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EXISTENCE OF POSITIVE SOLUTIONS FOR FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract In this paper, by introducing a new operator, improving and generating a p-Laplace operator for some $p > 1$, we discuss the existence and multiplicity of positive solutions to the four point boundary value problems of nonlinear fractional differential equations. Our results extend some recent works in the literature.

Keywords Positive solutions, fractional differential equations, fixed point theorems.

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1. Introduction

In this paper we'll consider the existence of multiplicity of positive solutions for the following problem

$$\begin{aligned}
&D^q\left(\varphi\left(D^r x(t)\right)\right)+f\left(t,x(t)\right)=0, \quad t\in(0,1),\\
&\alpha_1 x(0)-\beta_1 x'(0)=-\gamma_1 x(\xi_1),\\
&\alpha_2 x(1)+\beta_2 x'(1)=-\gamma_2 x(\xi_2),\\
&D^r x(0)=0,
\end{aligned} \tag{1.1}$$

where $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2$ are real constants with, $\alpha_1,\alpha_2,\beta_1,\beta_2>0$, $\beta_1>\gamma_1$, $\beta_2>\gamma_2$, $0<\xi_1\le\xi_2<1$, $f\in\mathcal{C}\left([0,1]\times\mathbb{R}^+,\mathbb{R}^+\right)$ and D^r and D^q are the standard Caputo fractional derivatives of fractional order r and q with $1<r\le 2, 0<q\le 1$, $\varphi : R\to R$ is an increasing homeomorphism and positive homomorphism with $\varphi(0)=0$.

A projection $\varphi : R\to R$ is called an increasing homeomorphism and positive homomorphism, if the following conditions are satisfied;

1) If $x\le y$, then $\varphi(x)\le\varphi(y)$, for all $x,y\in R$,
2) φ is a continuous bijection and its inverse mapping is also continuous,
3) $\varphi(xy)=\varphi(x)\varphi(y)$, for all $x,y\in R$.

Due to the development of the theory of fractional calculus and its applications, such as in the fields of physics, rheology, dynamical processes in self similar and porous structures, electrical networks, visco-elasticity, chemical physics, and many other branches of science, many works on the basic theory of fractional calculus and fractional order differential equations have been published. For details,

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see [5, 10, 12–16, 20]. Also, there have been many papers dealing with the existence and multiplicity of solutions of boundary value problems for nonlinear fractional differential equations, see [1–4, 6–9, 18, 19, 21–25] and the references therein. Very recently, some authors considered the nonlinear fractional differential equations with p-Laplacian operator ($\varphi(u) = |u|^{p-2}u, p > 1$) and two-point, three-point, multi-point boundary value conditions. It is well known that the p-Laplacian operator is odd. In this paper we'll use the operator which is not necessary odd improves and generalizes a p-Laplacian operator. Moreover, for the increasing homeomorphism and positive homomorphism operator, the research has proceeded very slowly. Especially for the existence of countable many positive solutions of boundary value problems for fractional differential equations still remain unknown.

In [24], Zhao *et al.* investigated following fractional boundary value problem:

$$\begin{aligned} &D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \\ &u'(0) - \beta u(\xi) = 0, \qquad u'(1) + \gamma u(\eta) = 0, \end{aligned}$$

where α is a real constants with, $1 < \alpha \le 2$, $0 \le \xi \le \eta \le 1$, $0 \le \beta, \gamma \le 1$ and D_{0+}^{α} is the Caputo fractional derivative.

In [9], Ji and Ge obtained positive solutions for the following four-point nonlocal boundary value problems of fractional order:

$$\begin{aligned} &D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \\ &u'(0) - \beta u'(\xi) = 0, \qquad u(1) + \gamma u'(\eta) = 0, \end{aligned}$$

where α is a real constants with, $1 < \alpha \le 2$, $0 \le \xi \le \eta \le 1$, $0 < \beta < 1$, $\gamma > 0$ and D_{0+}^{α} is the Caputo fractional derivative.

In [18], Lu *et al.* studied the following fractional differential equations with p-Laplacian operator:

$$\begin{aligned} &D^{\beta}\left(\varphi_p\left(D^{\alpha} u(t)\right)\right) = f(t, u(t)), \quad t \in [0, 1], \\ &u(0) = u'(0) = u'(1) = 0, \qquad D^{\alpha} u(0) = D^{\alpha} u(1) = 0, \end{aligned}$$

where α is a real constants with, $2 < \alpha \le 3$, $1 < \beta \le 2$ and D^{α}, D^{β} are the Caputo fractional derivatives.

In [22], Yang studied the following fractional differential equations with p-Laplacian operator:

$$\begin{aligned} &D^{\beta}\left(\varphi_p\left(D^{\alpha} x(t)\right)\right) = f(t, x(t)), \quad t \in [0, 1], \\ &x(0) = x(1) = 0, \qquad D^{\alpha} x(0) = D^{\alpha} x(1) = 0, \end{aligned}$$

where α is a real constants with, $0 < \alpha, \beta \le 1$, $1 < \alpha + \beta \le 2$ and D^{α}, D^{β} are the Caputo fractional derivatives.

Motivated by the above-mentioned works, using Krasnoselskiis and Legget-Williams fixed point theorems in a cone, we show that the problem (1.1) has at least one and three positive solutions. The remainder of the paper is organized as follows. In Section 2 we state some preliminary facts needed in the proof of the main results. We also state a version of the Krasnoselskiis and Legget-Williams fixed point theorems. In Section 3, we state the main results of the paper, that establish existence of at least one or multiple positive solutions for the problem (1.1).

2. Preliminaries

In this section we collect some preliminary definitions and results that will be used in subsequent section. Firstly, for convenience of the reader, we give some definitions and fundamental results of fractional calculus.

Definition 2.1. For a function f given on the interval $[a,b]$, the Caputo derivative of fractional order r is defined as

$$D^r f(t) = \frac{1}{\Gamma(n-r)} \int_0^t (t-s)^{n-r-1} f^{(n)}(s)ds, \quad n = [r]+1, \tag{2.1}$$

where $[r]$ denotes the integer part of r.

Definition 2.2. The Riemann-Liouville fractional integral of order r for a function f is defined as

$$I^r f(t) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s)ds, \quad r > 0. \tag{2.2}$$

Lemma 2.1. *Let $r > 0$. Then the differential equation $D^r x(t) = 0$ has solutions*

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{2.3}$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n$, $n = [r]+1$.

Lemma 2.2. *Let $r > 0$. Then*

$$I^r (D^r x)(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \tag{2.4}$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n$, $n = [r]+1$.

For finding a solution of the problem (1.1), we first consider the following fractional differential equation

$$\begin{aligned} &-D^r x(t) = y(t), \\ &\alpha_1 x(0) - \beta_1 x'(0) = -\gamma_1 x(\xi_1), \\ &\alpha_2 x(1) + \beta_2 x'(1) = -\gamma_2 x(\xi_2), \end{aligned} \tag{2.5}$$

where $y \in \mathcal{C}([0,1], \mathbb{R}^+)$.

Let we define $d := \alpha_1(\alpha_2 + \beta_2 + \gamma_2 \xi_2) + \gamma_1(\beta_2 + \alpha_2(1-\xi_1) + \gamma_2(\xi_2 - \xi_1)) + \beta_1(\alpha_2 + \gamma_2)$.

Lemma 2.3. *Let $r \in (1,2]$ and $y \in \mathcal{C}[0,1]$. The boundary value problem*

$$\begin{aligned} &-D^r x(t) = y(t), \quad 0 < t < 1, \\ &\alpha_1 x(0) - \beta_1 x'(0) = -\gamma_1 x(\xi_1), \\ &\alpha_2 x(1) + \beta_2 x'(1) = -\gamma_2 x(\xi_2) \end{aligned} \tag{2.6}$$

has a unique solution x in the form

$$x(t) = \int_0^1 G(t,s) y(s) ds \tag{2.7}$$

where

$$G(t,s)=\begin{cases} -\dfrac{1}{\Gamma(r)}(t-s)^{r-1}+\dfrac{\gamma_1}{d\Gamma(r)}\left[\alpha_2+\beta_2+\gamma_1\xi_2\right. & \\ \left.-t(\alpha_2+\gamma_2)\right](\xi_1-s)^{r-1}+\dfrac{1}{d\Gamma(r)}\left[\beta_1-\gamma_1\xi_1\right. & s\le\xi_1, s\le t,\\ \left.+(\alpha_1+\gamma_1)t\right]\left[\gamma_2(\xi_2-s)^{r-1}+\alpha_2(1-s)^{r-1}\right. & \\ \left.+r\beta_2(1-s)^r-2\right], & \\ \dfrac{\gamma_1}{d\Gamma(r)}\left[\alpha_2+\beta_2+\gamma_1\xi_2-t(\alpha_2+\gamma_2)\right](\xi_1-s)^{r-1} & \\ \dfrac{1}{d\Gamma(r)}\left[\beta_1-\gamma_1\xi_1+(\alpha_1+\gamma_1)t\right]\left[\gamma_2(\xi_2-s)^{r-1}\right. & s\le\xi_1, s\ge t,\\ \left.+\alpha_2(1-s)^{r-1}+r\beta_2(1-s)^{r-2}\right], & \\ -\dfrac{1}{\Gamma(r)}(t-s)^{r-1}+\dfrac{1}{d\Gamma(r)}\left[\beta_1-\gamma_1\xi_1+(\alpha_1+\gamma_1)t\right] & \xi_1\le s\le\xi_2, s\le t,\\ \left[\gamma_2(\xi_2-s)^{r-1}+\alpha_2(1-s)^{r-1}+r\beta_2(1-s)^{r-2}\right], & \\ \dfrac{1}{\Gamma(r)}\left[\beta_1-\gamma_1\xi_1+(\alpha_1+\gamma_1)t\right]\left[\gamma_2(\xi_2-s)^{r-1}\right. & \xi_1\le s\le\xi_2, s\ge t,\\ \left.+r\beta_2(1-s)^{r-2}\right], & \\ -\dfrac{1}{\Gamma(r)}(t-s)^{r-1}+\dfrac{1}{d\Gamma(r)}\left[\beta_1-\gamma_1\xi_1+(\alpha_1+\gamma_1)t\right] & \xi_2\le s, s\le t,\\ \left[\alpha_2(1-s)^{r-1}+r\beta_2(1-s)^{r-2}\right], & \\ \dfrac{1}{\Gamma(r)}\left[\beta_1-\gamma_1\xi_1+(\alpha_1+\gamma_1)t\right]\left[\alpha_2(1-s)^{r-1}\right. & \xi_2\le s, s\ge t.\\ \left.+r\beta_2(1-s)^{r-2}\right], & \end{cases} \tag{2.8}$$

Proof. The equation $D^r x(t)+y(t)=0$ has a unique solution

$$x(t)=-\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}y(s)ds+c_0+c_1t, \tag{2.9}$$

where $c_0,c_1\in\mathbb{R}$.

By $\alpha_1x(0)-\beta_1x'(0)=-\gamma_1x(\xi_1)$, $\alpha_2x(1)+\beta_2x'(1)=-\gamma_2x(\xi_2)$, we have

$$\begin{aligned} c_0=&\frac{\gamma_1(\alpha_2+\beta_2+\gamma_2\xi_2)}{d\Gamma(r)}\int_0^{\xi_1}(\xi_1-s)^{r-1}y(s)ds-\frac{1}{d}(\gamma_1\xi_1-\beta_1)\\ &\times\left[\frac{\alpha_2}{\Gamma(r)}\int_0^1(1-s)^{r-1}y(s)ds+\frac{\beta_2}{\Gamma(r)}\int_0^1(1-s)^{r-2}y(s)ds\right.\\ &\left.+\frac{\gamma_2}{\Gamma(r)}\int_0^{\xi_2}(\xi_2-s)^{r-1}y(s)ds\right] \end{aligned}$$

and

$$\begin{aligned} c_1=&\frac{\alpha_1+\gamma_1}{d}\left[\frac{\alpha_2}{\Gamma(r)}\int_0^1(1-s)^{r-1}y(s)ds+\frac{\beta_2}{\Gamma(r-1)}\int_0^1(1-s)^{r-2}y(s)ds\right.\\ &\left.+\frac{\gamma_2}{\Gamma(r)}\int_0^{\xi_2}(\xi_2-s)^{r-1}y(s)ds\right]-\frac{\gamma_1(\alpha_2+\gamma_2)}{d\Gamma(r)}\int_0^{\xi_1}(\xi_1-s)^{r-1}y(s)ds. \end{aligned}$$

Substituting c_0, c_1 into equation (2.9) we find,

$$\begin{aligned}
x(t) =& -\frac{1}{\Gamma(r)}\int_0^1 (t-s)^{r-1}y(s)ds + \frac{\gamma_1(\alpha_2+\beta_2+\gamma_2\xi_2)}{d\Gamma(r)}\int_0^{\xi_1}(\xi_1-s)^{r-1}y(s)ds \\
&- \frac{1}{d}(\gamma_1\xi_1-\beta_1)\Big[\frac{\alpha_2}{\Gamma(r)}\int_0^1(1-s)^{r-1}y(s)ds + \frac{\beta_2}{\Gamma(r)}\int_0^1(1-s)^{r-2}y(s)ds \\
&+ \frac{\gamma_2}{\Gamma(r)}\int_0^{\xi_2}(\xi_2-s)^{r-1}y(s)ds\Big] + \Big[\frac{\alpha_1+\gamma_1}{d}\Big[\frac{\alpha_2}{\Gamma(r)}\int_0^1(1-s)^{r-1}y(s)ds \\
&+ \frac{\beta_2}{\Gamma(r-1)}\int_0^1(1-s)^{r-2}y(s)ds + \frac{\gamma_2}{\Gamma(r)}\int_0^{\xi_2}(\xi_2-s)^{r-1}y(s)ds\Big] \\
&- \frac{\gamma_1(\alpha_2+\gamma_2)}{d\Gamma(r)}\int_0^{\xi_1}(\xi_1-s)^{r-1}y(s)ds\Big]t \\
=& \int_0^1 G(t,s)y(s)ds.
\end{aligned}$$

The proof is complete. □

Throughout this study we will assume the following condition is satisfied:

(H1) $(\alpha_2+(r-1)\beta_2)(\beta_1-\gamma_1\xi_1) \geq d$.

Lemma 2.4. *If (H1) holds, then there exist a constant N such that $0 \leq G(t,s) \leq N(1-s)^{r-2}$, $t,s \in [0,1]$, where*

$$N := \frac{1}{d\Gamma(r)}\left[\gamma_1(\alpha_2+\beta_2+\gamma_1\xi_2) + (\alpha_1+\beta_1)(\gamma_2+\alpha_2+(r-1)\beta_2)\right].$$

Proof. Obviously $G(t,s) \geq 0$,

$$\begin{aligned}
\max_{0\leq t\leq 1} G(t,s) \leq& \frac{\gamma_1}{d\Gamma(r)}\big(\alpha_2+\beta_2+\gamma_1\xi_2-(\alpha_2+\gamma_2)t\big)(\xi_1-s)^{r-1} \\
&+ \frac{\gamma_2}{d\Gamma(r)}\big(\beta_1-\gamma_1\xi_1+(\alpha_1+\gamma_1)t\big)(\xi_2-s)^{r-1} \\
&+ \frac{\alpha_2+(r-1)\beta_2}{d\Gamma(r)}\big(\beta_1-\gamma_1\xi_1+(\alpha_1+\gamma_1)t\big)(1-s)^{r-2} \\
\leq& \frac{\gamma_1}{d\Gamma(r)}(\alpha_2+\beta_2+\gamma_1\xi_2)(1-s)^{r-2} + \frac{\gamma_2}{d\Gamma(r)}\big(\beta_1-\gamma_1\xi_1+\alpha_1+\gamma_1\big) \\
&(1-s)^{r-2} + \frac{\alpha_2+(r-1)\beta_2}{d\Gamma(r)}\big(\beta_1-\gamma_1\xi_1+\alpha_1+\gamma_1\big)(1-s)^{r-2} \\
\leq& \frac{1}{d\Gamma(r)}\Big[\gamma_1(\alpha_2+\beta_2+\gamma_1\xi_2)+(\alpha_1+\beta_1) \\
&\big(\gamma_2+\alpha_2+(r-1)\beta_2\big)\Big](1-s)^{r-2} \\
\leq& N(1-s)^{r-2}.
\end{aligned}$$

The proof is completed. □

Lemma 2.5. *If $0 < s < 1$, $\theta \in (0, \frac{1}{2})$, then there exists a constant Ω such that*

$$G(t,s) \geq \Omega N(1-s)^{r-2}, \tag{2.10}$$

where

$$\Omega := \frac{-d + (\alpha_2 + (r-1)\beta_2)\left(\beta_1 - \gamma_1\xi_1 + min\{\theta, 1-\theta\}(\alpha_1+\gamma_1)\right)}{\gamma_1(\alpha_2+\beta_2+\gamma_1\xi_2) + (\alpha_1+\beta_1)\left(\gamma_2+\alpha_2+(r-1)\beta_2\right)}. \tag{2.11}$$

Proof. We have two cases:
Case 1. For $0 \le s \le t \le 1-\theta$, we get

$$G(t,s) \ge -\frac{1}{\Gamma(r)}(1-s)^{r-2} + \frac{\alpha_2+(r-1)\beta_2}{d\Gamma(r)}\left[\beta_1 - \gamma_1\xi_1 + (\alpha_1+\gamma_1)t\right](1-s)^{r-2}. \tag{2.12}$$

Case 2. For $\theta \le t \le s \le 1$, we get

$$G(t,s) \ge \frac{\alpha_2+(r-1)\beta_2}{d\Gamma(r)}\left[\beta_1 - \gamma_1\xi_1 + (\alpha_1+\gamma_1)t\right](1-s)^{r-2}. \tag{2.13}$$

Hence we have

$$G(t,s) \ge \frac{-d + (\alpha_2+(r-1)\beta_2)\left(\beta_1 - \gamma_1\xi_1 + (\alpha_1+\gamma_1)min\{\theta, 1-\theta\}\right)}{d\Gamma(r)}(1-s)^{r-2}. \tag{2.14}$$

$\square$

Lemma 2.6. *Let $f \in \mathcal{C}([0,1]\times[0,\infty])$, then the problem (1.1) has a unique solution*

$$x(t) = \int_0^1 G(t,s)\varphi^{-1}\left(I^q f(s,x(s))\right)ds. \tag{2.15}$$

Proof. Let $D^r x(t) = g(t)$ and $h = \varphi(g)$, then we have the following problem

$$\begin{aligned} &D^q h(t) + f\left(t, x(t)\right) = 0,\\ &h(0) = 0. \end{aligned} \tag{2.16}$$

By Lemma 2.1, we have

$$h(t) = c_1 t^{q-1} - I^q\left(f\left(t, x(t)\right)\right). \tag{2.17}$$

Since $h(0) = 0$, we get

$$h(t) = -I^q\left(f\left(t, x(t)\right)\right), \quad 0 < t < 1. \tag{2.18}$$

So, the problem

$$\begin{aligned} &D^r x(t) = \varphi^{-1}\left(I^q\left(f\left(t, x(t)\right)\right)\right) = -\varphi^{-1}\left(I^q\left(f(t, x(t))\right)\right),\\ &\alpha_1 x(0) - \beta_1 x'(0) = -\gamma_1 x(\xi_1),\\ &\alpha_2 x(1) + \beta_2 x'(1) = -\gamma_2 x(\xi_2) \end{aligned} \tag{2.19}$$

has a unique solution

$$x(t) = \int_0^1 G(t,s)\varphi^{-1}\left(I^q f(s,x(s))\right)ds. \tag{2.20}$$

$\square$

To prove our results, we need the following fixed point theorems.

Theorem 2.1 ([11]). *Let $E = (E, \| \, . \, \|)$ be a Banach space, $P \subset E$ be a cone in E. Suppose that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Suppose further that $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(1) $\|Tu\| \leq \|u\|$ for $u \in P \cap \partial\Omega_1, \|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$, or

(2) $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_1, \|Tu\| \leq \|u\|$ for $u \in P \cap \partial\Omega_2$

holds. Then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Define $P_c := \{x \in P : \| \, x \, \| < c\}$, $\quad P(\alpha, a, b) := \{x \in P : a \leq \alpha(x), \| \, x \, \| \leq b\}$ where $a, b, c > 0$.

Theorem 2.2 ([17]). *Let $E = (E, \| \, . \, \|)$ be a Banach space, $P \subset E$ a cone of E and $c > 0$ a constant. Suppose that there exists a nonnegative continuous concave functional α on P with $\alpha(x) \leq \| \, x \, \|$ for $x \in \overline{P}_c$ and let $T : \overline{P}_c \to \overline{P}_c$ be a completely continuous map. Assume that there exist a, b, c, d with $0 < a < b < d \leq c$ such that*

(S1) $\{x \in P(\alpha, b, d) : \alpha(x) > b\} \neq \emptyset$ and $\alpha(Tx) > b$ for all $x \in P(\alpha, b, d)$;

(S2) $\| \, Tu \, \| < \alpha$ for all $x \in \overline{P}_a$;

(S3) $\alpha(Tx) > b$ for all $x \in P(\alpha, b, c)$ with $\| \, Tu \, \| > d$.

Then T has at least three fixed points $x_1, x_2, x_3 \in P$ such that $\| \, x_1 \, \| < a, \alpha(x_2) > b, \| \, x_3 \, \| > a$ and $\alpha(x_3) < b$.

3. Main Result

In this section, we prove the existence of multiple positive solutions of the problem (1.1) by using Theorem 2.1 and Theorem 2.2. We consider the Banach space $E = \mathcal{C}([0, 1], \mathbb{R})$ endowed with the norm defined by $\|x\| = \sup_{0 \leq t \leq 1} |x(t)|$. Let $P = \{x \in E : \Omega \, \| \, x \, \| \leq \min_{t \in [\theta, 1-\theta]} x(t)\}$, then P is a cone in E.

Theorem 3.1. *Assume that*

(A1) There exist $t_1, t_2 \in (0, 1)$ such that $\lim_{x \to \infty} \frac{f(t,x)}{x} = \infty$ uniformly on $[t_1, t_2]$,

(A2) R_1 is a positive real number such that $R_1 \geq \varphi^{-1}(\frac{M}{\Gamma(q+1)}) \frac{N}{r-1}$ where $M = \max\{f(t, x) : (t, x) \in [0, 1] \times [0, R_1]\}$,

then the problem (1.1) has at least one positive solution such that $R_1 \leq \| \, x \, \| \leq R_2$.

Proof. It is well known that the existence of positive solution to the boundary value problem (1.1) is equivalent to the existence of fixed point of the operator T. So, we shall seek a fixed point of T in our cone P where the operator $T : E \to E$ is defined by

$$Tx(t) = \int_0^1 G(t, s)\varphi^{-1}\left(I^q\left(f\left(s, x(s)\right)\right)\right) ds, \quad t \in [0, 1]. \tag{3.1}$$

First it is obvious that T is completely continuous. Now we will prove that $T(P) \subset$

P.

$$\begin{aligned} Tx(t) &= \int_0^1 G(t,s)\varphi^{-1}\left(I^q\left(f(s,x(s))\right)\right)ds \\ &\leq \int_0^1 N(1-s)^{r-2}\varphi^{-1}\left(I^q\left(f(s,x(s))\right)\right)ds \\ &\leq \min_{t\in[\theta,1-\theta]}\int_0^1 \frac{1}{\Omega}G(t,s)\varphi^{-1}\left(I^q\left(f(s,x(s))\right)\right)ds. \end{aligned}$$

Therefore

$$\| Tx \| = \frac{1}{\Omega}\int_0^1 \min_{t\in[\theta,1-\theta]} G(t,s)\varphi^{-1}\left(I^q\left(f(s,x(s))\right)\right)ds, \quad t\in[0,1]. \tag{3.2}$$

Thus, we get

$$\Omega \| Tx \| \leq \min_{t\in[\theta,1-\theta]} Tx(t). \tag{3.3}$$

This shows that $T(P)\subset P$.

Let $\Omega_{R_1}=\{x\in E : \| x \| < R_1\}$. We shall prove that $\| Tx \| \leq \| x \|$, for $x\in P\bigcap\partial\Omega_{R_1}$. Then $\| x \| = R_1$. Then, we find for $t\in[0,1]$,

$$\begin{aligned} Tx(t) &= \int_0^1 G(t,s)\varphi^{-1}\left(I^q\left(f(s,x(s))\right)\right)ds \\ &\leq \int_0^1 G(t,s)\varphi^{-1}\left(I^q\left(M\right)\right)ds \\ &\leq \int_0^1 G(t,s)\varphi^{-1}(M)\varphi^{-1}\left(I^q\left(1\right)\right)ds \\ &\leq \int_0^1 N(1-s)^{r-2}\varphi^{-1}(M)\varphi^{-1}\left(I^q\left(1\right)\right)ds \\ &\leq \varphi^{-1}(M)N\int_0^1 (1-s)^{r-2}\varphi^{-1}\left(\frac{1}{\Gamma(q+1)}\right)ds \\ &= \varphi^{-1}(\frac{M}{\Gamma(q+1)})\frac{N}{r-1} \leq R_1, \end{aligned}$$

since

$$I^q\left(1\right) = \frac{1}{\Gamma(q)}\int_0^1 (t-s)^{q-1}ds = \frac{t^q}{\Gamma(q+1)} \leq \frac{1}{\Gamma(q+1)}.$$

Therefore $\| Tx \| \leq R_1 = \| x \|$ for $y\in P\bigcap\partial\Omega_{R_1}$.

Let K be a positive real number such that

$$\frac{\Omega N}{r-1}\varphi^{-1}\left(\frac{KL}{\Gamma(q+1)}\right)R_2^{-1} \geq 1. \tag{3.4}$$

In the view of (A2), there is a constant $L>0$ such that $f(t,x)\geq Kx$, $\forall x\geq L$ and $t\in[t_1,t_2]$.

Now set, $R_2 := R_1 + L$ and define $\Omega_{R_2}=\{x\in E : \| x \| < R_2\}$. Therefore for $x\in P\bigcap\partial\Omega_{R_2}$, we have

$$f(t,x(t)) \geq Kx(t) \geq KL, \quad t\in[t_1,t_2]. \tag{3.5}$$

(3.26) implies that

$$\begin{aligned} Tx(t) &= \int_0^1 G(t,s)\varphi^{-1}\left(I^q\left(f(s,x(s))\right)\right) ds \\ &\geq \int_0^1 \Omega N(1-s)^{r-2}\varphi^{-1}\left(I^q\left(KL\right)\right) ds \\ &= \frac{\Omega N}{r-1}\varphi^{-1}\left(\frac{KL}{\Gamma(q+1)}\right) \geq R_2 = \parallel x \parallel, \end{aligned}$$

so we get $\parallel Tx \parallel \geq R_2 = \parallel x \parallel$.

Then it follows from Theorem 2.1 that T has a fixed point x with $R_1 \leq \parallel x \parallel \leq R_2$.

Hence, x is a positive solution of the problem (1.1) such that $R_1 \leq \parallel x \parallel \leq R_2$. □

Theorem 3.2. *Assume that there exist constants a, b, c, d with $0 < a < b < c = d$ such that the following conditions hold:*

(B1) $f(t,x) \leq \varphi\left(\frac{a(r-1)}{N}\right)\Gamma(q+1), \quad 0 \leq x \leq a,$

(B2) $f(t,x) \geq \varphi\left(\frac{b(r-1)}{\Omega N}\right)\Gamma(q+1), \quad b \leq x \leq c,$

(B3) $f(t,x) \leq \varphi\left(\frac{c(r-1)}{N}\right)\Gamma(q+1), \quad 0 \leq x \leq c.$

Then the boundary value problem (1.1) has at least three positive solutions x_1, x_2, x_3 such that

$$\begin{aligned} &\max_{t\in[0,1]} |x_1(t)| < a, \quad b < \min_{t\in[\theta,1-\theta]} |x_2(t)| < \max_{t\in[0,1]} |x_2(t)| \leq c, \\ &a < \max_{t\in[0,1]} |x_3(t)| \leq c, \quad \min_{t\in[\theta,1-\theta]} |x_3(t)| < b. \end{aligned}$$

Proof. Let we define the nonnegative, continuous concave functional $\alpha : P \to [0,\infty)$ by $\alpha(x) = \min_{t\in[0,1]} |x(t)|$. For each $x \in P$, it is easy to see $\alpha(x) \leq \parallel x \parallel$.

First we show that (S1) of Theorem 2.2 holds. To check the condition (1) of Theorem 2.2, we choose $x_0(t) = \dfrac{b+c}{2}$, for $t \in [0,1]$. It is easy to see that $x_0 \in P$, $\parallel x_0 \parallel = \dfrac{b+c}{2} \leq c$ and $\alpha(x_0) = \dfrac{b+c}{2} > b$. That is $x_0 \in \{x \in P(\alpha, b, d) : \alpha(x) > b\} \neq \emptyset$. Moreover, if $x \in P(\alpha, b, d)$, we have $b \leq x(t) \leq c$ for $t \in [0,1]$. By (B2) and Lemma 2.5, we have

$$\begin{aligned} \alpha(Tx) &= \min_{t\in[\theta,1-\theta]} |(Tx)(t)| \\ &\geq \int_0^1 \Omega N(1-s)^{r-2}\varphi^{-1}\left(I^q\left(f\left(s,x(s)\right)\right)\right) ds \\ &\geq \int_0^1 \Omega N(1-s)^{r-2}\varphi^{-1}\left(I^q\left(\varphi\left(\frac{b(r-1)}{\Omega N}\right)\Gamma(q+1)\right)\right) ds \\ &= \Omega N \frac{b(r-1)}{\Omega N \Gamma(q+1)}\Gamma(q+1)\frac{1}{r-1} = b. \end{aligned}$$

Hence condition (S1) of Theorem 2.2 is satisfied.

If $d = c$, then the condition (S1) of Theorem 2.2 implies the condition (S3) of Theorem 2.2. So condition (S3) of Theorem 2.2 is satisfied.

Next we show that (S2) of Theorem 2.2 holds. If $x \in \overline{P_a}$, then $\| x \| \leq a$. By Lemma 2.4 and (B3), we get

$$\begin{aligned}
\| Tx \| &= \max_{t\in[0,1]} \int_0^1 G(t,s)\varphi^{-1}\left(I^q\left(f\left(s,x(s)\right)\right)\right) ds \\
&\leq \int_0^1 N(1-s)^{r-2}\varphi^{-1}\left(I^q\left(\varphi\left(\frac{a(r-1)}{N}\right)\Gamma(q+1)\right)\right) ds \\
&\leq \int_0^1 N(1-s)^{r-2}\varphi^{-1}\left(\frac{\varphi\left(\frac{a(r-1)}{N}\right)\Gamma(q+1)}{\Gamma(q+1)}\right) ds \\
&= N\frac{c(r-1)}{N\Gamma(q+1)}\Gamma(q+1)\frac{1}{r-1} = a.
\end{aligned}$$

Hence condition (S2) of Theorem 2.2 is satisfied.

In the same way, we can show that if (B3) holds, then $T(\overline{P_c}) \subseteq \overline{P_c}$.

To sum up, all the hypotheses of Theorem 2.2 are satisfied. The proof is completed. □

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