameter space into eight categories containing eleven cases. In each case, we show that every population is either stable in time average or extinct, depending on the parameters of the model. Finally, we work out some simulation figures to illustrate the theoretical results.

**Keywords** Two-predators one-prey model, stochastic noise, stability in time average, extinction, Itô's formula.

**MSC(2010)** 60H10, 60H30, 92D25.

## 1. Introduction

The famous predator-prey model with modified Leslie-Gower and Holling-type II schemes can be denoted as follows (Aziz-Alaoui and Daher Okiye [1]):

$$\frac{dx(t)}{dt} = x(t)\left(r_1 - ax(t) - \frac{cy(t)}{h + x(t)}\right), \quad \frac{dy(t)}{dt} = y(t)\left(r_2 - \frac{fy(t)}{h + x(t)}\right), \tag{1.1}$$

where $x(t)$ and $y(t)$ represent the population sizes of the prey and the predator respectively. $r_1,\ r_2, a, c, f$ and $h$ are positive constants. $r_1$ and $r_2$ is the growth rates of the prey and the the predator respectively, $a$ represents the competitive strength among individuals of the prey, $c$ stands for the per capita reduction rate, $h$ describes the protection of the environment, the meaning of $f$ is similar to $c$. Recently, many authors have paid attention to model (1.1) and its generalized forms, see e.g. [2, 6–8, 11, 12, 16, 22–28]. Aziz-Alaoui and Okiye [1] considered the boundedness and stability of the positive equilibrium of model (1.1). Nindjin et al. [23] introduced time delay into Eq. (1.1) and studied the stability of the positive equilibrium of

†the corresponding author. Email address:liumeng0557@sina.com(M. Liu)
1School of Mathematical Science, Huaiyin Normal University, Huaian 223300, China
2School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China
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their model; Eq. (1.1) with impulse was investigated by Guo and Song [8], Song and Li [24] and Nie et al. [22]; Ji et al. [11,12] considered model (1.1) with white noise and studied the persistence, extinction, and stationary distribution to the corresponding system. System (1.1) with reaction-diffusion was explored in [2, 7, 28].

The above studies have focused on two-species models. However, in the nature world it is a common phenomenon that several predators compete for a prey. At the same time, the growth of population is the real world is inevitably affected by environmental fluctuations ( [21]). And several authors have revealed that the environmental fluctuations may change the properties of population models greatly. For example, Mao, Marion and Renshaw [20] revealed that the environmental fluctuations can suppress a potential population explosion. Therefore it is useful to study how the environmental fluctuations affects the multi-predators one-prey model with modified Leslie-Gower and Holling-type II schemes. However, to the best of our knowledge, no result of this aspect has previously been reported. Suppose that the growth rate $r_i$ is affected by white noise (see, e.g., [3–5, 11–19, 29]), with $r_i \to r_i + \sigma_i \dot{W}_i(t)$, then we obtain the following stochastic two-predators one-prey system with modified Leslie-Gower and Holling-type II schemes:

$$\begin{cases} dx(t) = x(t)\left(r_1 - ax(t) - \dfrac{c_1 y_1(t)}{h_1 + x(t)} - \dfrac{c_2 y_2(t)}{h_2 + x(t)}\right)dt + \sigma_1 x(t) dW_1(t), \\ dy_1(t) = y_1(t)\left(r_2 - \dfrac{f_1 y_1(t)}{h_1 + x(t)}\right)dt + \sigma_2 y_1(t) dW_2(t), \\ dy_2(t) = y_2(t)\left(r_3 - \dfrac{f_2 y_2(t)}{h_2 + x(t)}\right)dt + \sigma_3 y_2(t) dW_3(t), \end{cases} \tag{1.2}$$

where $W_i(t)$ is a standard Wiener process and $\sigma_i^2$ stands for the intensity of the noise.

For model (1.2), some interesting and important problems arise naturally.

(Q1): System (1.2) is a population model, then when the populations will be extinct and when will be not?

(Q2): When investigating deterministic population models, an interesting topic is to seek the positive equilibrium point and to analyze its stability. However, Eq. (1.2) does not have positive equilibrium point. Consequently, its solution can not tend to a positive point. Then whether model (1.2) still has some stability properties around some positive point?

(Q3): Do white noises have effects on the stability and extinction of model (1.2)?

The aim of this paper is to consider these questions. In Section 2, the almost complete parameters analysis is carried out. In each case, it is shown that every population is either stable in time average or extinct, depending on the coefficients of model (1.2), especially, depending on $\sigma_1^2$, $\sigma_2^2$, $\sigma_3^2$, the intensities of the white noises. In Section 3, we work out some figures to support the theoretical findings. Section 4 gives to the concluding remarks.

## 2. Main results

For the sake of convenience, we define some notations.

$$R_+^3 = \{z \in R^3 | z_i > 0,\ i = 1, 2, 3\},\ b_i = r_i - 0.5\sigma_i^2,\quad i = 1, 2, 3.$$

Before we state and prove our main results, we prepare some lemmas.

**Lemma 2.1.** *For any initial data $(x(0), y_1(0), y_2(0)) \in R^3_+$, there is a unique global positive solution $(x(t), y_1(t), y_2(t))$ to Eq. (1.2) a.s. (almost surly).*

**Proof.** Our proof is motivated by Ji et al. [11]. Consider the equations:

$$\begin{cases} du(t) = \left(r_1 - ae^{u(t)} - \dfrac{c_1 e^{v_1(t)}}{h_1 + e^{u(t)}} - \dfrac{c_2 e^{v_2(t)}}{h_2 + e^{u(t)}}\right)dt + \sigma_1 dW_1(t), \\ dv_1(t) = \left(r_2 - \dfrac{f_1 e^{v_1(t)}}{h_1 + e^{u(t)}}\right)dt + \sigma_2 dW_2(t), \\ dv_2(t) = \left(r_3 - \dfrac{f_2 e^{v_2(t)}}{h_2 + e^{u(t)}}\right)dt + \sigma_3 dW_3(t), \end{cases} \tag{2.1}$$

and $u(0) = \ln x(0), v_1(0) = \ln y_1(0), v_2(0) = \ln y_2(0)$. It is easy to see that the coefficients of model (2.1) obey the local Lipschitz condition, hence model (2.1) has a unique local solution $(u(t),\ v_1(t),\ v_2(t))$ on $[0, \tau_e)$, where $\tau_e$ stands for the explosion time. In view of Itô's formula, $(x(t) = e^{u(t)},\ y_1(t) = e^{v_1(t)},\ y_2(t) = e^{v_2(t)})$ is the unique local positive solution to model (1.2). To complete the proof, we need only to show $\tau_e = +\infty$. To this end, we construct the following auxiliary equations:

$$d\Phi(t) = \Phi(t)\big(r_1 - a\Phi(t)\big)dt + \sigma_1\Phi(t)dW_1(t), \quad \Phi(0) = x(0); \tag{2.2}$$

$$d\psi_1(t) = \psi_1(t)\big(r_2 - \frac{f_1}{h_1}\psi_1(t)\big)dt + \sigma_2\psi_1(t)dW_2(t), \quad \psi_1(0) = y_1(0); \tag{2.3}$$

$$d\Psi_1(t) = \Psi_1(t)\left(r_2 - \frac{f_1}{h_1 + \Phi(t)}\Psi_1(t)\right)dt + \sigma_2\Psi_1(t)dW_2(t), \quad \Psi_1(0) = y_1(0);$$

$$d\psi_2(t) = \psi_2(t)\left(r_3 - \frac{f_2}{h_2}\psi_2(t)\right)dt + \sigma_3\psi_2(t)dW_3(t), \quad \psi_2(0) = y_2(0);$$

$$d\Psi_2(t) = \Psi_2(t)\left(r_3 - \frac{f_2}{h_2 + \Phi(t)}\Psi_2(t)\right)dt + \sigma_3\Psi_2(t)dW_3(t), \quad \Psi_2(0) = y_2(0).$$

Making use of the famous stochastic comparison theorem ( [10]), one can see that for $t \in [0, \tau_e)$,

$$x(t) \le \Phi(t),\ \psi_1(t) \le y_1(t) \le \Psi_1(t),\ \psi_2(t) \le y_2(t) \le \Psi_2(t),\ a.s. \tag{2.4}$$

By Theorem 2.2 in Jiang and Shi [13], Eq. (2.2) has the explicit solution

$$\Phi(t) = \frac{\exp\{b_1 t + \sigma_1 W_1(t)\}}{x^{-1}(0) + a\int_0^t \exp\{b_1 s + \sigma_1 W_1(s)\}ds}. \tag{2.5}$$

Similarly,

$$\psi_1(t) = \frac{\exp\{b_2 t + \sigma_2 W_2(t)\}}{y_1^{-1}(0) + \frac{f_1}{h_1}\int_0^t \exp\{b_2 s + \sigma_2 W_2(s)\}ds}, \tag{2.6}$$

$$\Psi_1(t) = \frac{\exp\{b_2 t + \sigma_2 W_2(t)\}}{y_1^{-1}(0) + \int_0^t \frac{f_1}{h_1 + \Phi(s)}\exp\{b_2 s + \sigma_2 W_2(s)\}ds}, \tag{2.7}$$

$$\psi_2(t) = \frac{\exp\{b_3 t + \sigma_3 W_3(t)\}}{y_2^{-1}(0) + \frac{f_2}{h_2}\int_0^t \exp\{b_3 s + \sigma_3 W_3(s)\}ds}, \tag{2.8}$$

$$\Psi_2(t) = \frac{\exp\{b_3 t + \sigma_3 W_3(t)\}}{y_2^{-1}(0) + \int_0^t \frac{f_2}{h_2+\Phi(s)} \exp\{b_3 s + \sigma_3 W_3(s)\} ds}. \tag{2.9}$$

Note that $\Phi(t)$, $\psi_1(t)$, $\Psi_1(t)$, $\psi_2(t)$ and $\Psi_2(t)$ are existent on $t \geq 0$, therefore $\tau_e = +\infty$. □

**Lemma 2.2** ( [15])**.** *Let* $Y(t) \in C(\Omega \times [0, +\infty), (0, +\infty))$.
*(I) If there exist three positive constants* $T$, $\tau$ *and* $\tau_0$ *such that for all* $t \geq T$

$$\ln Y(t) \leq \tau t - \tau_0 \int_0^t Y(s) ds + \sum_{i=1}^n \alpha_i W_i(t),$$

*where* $\alpha_i$ $(1 \leq i \leq n)$ *are constants, then*

$$\limsup_{t\to+\infty} t^{-1} \int_0^t Y(s) ds \leq \tau/\tau_0, \quad a.s.$$

*(II) If there exist three positive constants* $T$, $\tau$ *and* $\tau_0$ *such that for all* $t \geq T$,

$$\ln Y(t) \geq \tau t - \tau_0 \int_0^t Y(s) ds + \sum_{i=1}^n \alpha_i W_i(t),$$

*then* $\liminf_{t\to+\infty} t^{-1} \int_0^t Y(s) ds \geq \tau/\tau_0, \quad a.s.$

**Lemma 2.3.** *Let* $b_1 > 0$. *If* $b_2 > 0$ *(respectively,* $b_3 > 0$*), then*

$$\lim_{t\to+\infty} t^{-1} \ln y_1(t) = 0 \ (\textit{respectively}, \lim_{t\to+\infty} t^{-1} \ln y_2(t) = 0), \quad a.s. \tag{2.10}$$

**Proof.** Without loss of generality, we only prove the case $b_2 > 0$. Let $T$ be sufficiently large satisfying $0.5 \exp\{b_1 t\} \geq 1$ for $t \geq T$. Hence for $t \geq T$, it follows from (2.5) that

$$\begin{aligned}
\Phi(t) &= \frac{\exp\{b_1 t + \sigma_1 W_1(t)\}}{x^{-1}(0) + a\int_0^t \exp\{b_1 s + \sigma_1 W_1(s)\} ds} \\
&\leq \frac{\exp\{b_1 t + \sigma_1 W_1(t)\}}{a\int_0^t \exp\{b_1 s + \sigma_1 W_1(s)\} ds} \\
&\leq \frac{\exp\{b_1 t + \sigma_1 W_1(t)\}}{a \exp\{\min_{0\leq\nu\leq t} \sigma_1 W_1(\nu)\} \int_0^t \exp\{b_1 s\} ds} \\
&= \frac{b_1}{a} \frac{\exp\{b_1 t + \sigma_1 W_1(t)\}}{\exp\{\min_{0\leq\nu\leq t} \sigma_1 W_1(\nu)\} [\exp\{b_1 t\} - 1]} \\
&\leq \frac{2b_1}{a} \frac{\exp\{b_1 t + \sigma_1 W_1(t)\}}{\exp\{\min_{0\leq\nu\leq t} \sigma_1 W_1(\nu)\} \exp\{b_1 t\}} \\
&= \frac{2b_1}{a} \exp\left\{\sigma_1 \left[W_1(t) - \min_{0\leq\nu\leq t} W_1(\nu)\right]\right\} \\
&\leq \frac{2b_1}{a} \exp\left\{|\sigma_1| \left[W_1(t) - \min_{0\leq\nu\leq t} W_1(\nu)\right]\right\}.
\end{aligned}$$

Clearly,

$$\exp\left\{|\sigma_1| \left[W_1(t) - \min_{0\leq\nu\leq t} W_1(\nu)\right]\right\} > 1.$$

Then we obtain

$$\begin{aligned}
&\int_T^t \frac{f_1}{h_1+\Phi(s)}\exp\{b_2 s+\sigma_2 W_2(s)\}ds\\
\geq&\int_T^t \frac{f\exp\{b_2 s+\sigma_2 W_2(s)\}}{h_1+\frac{2b_1}{a}\exp\{|\sigma_1|\big[W_1(s)-\min_{0\le\nu\le s}W_1(\nu)\big]\}}ds\\
\geq&\int_T^t \frac{f_1\exp\{b_2 s+\sigma_2 W_2(s)\}}{\big[h_1+\frac{2b_1}{a}\big]\exp\{|\sigma_1|\big[W_1(s)-\min_{0\le\nu\le s}W_1(\nu)\big]\}}ds\\
=&\frac{af_1}{ah_1+2b_1}\int_T^t \exp\bigg\{-|\sigma_1|\big[W_1(s)-\min_{0\le\nu\le s}W_1(\nu)\big]\bigg\}\exp\bigg\{b_2 s+\sigma_2 W_2(s)\bigg\}ds\\
\geq&\frac{af_1}{ah_1+2b_1}\exp\bigg\{|\sigma_1|\min_{0\le\nu\le t}W_1(\nu)-|\sigma_1|\max_{0\le\nu\le t}W_1(\nu)\bigg\}\\
&\times\exp\bigg\{\min_{0\le\nu\le t}\sigma_2 W_2(\nu)\bigg\}\int_T^t\exp\bigg\{b_2 s\bigg\}ds\\
=&K_1(t)\bigg[\exp\{b_2 t\}-\exp\{b_2 T\}\bigg],
\end{aligned}$$

where

$$K_1(t)=\frac{af_1}{am_1+2b_1}\exp\bigg\{|\sigma_1|\min_{0\le\nu\le t}W_1(\nu)-|\sigma_1|\max_{0\le\nu\le t}W_1(\nu)+\min_{0\le\nu\le t}\sigma_2 W_2(\nu)\bigg\}.$$

When this inequality is used in (2.7), we can derive that

$$\begin{aligned}
\frac{1}{\Psi_1(t)}\geq&\exp\bigg\{-b_2(t-T)-\sigma_2\Big(W_2(t)-W_2(T)\Big)\bigg\}\\
&\times\bigg[y_1^{-1}(0)(T)+K_1(t)\Big(\exp\{b_2 t\}-\exp\{b_2 T\}\Big)\bigg]\\
\geq&\exp\bigg\{b_2 T+\sigma_2 W_2(T)\bigg\}\Big(1-\exp\{-b_2(t-T)\}\Big)\\
&\times K_1(t)\exp\bigg\{-\max_{0\le\nu\le t}\sigma_2 W_2(\nu)\bigg\}\\
=:&K_2(t)\times K_3(t),
\end{aligned}$$

where

$$\begin{aligned}
K_2(t)&=\exp\bigg\{b_2 T+\sigma_2 W_2(T)\bigg\}\Big(1-\exp\{-b_2(t-T)\}\Big),\\
K_3(t)&=K_1(t)\exp\bigg\{-\max_{0\le\nu\le t}\sigma_2 W_2(\nu)\bigg\}.
\end{aligned}$$

Consequently,

$$t^{-1}\ln\Psi(t)\le -t^{-1}\ln K_2(t)-t^{-1}\ln K_3(t). \tag{2.11}$$

Since $\lim_{t\to+\infty}W_i(t)/t=0\ \ a.s.,\ \ i=1,2,3$, then if $b_2>0$, we have

$$\lim_{t\to+\infty}t^{-1}\ln K_2(t)=0,\quad \lim_{t\to+\infty}t^{-1}\ln K_3(t)=0,\quad a.s.$$

When these identities are used in (2.11), we can observe that

$$\limsup_{t\to+\infty}t^{-1}\ln y_1(t)\le\limsup_{t\to+\infty}t^{-1}\ln\Psi_1(t)\le 0,\ \ a.s.$$

To complete the proof, it suffices to show $\liminf_{t\to+\infty} t^{-1}\ln y_1(t) \geq 0$, a.s. An application of Itô's formula to (2.3) results in

$$d\ln\psi_1(t) = \left[b_2 - \frac{f_1}{h_1}\psi_1(t)\right]dt + \sigma_2 dW_2(t).$$

In other words,

$$t^{-1}\ln\psi_1(t) = t^{-1}\ln y_1(0) + b_2 - \frac{f_1}{h_1}t^{-1}\int_0^t \psi_1(s)ds + \sigma_2 t^{-1}W_2(t). \tag{2.12}$$

Clearly, for arbitrary $\varepsilon > 0$, there exists $T > 0$ such that for $t \geq T$,

$$-\varepsilon/2 \leq t^{-1}f\ln y_1(0) \leq \varepsilon/2.$$

Substituting this inequality into (2.12) yields that for $t \geq T$,

$$t^{-1}\ln\psi_1(t) \leq b_2 + \varepsilon - t^{-1}\frac{f_1}{h_1}\int_0^t \psi_1(s)ds + \sigma_2 t^{-1}W_2(t), \tag{2.13}$$

$$t^{-1}\ln\psi_1(t) \geq b_2 - \varepsilon - t^{-1}\frac{f_1}{h_1}\int_0^t \psi_1(s)ds + \sigma_2 t^{-1}W_2(t). \tag{2.14}$$

We can choose $\varepsilon$ be sufficiently small satisfying $b_2 - \varepsilon > 0$. Now applying (I) and (II) in Lemma 2.2 to (2.13) and (2.14) respectively, one can derive that

$$\frac{h_1(b_2-\varepsilon)}{f_1} \leq \liminf_{t\to+\infty} t^{-1}\int_0^t \psi_1(s)ds \leq \limsup_{t\to+\infty} t^{-1}\int_0^t \psi_1(s)ds \leq \frac{h_1(b_2+\varepsilon)}{f_1}, \quad a.s.$$

It therefore follows from the arbitrariness of $\varepsilon$ that

$$\lim_{t\to+\infty} t^{-1}\int_0^t \psi_1(s)ds = \frac{h_1 b_2}{f_1}, \ a.s. \tag{2.15}$$

When this identity is used in (2.12), then by

$$\lim_{t\to+\infty} t^{-1}\ln y_1(0) = 0 \text{ and } \lim_{t\to+\infty} W_2(t)/t = 0,$$

we obtain $\lim_{t\to+\infty} t^{-1}\ln\psi_1(t) = 0, \quad a.s.$ In view of (2.4), one can see that

$$\liminf_{t\to+\infty} t^{-1}\ln y_1(t) \geq \lim_{t\to+\infty} t^{-1}\ln\psi_1(t) = 0, \quad a.s.$$

□

Now we are in the position to give our main result.

**Theorem 2.1.** *For model (1.2),*

*(i) If $b_1 < 0$, $b_2 < 0$ and $b_3 < 0$, then $x$, $y_1$ and $y_2$ go to extinction, i.e., $\lim_{t\to+\infty} x(t) = 0$, $\lim_{t\to+\infty} y_1(t) = 0$, $\lim_{t\to+\infty} y_2(t) = 0$, a.s.;*

*(ii) If $b_1 < 0$, $b_2 > 0$ and $b_3 < 0$, then $x$ and $y_2$ go to extinction and $y_1$ is stable in time average, i.e.,*

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds = \frac{h_1 b_2}{f_1}, \quad a.s.;$$

*(iii) If $b_1 < 0$, $b_2 < 0$ and $b_3 > 0$, then $x$ and $y_1$ go to extinction and $y_2$ is stable in time average, i.e.,*

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)ds = \frac{h_2 b_3}{f_2}, \quad a.s.;$$

*(iv) If $b_1 < 0$, $b_2 > 0$ and $b_3 > 0$, then $x$ goes to extinction and both $y_1$ and $y_2$ are stable in time average:*

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds = \frac{h_1 b_2}{f_1}, \quad \lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)ds = \frac{h_2 b_3}{f_2}, \ a.s.; \tag{2.16}$$

*(v) If $b_1 > 0$, $b_2 < 0$ and $b_3 < 0$, then $y_1$ and $y_2$ go to extinction and $x$ is stable in time average:*

$$\lim_{t\to+\infty} t^{-1}\int_0^t x(s)ds = \frac{b_1}{a}, \quad a.s.; \tag{2.17}$$

*(vi) If $b_1 > 0$, $b_2 > 0$ and $b_3 < 0$, then $y_2$ goes to extinction and moreover*

*(a) If $b_1 < \frac{c_1}{f_1} b_2$, then $x$ goes to extinction and $y_1$ is stable in time average:*

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds = \frac{h_1 b_2}{f_1}, \ a.s.$$

*(b) If $b_1 > \frac{c_1}{f_1} b_2$, then*

$$\lim_{t\to+\infty} t^{-1}\int_0^t x(s)ds = \frac{b_1}{a} - \frac{c_1 b_2}{a f_1}, \quad \lim_{t\to+\infty} t^{-1}\int_0^t \frac{y_1(s)}{h_1 + x(s)}ds = \frac{b_2}{f_1}, \quad a.s.$$

*(vii) If $b_1 > 0$, $b_2 < 0$ and $b_3 > 0$, then $y_1$ goes to extinction and moreover*

*(c) If $b_1 < \frac{c_2}{f_2} b_3$, then $x$ goes to extinction and $y_2$ is stable in time average:*

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)ds = \frac{h_2 b_3}{f_2}, \ a.s.$$

*(d) If $b_1 > \frac{c_2}{f_2} b_3$, then*

$$\lim_{t\to+\infty} t^{-1}\int_0^t x(s)ds = \frac{b_1}{a} - \frac{c_2 b_3}{a f_2}, \quad \lim_{t\to+\infty} t^{-1}\int_0^t \frac{y_2(s)}{h_2 + x(s)}ds = \frac{b_3}{f_2}, \quad a.s.$$

*(viii) If $b_1 > 0$, $b_2 > 0$ and $b_3 > 0$, then*

*(e) If $b_1 < \frac{c_1}{f_1} b_2 + \frac{c_2}{f_2} b_3$, then $x$ goes to extinction and both $y_1$ and $y_2$ are stable in time average:*

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds = \frac{h_1 b_2}{f_1}, \quad \lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)ds = \frac{h_2 b_3}{f_2}, \ a.s.$$

*(f) If $b_1 > \frac{c_1}{f_1} b_2 + \frac{c_2}{f_2} b_3$, then*

$$\begin{aligned} &\lim_{t\to+\infty} t^{-1}\int_0^t x(s)ds = \frac{b_1}{a} - \frac{c_1 b_2}{a f_1} - \frac{c_2 b_3}{a f_2}, \ a.s. \\ &\lim_{t\to+\infty} t^{-1}\int_0^t \frac{y_1(s)}{h_1 + x(s)}ds = \frac{b_2}{f_1}, \quad a.s. \\ &\lim_{t\to+\infty} t^{-1}\int_0^t \frac{y_2(s)}{h_2 + x(s)}ds = \frac{b_3}{f_2}, \quad a.s. \end{aligned} \tag{2.18}$$

**Proof.** Applying Itô's formula to model (1.2) leads to

$$
\begin{aligned}
d\ln x(t) &= \left[b_1 - ax(t) - \frac{c_1y_1(t)}{h_1+x(t)} - \frac{c_2y_2(t)}{h_2+x(t)}\right]dt + \sigma_1 dW_1(t),\\
d\ln y_1(t) &= \left[b_2 - \frac{f_1y_1(t)}{h_1+x(t)}\right]dt + \sigma_2 dW_2(t),\\
d\ln y_2(t) &= \left[b_3 - \frac{f_3y_3(t)}{m_3+x(t)}\right]dt + \sigma_3 dW_3(t).
\end{aligned}
$$

That is to say

$$
\begin{aligned}
\ln x(t) - \ln x(0) &= b_1t - a\int_0^t x(s)ds - c_1\int_0^t \frac{y_1(s)}{h_1+x(s)}ds - c_2\int_0^t \frac{y_2(s)}{h_2+x(s)}ds\\
&\quad + \sigma_1 W_1(t), \qquad (2.19)\\
\ln y_1(t) - \ln y_1(0) &= b_2t - f_1\int_0^t \frac{y_1(s)}{h_1+x(s)}ds + \sigma_2 W_2(t), \qquad (2.20)\\
\ln y_2(t) - \ln y_2(0) &= b_3t - f_2\int_0^t \frac{y_2(s)}{h_2+x(s)}ds + \sigma_3 W_3(t). \qquad (2.21)
\end{aligned}
$$

The proof of (i): by virtue of (2.19),

$$
t^{-1}\ln\frac{x(t)}{x(0)} \le b_1 + \sigma_1 t^{-1}W_1(t).
$$

Since $\lim\limits_{t\to+\infty} W_1(t)/t = 0$ and $b_1 < 0$, then $\lim\limits_{t\to+\infty} x(t) = 0$, $a.s.$ Similarly, if $b_2 < 0$ (respectively, $b_3 < 0$), it then follow from (2.20) (respectively, (2.21)) that $\lim\limits_{t\to+\infty} y_1(t) = 0$ $a.s.$ (respectively, $\lim\limits_{t\to+\infty} y_2(t) = 0$ $a.s.$).

(ii): Note that $b_1 < 0$ and $b_3 < 0$, hence (i) means $\lim\limits_{t\to+\infty} x(t) = 0$, $\lim\limits_{t\to+\infty} y_2(t) = 0$, $a.s.$ Consequently, for sufficiently large $t$,

$$
\ln y_1(t) - \ln y_1(0) \le b_2t - \frac{f_1}{h_1+\varepsilon}\int_0^t y_1(s)ds + \sigma_2 W_2(t), \qquad (2.22)
$$

$$
\ln y_1(t) - \ln y_1(0) \le b_2t - \frac{f_1}{h_1-\varepsilon}\int_0^t y_1(s)ds + \sigma_2 W_2(t). \qquad (2.23)
$$

Applying of (I) and (II) in Lemma 2.2 to (2.22) and (2.23) respectively, one can see that

$$
\begin{aligned}
&\limsup_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds \le \frac{(h_1+\varepsilon)b_2}{f_1}, \quad a.s.\\
&\liminf_{t\to+\infty} t^{-1}\int_0^t y(s)ds \ge \frac{(h_1-\varepsilon)b_2}{f_1}, \ a.s.
\end{aligned}
$$

In view of the arbitrariness of $\varepsilon$, we obtain $\lim\limits_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds = \frac{h_1b_2}{f_1}$, $a.s.$

The proof of (iii) is similar to (ii) and hence is omitted.

(iv): Since $b_1 < 0$, it then follows from (i) that $\lim\limits_{t\to+\infty} x(t) = 0$. The proof of (2.16) is similar to (ii) and hence is omitted.

(v): Since $b_2 < 0$ and $b_3 < 0$, then similar to the proof of (i), we can show that $\lim_{t\to+\infty} y_1(t) = 0$, $\lim_{t\to+\infty} y_2(t) = 0$. The proof of (2.17) is similar to (ii) and hence is omitted.

We are in the position to show (vi). Clearly, $b_3 < 0 \Rightarrow \lim_{t\to+\infty} y_2(t) = 0$.

(a): By (2.19)$\times f_1$ − (2.20) $\times c_1$, we get

$$\begin{aligned}
t^{-1} f_1 \ln \frac{x(t)}{x(0)} =& c_1 t^{-1} \ln \frac{y_1(t)}{y_1(0)} + f_1 b_1 - c_1 b_2 - a f_1 t^{-1} \int_0^t x(s)ds \\
& - c_2 f_1 t^{-1} \int_0^t \frac{y_2(s)}{h_2 + x(s)} ds + f_1\sigma_1 t^{-1} W_1(t) - c_1\sigma_2 t^{-1} W_2(t) \\
\le& c_1 t^{-1} \ln \frac{y_1(t)}{y_1(0)} + f_1 b_1 - c_1 b_2 - a f_1 t^{-1} \int_0^t x(s)ds \\
& + f_1\sigma_1 t^{-1} W_1(t) - c_1\sigma_2 t^{-1} W_2(t).
\end{aligned} \tag{2.24}$$

According to (2.10), for arbitrary $\varepsilon > 0$, there exists $T > 0$ such that for $t \ge T$,

$$t^{-1} f_1 \ln x(0) \le \varepsilon/3,\ c_1 t^{-1} \ln \frac{y_1(t)}{y_1(0)} \le \varepsilon/3,\ f_1\sigma_1 t^{-1} W_1(t) - c_1\sigma_2 t^{-1} W_2(t) \le \varepsilon/3.$$

Substituting these inequalities into (2.24), we can observe that for $t \ge T$

$$t^{-1} f_1 \ln x(t) \le \varepsilon + f_1 b_1 - c_1 b_2. \tag{2.25}$$

It then follows from $\frac{b_1}{c_1} < \frac{b_2}{f_1}$ that we can let $\varepsilon$ be sufficiently small such that $\varepsilon + f_1 b_1 - c_1 b_2 < 0$. Hence $\lim_{t\to+\infty} x(t) = 0$, $a.s.$ The proof of $\lim_{t\to+\infty} t^{-1} \int_0^t y_1(s)ds = \frac{h_1 b_2}{f_1}$ is similar to that of (ii) and hence is omitted. This completes the proof of (a).

(b): According to (2.20),

$$t^{-1} \ln y_1(t) - t^{-1} \ln y_1(0) = b_2 - f_1 t^{-1} \int_0^t \frac{y_1(s)}{h_1 + x(s)} ds + \sigma_2 t^{-1} W_2(t).$$

In view of (2.10) and $\lim_{t\to+\infty} t^{-1} W_2(t) = 0$, we can see that

$$\lim_{t\to+\infty} t^{-1} \int_0^t \frac{y_1(s)}{h_1 + x(s)} ds = \frac{b_2}{f_1}, \quad a.s. \tag{2.26}$$

At the same time, by virtue of (2.19),

$$\begin{aligned}
t^{-1} \ln x(t) =& b_1 - a t^{-1} \int_0^t x(s)ds - c_1 t^{-1} \int_0^t \frac{y_1(s)}{h_1 + x(s)} ds \\
& + t^{-1} \ln x(0) - c_2 t^{-1} \int_0^t \frac{y_2(s)}{h_2 + x(s)} ds + \sigma_1 W_1(t)/t.
\end{aligned} \tag{2.27}$$

It therefore follows from $\lim_{t\to+\infty} y_2(t) = 0$ and (2.26) that for arbitrary $\varepsilon > 0$, there exists $T > 0$ such that for $t \ge T$,

$$-\frac{c_1 b_2}{f_1} - \varepsilon \le -c_1 t^{-1} \int_0^t \frac{y_1(s)}{h_1 + x(s)} ds + t^{-1} \ln x(0) - c_2 t^{-1} \int_0^t \frac{y_2(s)}{h_2 + x(s)} ds \le -\frac{c_1 b_2}{f_1} + \varepsilon.$$

When these inequalities are used in (2.27), one can derive that for $t \geq T$,

$$t^{-1}\ln x(t) \geq b_1 - \frac{c_1 b_2}{f_1} - \varepsilon - at^{-1}\int_0^t x(s)ds + \sigma_1 t^{-1}W_1(t), \tag{2.28}$$

$$t^{-1}\ln x(t) \leq b_1 - \frac{c_1 b_2}{f_1} + \varepsilon - at^{-1}\int_0^t x(s)ds + \sigma_1 t^{-1}W_1(t). \tag{2.29}$$

Let $\varepsilon$ be sufficiently small such that $b_1 - \frac{c_1 b_2}{f_1} - \varepsilon > 0$, and then using (I) and (II) in Lemma 2.2 to (2.28) and (2.29) respectively, one derives

$$\frac{b_1}{a} - \frac{c_1 b_2}{af_1} + \frac{\varepsilon}{a} \leq \liminf_{t\to+\infty} t^{-1}\int_0^t x(s)ds \leq \limsup_{t\to+\infty} t^{-1}\int_0^t x(s)ds \leq \frac{b_1}{a} - \frac{c_1 b_2}{af_1} + \frac{\varepsilon}{a},\ a.s.$$

According to the arbitrariness of $\varepsilon$, we have

$$\lim_{t\to+\infty} t^{-1}\int_0^t x(s)ds = \frac{b_1}{a} - \frac{c_1 b_2}{af_1},\ a.s.$$

This completes the proof of (vi).

The proof of (vii) is similar to (vi) and hence is omitted.

To completes the proof, we need only to show (viii).

(e): By (2.24)$\times f_2$ − (2.21) $\times c_1 f_1$,

$$\begin{aligned} t^{-1}f_1 f_2 \ln\frac{x(t)}{x(0)} =& c_1 f_2 t^{-1}\ln\frac{y_1(t)}{y_1(0)} + c_2 f_1 t^{-1}\ln\frac{y_2(t)}{y_2(0)} \\ &+ f_1 b_1 f_2 - c_1 b_2 f_2 - c_2 f_1 b_3 - af_1 f_2 t^{-1}\int_0^t x(s)ds \\ &+ f_1 f_2 \sigma_1 t^{-1}W_1(t) - c_1 f_2 \sigma_2 t^{-1}W_2(t) - c_2 f_1 \sigma_3 t^{-1}W_3(t). \end{aligned} \tag{2.30}$$

It follows from (2.10) that for arbitrary $\varepsilon > 0$, there exists $T > 0$ such that for $t \geq T$,

$$\begin{aligned} &t^{-1}f_1 f_2 \ln x(0) \leq \varepsilon/4,\ c_1 f_2 t^{-1}\ln\frac{y_1(t)}{y_1(0)} \leq \varepsilon/4,\ c_2 f_1 t^{-1}\ln\frac{y_2(t)}{y_2(0)} \leq \varepsilon/4, \\ &f_1 f_2 \sigma_1 t^{-1}W_1(t) - c_1 f_2 \sigma_2 t^{-1}W_2(t) - c_2 f_1 \sigma_3 t^{-1}W_3(t) \leq \varepsilon/4. \end{aligned}$$

When these inequalities are used in (2.30), we obtain that for $t \geq T$

$$t^{-1}f_1 \ln x(t) \leq \varepsilon + f_1 b_1 f_2 - c_1 b_2 f_2 - c_2 f_1 b_3. \tag{2.31}$$

Note that $b_1 < \frac{c_1}{f_1}b_2 + \frac{c_2}{f_2}b_3$, thus we can choose $\varepsilon$ sufficiently small such that $f_1 b_1 f_2 - c_1 b_2 f_2 - c_2 f_1 b_3 + \varepsilon < 0$. Consequently, $\lim_{t\to+\infty} x(t) = 0,\ a.s.$ The proof of

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds = \frac{h_1 b_2}{f_1},\ \lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)ds = \frac{h_2 b_3}{f_2}$$

is similar to that of (ii) and hence is omitted.

(f): Similar to (2.26), we can show that

$$\lim_{t\to+\infty} t^{-1}\int_0^t \frac{y_1(s)}{h_1 + x(s)}ds = \frac{b_2}{f_1},\ \lim_{t\to+\infty} t^{-1}\int_0^t \frac{y_2(s)}{h_2 + x(s)}ds = \frac{b_3}{f_2},\ \ a.s.$$

It then follows from (2.27) that for sufficiently large $t$,

$$t^{-1}\ln x(t) \geq b_1 - \frac{c_1 b_2}{f_1} - \frac{c_2 b_3}{f_2} - \varepsilon - at^{-1}\int_0^t x(s)ds + \sigma_1 t^{-1}W_1(t),$$
$$t^{-1}\ln x(t) \leq b_1 - \frac{c_1 b_2}{f_1} - \frac{c_2 b_3}{f_2} + \varepsilon - at^{-1}\int_0^t x(s)ds + \sigma_1 t^{-1}W_1(t).$$

Then using Lemma 2.2 and the arbitrariness of $\varepsilon$, we can derive the desired assertion (2.18). □ □

# 3. Numerical simulations

In this section, we shall work out some figures to validate our analytical results by using the famous Milstein method (see e.g. [9]). Without loss of generality, we suppose that $b_2 = r_2 - 0.5\sigma_2^2 > b_3 = r_3 - 0.5\sigma_3^2$. Consider the discretization equation:

$$\begin{aligned} x^{(k+1)} =& x^{(k)} + x^{(k)}\left[r_1 - ax^{(k)} - \frac{c_1 y_1^{(k)}}{h_1 + x^{(k)}} - \frac{c_2 y_2^{(k)}}{h_2 + x^{(k)}}\right]\Delta t \\ & + \sigma_1 x^{(k)}\sqrt{\Delta t}\xi^{(k)} + \frac{\sigma_1^2}{2}x^{(k)}((\xi^{(k)})^2\Delta t - \Delta t), \\ y_1^{(k+1)} =& y_1^{(k)} + y_1^{(k)}\left[r_2 - \frac{f_1 y_1^{(k)}}{h_1 + x^{(k)}}\right]\Delta t + \sigma_2 y_1^{(k)}\sqrt{\Delta t}\eta_1^{(k)} + \frac{\sigma_2^2}{2}y_1^{(k)}((\eta_1^{(k)})^2\Delta t - \Delta t), \\ y_2^{(k+1)} =& y_2^{(k)} + y_2^{(k)}\left[r_3 - \frac{f_2 y_2^{(k)}}{h_2 + x^{(k)}}\right]\Delta t + \sigma_3 y_2^{(k)}\sqrt{\Delta t}\eta_2^{(k)} + \frac{\sigma_3^2}{2}y_2^{(k)}((\eta_2^{(k)})^2\Delta t - \Delta t), \end{aligned}$$

where $\xi^{(k)}$, $\eta_1^{(k)}$ and $\eta_2^{(k)}$,$k = 1, 2, \ldots, K$, are the Gaussian random variables.

In Fig.1, we set $r_1 = 0.8,\ r_2 = 0.25,\ r_3 = 0.1,\ a = 0.4,\ c_1 = 0.8,\ c_2 = 0.6,\ h_1 = h_2 = 1,\ f_1 = 0.4,\ f_2 = 0.3$. The only difference between the conditions of Fig.1(a)-Fig.1(h) is that the values of $\sigma_1^2$, $\sigma_2^2$ and $\sigma_3^2$ are different.

(a) In Fig.1(a), we choose $\sigma_1^2/2 = 0.85,\ \sigma_2^2/2 = 0.26,\ \sigma_3^2/2 = 0.15$. Then $b_1 = -0.05,\ b_2 = -0.01 > b_3 = -0.05$. According to (i) in Theorem 2.1, all the population become extinct. Fig.1(a) confirms these.

(b) In Fig.1(b), we choose $\sigma_1^2/2 = 0.85,\ \sigma_2^2/2 = 0.2,\ \sigma_3^2/2 = 0.15$. Then $b_1 = -0.05,\ b_2 = 0.05 > b_3 = -0.05$. In view of (ii) in Theorem 2.1, both $x$ and $y_2$ go to extinction and

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds = \frac{h_1 b_2}{f_1} = \frac{0.05}{0.4} = 0.125.$$

See Fig.1(b).

(c) In Fig.1(c), we choose $\sigma_1^2/2 = 0.85,\ \sigma_2^2/2 = 0.2,\ \sigma_3^2/2 = 0.08$. Then $b_1 = -0.05,\ b_2 = 0.05 > b_3 = 0.02$. By virtue of (iv) in Theorem 2.1, $x$ is extinct and

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds = \frac{h_1 b_2}{f_1} = 0.125, \quad \lim_{t\to+\infty} t^{-1}\int_0^t y_2(s)ds = \frac{h_2 b_3}{f_2} = 0.0667.$$

Fig.1(c) confirms these.

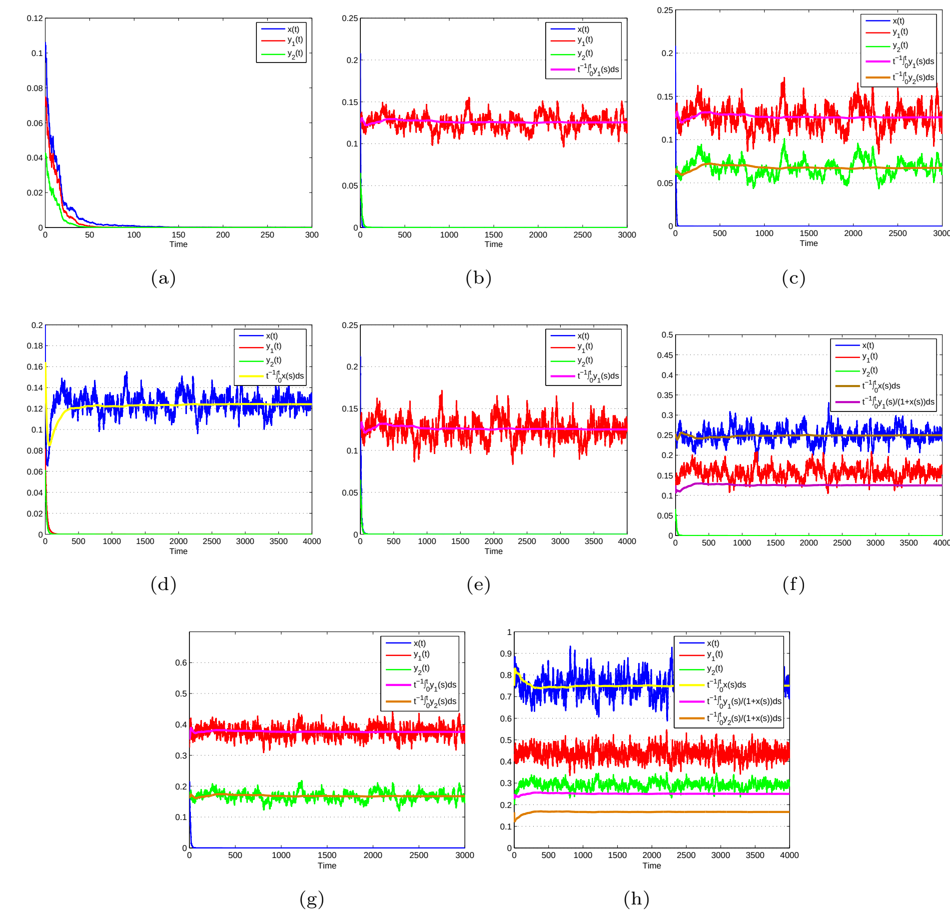


**Figure 1.** Solutions of (1.2) for $r_1 = 0.8,\ r_2 = 0.25,\ r_3 = 0.1,\ a = 0.4,\ c_1 = 0.8,\ c_2 = 0.6,\ h_1 = h_2 = 1,\ f_1 = 0.4,\ f_2 = 0.3$, step size $\Delta t = 0.001$. (a) is with $\sigma_1^2/2 = 0.85,\ \sigma_2^2/2 = 0.26,\ \sigma_3^2/2 = 0.15$; (b) is with $\sigma_1^2/2 = 0.85,\ \sigma_2^2/2 = 0.2,\ \sigma_3^2/2 = 0.15$; (c) is with $\sigma_1^2/2 = 0.85,\ \sigma_2^2/2 = 0.2,\ \sigma_3^2/2 = 0.08$; (d) is with $\sigma_1^2/2 = 0.75,\ \sigma_2^2/2 = 0.26,\ \sigma_3^2/2 = 0.15$; (e) is with $\sigma_1^2/2 = 0.75,\ \sigma_2^2/2 = 0.2,\ \sigma_3^2/2 = 0.15$; (f) is with $\sigma_1^2/2 = 0.6,\ \sigma_2^2/2 = 0.2,\ \sigma_3^2/2 = 0.15$; (g) is with $\sigma_1^2/2 = 0.5,\ \sigma_2^2/2 = 0.1,\ \sigma_3^2/2 = 0.05$; (h) is with $\sigma_1^2/2 = 0.2,\ \sigma_2^2/2 = 0.15,\ \sigma_3^2/2 = 0.05$.

(d) In Fig.1(d), we choose $\sigma_1^2/2 = 0.75,\ \sigma_2^2/2 = 0.26,\ \sigma_3^2/2 = 0.15$. Then $b_1 = 0.05,\ b_2 = -0.01 > b_3 = -0.05$. Using (v) in Theorem 2.1 results in that $y_1$ and $y_2$ go to extinction and

$$\lim_{t\to+\infty} t^{-1}\int_0^t x(s)ds = \frac{b_1}{a} = 0.125.$$

See Fig.1(d).

(e) In Fig.1(e), we choose $\sigma_1^2/2 = 0.75,\ \sigma_2^2/2 = 0.2,\ \sigma_3^2/2 = 0.15$. Then $b_1 = 0.05,\ b_2 = 0.05 > b_3 = -0.05,\ b_1 < \frac{c_1}{f_1}b_2 = 0.1$. Applying (a) in Theorem 2.1 gives that $x$ and $y_2$ go to extinction and

$$\lim_{t\to+\infty} t^{-1}\int_0^t y_1(s)ds = \frac{h_1 b_2}{f_1} = 0.125.$$

Fig.1(e) confirms these.

(f) In Fig.1(f), we choose $\sigma_1^2/2 = 0.6$, $\sigma_2^2/2 = 0.2$, $\sigma_3^2/2 = 0.15$. Then $b_1 = 0.2$, $b_2 = 0.05 > b_3 = -0.05$, $b_1 > \frac{c_1}{f_1} b_2 = 0.1$. It then follows from (b) in Theorem 2.1 that

$$\lim_{t\to+\infty} t^{-1} \int_0^t x(s)ds = \frac{b_1}{a} - \frac{c_1 b_2}{a f_1} = 0.25,$$
$$\lim_{t\to+\infty} t^{-1} \int_0^t \frac{y_1(s)}{1+x(s)} ds = \frac{b_2}{f_1} = 0.125.$$

See Fig.1(f).

(g) In Fig.1(g), we choose $\sigma_1^2/2 = 0.5$, $\sigma_2^2/2 = 0.1$, $\sigma_3^2/2 = 0.05$. Then $b_1 = 0.3$, $b_2 = 0.15 > b_3 = 0.05$, $b_1 < \frac{c_1}{f_1} b_2 + \frac{c_2}{f_2} b_3 = 0.4$. By (e) in Theorem 2.1, $x$ goes to extinction and

$$\lim_{t\to+\infty} t^{-1} \int_0^t y_1(s)ds = \frac{h_1 b_2}{f_1} = 0.375, \quad \lim_{t\to+\infty} t^{-1} \int_0^t y_2(s)ds = \frac{h_2 b_3}{f_2} = 0.1667.$$

Fig.1(g) confirms these.

(h) In Fig.1(h), we choose $\sigma_1^2/2 = 0.2$, $\sigma_2^2/2 = 0.15$, $\sigma_3^2/2 = 0.05$. Then $b_1 = 0.6$, $b_2 = 0.1 > b_3 = 0.05$, $b_1 > \frac{c_1}{f_1} b_2 + \frac{c_2}{f_2} b_3 = 0.3$. According to (f) in Theorem 2.1,

$$\lim_{t\to+\infty} t^{-1} \int_0^t x(s)ds = \frac{b_1}{a} - \frac{c_1 b_2}{a f_1} - \frac{c_2 b_3}{a f_2} = 0.75,$$
$$\lim_{t\to+\infty} t^{-1} \int_0^t \frac{y_1(s)}{1+x(s)} ds = \frac{b_2}{f_1} = 0.25,$$
$$\lim_{t\to+\infty} t^{-1} \int_0^t \frac{y_2(s)}{1+x(s)} ds = \frac{b_3}{f_2} = 0.1667.$$

Fig.1(h) confirms these.

## 4. Concluding remarks

This paper is devoted to the asymptotic properties of a stochastic two-predators one-prey model with modified Leslie-Gower and Holling-type II schemes. We have carried out the almost complete parameters analysis of the model. From these results, one can see that the stochastic noise play a key role in determining the extinction and stability in time average of the species.

Some interesting topics deserve further investigation. It is interesting to consider more realistic but complex models, for example, Markovian-switching (see e.g. [29]) or Lévy jumps (see e.g. [3]). The motivation is that the growth of population in the natural world often suffer sudden-environmental shocks, e.g., epidemics, waterflood, drought, etc. Moreover, it is interesting to study the stochastic one-predator two-preys model.

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