Journal of Applied Analysis and Computation
Volume 7, Number 2, May 2017, 617–631
Website:http://jaac-online.com/
DOI:10.11948/2017038

QUADRATIC TRIGONOMETRIC B-SPLINE GALERKIN METHODS FOR THE REGULARIZED LONG WAVE EQUATION*

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Abstract In this study, a numerical solution of the Regularized Long Wave (RLW) equation is obtained using Galerkin finite element method, based on two and three steps Adams Moulton method for the time integration and quadratic trigonometric B-spline functions for the space integration. After two different linearization techniques are applied, the proposed algorithms are tested on the problems of propagation of a solitary wave and interaction of two solitary waves. For the first test problem, the rate of convergence and the running time of the proposed algorithms are computed and the error norm L_∞ is used to measure the differences between exact and numerical solutions. The three conservation quantities of the motion are calculated to determine the conservation properties of the proposed algorithms for both of the test problems.

Keywords Finite element method, spline approximation, solitary waves equation.

MSC(2010) 65M60, 41A15, 74J35.

1. Introduction

The first observation of solitary wave or wave of translation in shallow water was made in 1834 by John Scott Russell [15]. The solitary wave solutions are assumed to be of the form

$$u(x,t) = f(x - vt),$$

where v is the speed of the wave propagation, and $f(z)$, $f'(z)$, $f''(z) \to 0$ as $z \to \pm\infty$, $z = x - vt$ [21].

In 1895, Korteweg and de Vries studied a partial differential equation (PDE) known as the KdV equation to provide an explanation of the phenomenon observed by Russell [10]. The KdV equation

$$u_t + \varepsilon u u_x + u_{xxx} = 0, \tag{1.1}$$

exhibits special solutions, describing the theory of water waves in shallow channels. These water waves are known as solitons, which are stable and do not disperse with time and they are also solitary waves that are not deformed after collision with other solitons.

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*This work was supported by the Scientific Research Council of Eskişehir Osmangazi University under Project No. 2014-581.

The RLW equation was formulated by Peregrine as an alternative to KdV equation for studying soliton phenomenon with boundary conditions $u \to 0$ as $x \to \pm\infty$ [14]. The regularized long wave (RLW) equation which can be shown in the form

$$u_t + u_x + \varepsilon u u_x - \mu u_{xxt} = 0, \tag{1.2}$$

where ε and μ are positive constants and u, x, t denote the amplitude, spatial and time coordinate, respectively.

The spatial variable x of the problem is restricted over the finite region $a \leq x \leq b$ for the numerical treatment. Note that the space interval must be chosen to fit $u(a,t) = u(b,t) \approx 0$ because of the boundary conditions $u \to 0$ as $x \to \pm\infty$. In this study the RLW equation will be considered with boundary condition over the space region as

$$\begin{aligned} u(a,t) = u(b,t) = 0, \\ u_x(a,t) = u_x(b,t) = 0, \end{aligned} \quad t \in (0,T] \tag{1.3}$$

and the initial condition

$$u(x,0) = f(x) \tag{1.4}$$

will be defined in the numerical experiments section.

Since the RLW equation has been solved analytically only for restricted set of boundary and initial conditions [2], the numerical solution of the RLW equation has been the subject of many papers over the last few years. Galerkin and Petrov Galerkin methods based on different degrees B-spline finite elements have been widely used to find numerical solutions of the RLW or similar equation [1, 3–9, 16–19, 22]. As far as we know, no paper has been found for the numerical solutions of the nonlinear partial differential equation using trigonometric B-spline Galerkin method.

In this paper, the purpose of this study is to obtain numerical solution of the RLW equation based on the application of two Adams Moulton methods for the time discretization and quadratic trigonometric B-spline Galerkin method for the space discretization using different two linearization techniques. After a description of the proposed method is outlined in section 2, the propagation of a solitary wave and interaction of two positive solitary waves test problems are investigated and results are compared with published numerical solutions in terms of norm L_∞ and conservative quantities in the third section.

For computational work, the space-time plane is discretized by grid with the time step Δt and space step h. The exact solution of the unknown function at the grid points is denoted by

$$u(x_m, t_n) = u_m^n, \ m = 0, 1, \ldots, N; \quad n = 0, 1, 2, \ldots,$$

where $x_m = a + mh$, $t_n = n\Delta t$ and the notation U_m^n is used to represent the numerical value of u_m^n.

1.1. Time Discretization

We consider the RLW equation of the form

$$v_t = (u - \mu u_{xx})_t = -(u_x + \varepsilon u u_x). \tag{1.5}$$

For the time discretization of the Eq. (1.5), we use following one-step and two-step Adams-Moulton methods:

$$v^{n+1} = v^n + \frac{\Delta t}{2}\left((v_t)^{n+1} + (v_t)^n\right) + \mathcal{O}(\Delta t^3), \tag{1.6}$$

$$v^{n+1} = v^n + \Delta t\left(\frac{5}{12}(v_t)^{n+1} + \frac{2}{3}(v_t)^n - \frac{1}{12}(v_t)^{n-1}\right) + \mathcal{O}(\Delta t^4). \tag{1.7}$$

The first Adams-Moulton method is also called Crank–Nicolson or trapezium method. The second Adams-Moulton method is typically more accurate than the first method because its order is bigger than the Crank-Nicolson method.

The general form of the above two methods can be written as follows:

$$v^{n+1} = v^n + \Delta t\left(\theta_1(v_t)^{n+1} + \theta_2(v_t)^n + \theta_3(v_t)^{n-1}\right). \tag{1.8}$$

If $\theta_1 = \theta_2 = 1/2$, $\theta_3 = 0$, the method is of order 2 and is called Crank-Nicolson method (CN method) and then if $\theta_1 = 5/12$, $\theta_2 = 2/3$, $\theta_3 = -1/12$, the method is of order 3 and is called Adams Moulton method (AM method). Using the (1.8) for the time discretization of the RLW equation, we have

$$\begin{aligned} &u^{n+1} + \theta_1\Delta t\,(u_x)^{n+1} + \theta_1\Delta t\varepsilon u^{n+1}\,(u_x)^{n+1} - \mu\,(u_{xx})^{n+1} \\ &= u^n - \mu\,(u_{xx})^n - \theta_2\Delta t\,(u_x)^n - \theta_2\Delta t\varepsilon u^n\,(u_x)^n - \theta_3\Delta t\,(u_x)^{n-1} \\ &\quad - \theta_3\Delta t\varepsilon u^{n-1}\,(u_x)^{n-1}. \end{aligned} \tag{1.9}$$

1.2. Quadratic Trigonometric B-spline Galerkin Methods

For the space discretization, the space domain $[a, b]$ is discretized into partitions of N finite elements of equal length h by the knots

$$a = x_0 < x_1 < \ldots < x_{N-1} < x_N = b.$$

Using the recurrence relation given in [11,20], the quadratic trigonometric B-spline functions are defined at the knots as [12]

$$T_m(x) = \frac{1}{\theta}\begin{cases} g^2(x_{m-1}), & x \in [x_{m-1}, x_m)\,, \\ -g(x_{m-1})g(x_{m+1}) - g(x_{m+2})g(x_m), & x \in [x_m, x_{m+1})\,, \\ g^2(x_{m+2}), & x \in [x_{m+1}, x_{m+2})\,, \\ 0, & \text{otherwise}, \end{cases} \tag{1.10}$$

where

$$\theta = \sin\left(\frac{h}{2}\right)\sin(h),$$
$$g(x_m) = \sin\left(\frac{x - x_m}{2}\right).$$

The global approximation to the analytical solution of the problem can be defined by using quadratic trigonometric B-splines as

$$u(x,t) \approx U(x,t) = \sum_{m=-1}^{N} T_m(x)\delta_m(t), \tag{1.11}$$

where the coefficients δ_m are time dependent parameters to be determined from boundary conditions and the quadratic trigonometric B-spline Galerkin form of the RLW equation. T_m $(m = -1(1)N)$ and their first derivatives with respect to space variable x vanish outside the interval $[x_{m-1}, x_{m+2}]$. Since from (1.10) each quadratic trigonometric B-splines covers 3 intervals, each element $[x_m, x_{m+1}]$ is covered by three splines. Therefore over the element $[x_m, x_{m+1}]$, an approximation to the exact solution $u(x,t)$ in terms of quadratic trigonometric B-splines can be written as

$$U(x,t) = \sum_{j=m-1}^{m+1} T_j(x)\delta_j(t) = T_{m-1}\delta_{m-1} + T_m\delta_m + T_{m+1}\delta_{m+1}. \tag{1.12}$$

Using (1.10) and the trial solution (1.12) the values of $U_m = U(x_m, t)$ and their first space derivatives at knots can be written in terms of the parameters δ_m as

$$U_m = \sin(h/2)\csc(h)\left(\delta_{m-1} + \delta_m\right), \tag{1.13}$$

$$U'_m = \frac{1}{2}\csc(h/2)\left(-\delta_{m-1} + \delta_m\right). \tag{1.14}$$

1.2.1. Linearization Technique 1

Applying Galerkin method to Eq. (1.9) with weight function $W(x)$ and then integrating by parts lead to the equation:

$$\begin{aligned}
&\int_a^b W(x)\left(U^{n+1} + \theta_1\Delta t\left(U_x\right)^{n+1} + \theta_1\Delta t\varepsilon U^{n+1}\left(U_x\right)^{n+1}\right)dx \\
&+\mu\int_a^b W_x(x)\left(U_x\right)^{n+1}dx \\
=&\int_a^b W(x)\left(U^n - \theta_2\Delta t\left(U_x\right)^n - \theta_2\Delta t\varepsilon U^n\left(U_x\right)^n - \theta_3\Delta t\left(U_x\right)^{n-1}\right. \\
&\left.-\theta_3\Delta t\varepsilon U^{n-1}\left(U_x\right)^{n-1}\right)dx + \mu\int_a^b W_x(x)\left(U_x\right)^n dx.
\end{aligned} \tag{1.15}$$

Now, identifying the weight function $W(x)$ with the quadratic trigonometric B-spline T_m and using the expression (1.12) in equation (1.15), a fully discrete approximation is obtained over the element $[x_m, x_{m+1}]$ as

$$\begin{aligned}
&\sum_{j=m-1}^{m+1}\left\{\left(\int_{x_m}^{x_{m+1}} T_iT_j dx\right)\delta_j^{n+1} + \mu\left(\int_{x_m}^{x_{m+1}} T'_iT'_j dx\right)\delta_j^{n+1} + \theta_1\Delta t\left(\int_{x_m}^{x_{m+1}} T_iT'_j dx\right)\right. \\
&\left.\times\ \delta_j^{n+1} + \theta_1\Delta t\varepsilon\sum_{k=m-1}^{m+1}\left(\int_{x_m}^{x_{m+1}} T_iT_k\left(\delta_k^{n+1}\right)T'_j dx\right)\delta_j^{n+1}\right\} \\
&-\sum_{j=m-1}^{m+1}\left\{\left(\int_{x_m}^{x_{m+1}} T_iT_j dx\right)\delta_j^n + \mu\left(\int_{x_m}^{x_{m+1}} T'_iT'_j dx\right)\delta_j^n - \theta_2\Delta t\left(\int_{x_m}^{x_{m+1}} T_iT'_j dx\right)\delta_j^n\right. \\
&-\theta_2\Delta t\varepsilon\sum_{k=m-1}^{m+1}\left(\int_{x_m}^{x_{m+1}} T_iT_k\left(\delta_k^n\right)T'_j dx\right)\delta_j^n - \theta_3\Delta t\left(\int_{x_m}^{x_{m+1}} T_iT'_j dx\right)\delta_j^{n-1} \\
&\left.-\theta_3\Delta t\varepsilon\sum_{k=m-1}^{m+1}\left(\int_{x_m}^{x_{m+1}} T_iT_k\left(\delta_k^{n-1}\right)T'_j dx\right)\delta_j^{n-1}\right\},
\end{aligned} \tag{1.16}$$

where i, j and k take only the values $m-1$, m, $m+1$ for this typical element $[x_m, x_{m+1}]$. (1.16) can be written in the matrices form as

$$\begin{aligned}&\left[\boldsymbol{A}^e+\mu\boldsymbol{D}^e+\theta_1\Delta t\boldsymbol{B}^e+\theta_1\Delta t\varepsilon\boldsymbol{C}^e\left(\left(\boldsymbol{\delta}^e\right)^{n+1}\right)\right]\left(\boldsymbol{\delta}^e\right)^{n+1}\\&-\left[\boldsymbol{A}^e+\mu\boldsymbol{D}^e-\theta_2\Delta t\boldsymbol{B}^e-\theta_2\Delta t\varepsilon\boldsymbol{C}^e\left(\left(\boldsymbol{\delta}^e\right)^{n}\right)\right]\left(\boldsymbol{\delta}^e\right)^{n}\\&+\left[\theta_3\Delta t\boldsymbol{B}^e+\theta_3\Delta t\varepsilon\boldsymbol{C}^e\left(\left(\boldsymbol{\delta}^e\right)^{n-1}\right)\right]\left(\boldsymbol{\delta}^e\right)^{n-1},\end{aligned}\tag{1.17}$$

where the element matrices and element parameters are

$$\begin{aligned}&A_{ij}^e=\int_{x_m}^{x_{m+1}}T_iT_jdx,\quad B_{ij}^e=\int_{x_m}^{x_{m+1}}T_iT_j'dx,\\&C_{ij}^e\left(\left(\boldsymbol{\delta}^e\right)^{n+1}\right)=\int_{x_m}^{x_{m+1}}T_iT_k\left(\boldsymbol{\delta}_k^{n+1}\right)T_j'dx,\ D_{ij}^e=\int_{x_m}^{x_{m+1}}T_i'T_j'dx,\\&\left(\boldsymbol{\delta}^e\right)^{n+1}=(\delta_{m-1}^{n+1},\delta_m^{n+1},\delta_{m+1}^{n+1})^T.\end{aligned}$$

The 3×3 element matrices $\boldsymbol{A}^e$, $\boldsymbol{B}^e$ and $\boldsymbol{D}^e$ are independent of the parameters $\boldsymbol{\delta}^e$ and $3\times3\times3$ element matrices $\boldsymbol{C}^e$ depends on the parameters $\boldsymbol{\delta}^e$.

Combining contributions from all elements lead to the nonlinear matrix equation

$$\begin{aligned}&\left[\mathbf{A}+\mu\mathbf{D}+\theta_1\Delta t\mathbf{B}+\theta_1\Delta t\varepsilon\mathbf{C}\left(\boldsymbol{\delta}^{n+1}\right)\right]\boldsymbol{\delta}^{n+1}\\&=\left[\mathbf{A}+\mu\mathbf{D}-\theta_2\Delta t\mathbf{B}-\theta_2\Delta t\varepsilon\mathbf{C}\left(\boldsymbol{\delta}^{n}\right)\right]\boldsymbol{\delta}^{n}\\&\quad-\left[\theta_3\Delta t\mathbf{B}+\theta_3\Delta t\varepsilon\mathbf{C}\left(\boldsymbol{\delta}^{n-1}\right)\right]\boldsymbol{\delta}^{n-1},\end{aligned}\tag{1.18}$$

where global element parameters

$$\boldsymbol{\delta}=\left(\delta_{-1},\delta_0,\ldots,\delta_{N-1},\delta_N\right)^T$$

and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are calculated from the corresponding element matrices $\boldsymbol{A}^e$, $\boldsymbol{B}^e$, $\boldsymbol{C}^e$ and $\boldsymbol{D}^e$.

The pentadiagonal system of equations (1.18) consists of $(N+2)$ equations of $(N+2)$ unknown parameters $(\delta_{-1},\delta_0,\ldots,\delta_{N-1},\delta_N)$. After the first and last equations are deleted in the system (1.18), imposition of the boundary conditions $U(a,x)=U(b,x)=0$ at the both ends of the region yields to eliminate δ_{-1}^{n+1} and δ_N^{n+1} from the above system. Therefore the solution of the pentadiagonal matrix equations with the dimensions $N\times N$ is obtained by way of Thomas algorithms. After initial vector $\mathbf{d}^0=(\delta_{-1}^0,\ldots,\delta_{N-1}^0,\delta_N^0)$ is found with the help of the boundary and initial conditions, $\mathbf{d}^1=(\delta_{-1}^1,\ldots,\delta_{N-1}^1,\delta_N^1)$ unknown vector is obtained using Crank-Nicolson method (CN1). Therefore $\mathbf{d}^{n+1},(n=1,2,\ldots)$ unknown vectors can be found repeatedly by solving the recurrence relation (1.18) using two previous $\mathbf{d}^n$ and $\mathbf{d}^{n-1}$ unknown vectors (AM1). Note that since the system (1.18) is an implicit system, we have proposed the following inner iteration algorithm for all time steps to increase the accuracy of the system:

Step 1: Set $error = 1$ and $\delta_m^* = \delta_m^{n+1}$ in $\mathbf{C}\left(\boldsymbol{\delta}^{n+1}\right)$ and taking $\delta_m^* = \delta_m^n$ find U_m^*,

Step 2: While $error \geq 10^{-10}$ do Steps 3–4,

Step 3: Find U_m^{n+1},

Step 4: Find $\max\limits_m \left|U_m^{n+1} - U_m^*\right|$ and set $\delta_m^* = \delta_m^{n+1}$,

Step 5: Stop and go to next time step.

1.2.2. Linearization Technique 2

One of the main purposes of this work is to use the new linearization technique for the linearization of the terms $(UU_x)^{n+1}$ in the following manner

$$\begin{aligned}(UU_x)^{n+1} \approx & 2\lambda_1\left(U^{n+1}U_x^n + U^nU_x^{n+1} - 2\lambda_1U^nU_x^n\right)\\ & + \lambda_2\left(-U^{n+1}U_x^{n-1} - U^{n-1}U_x^{n+1} + 2U^nU_x^{n-1} + 2U^{n-1}U_x^n - U^{n-1}U_x^{n-1}\right).\end{aligned} \tag{1.19}$$

Using (1.19) in Eq. (1.9), we have

$$\begin{aligned}& U^{n+1} + \theta_1\Delta t\,(U_x)^{n+1} + \theta_1\Delta t\varepsilon\left[2\lambda_1(U_x)^nU^{n+1} - \lambda_2(U_x)^{n-1}U^{n+1}\right.\\ & \left.+2\lambda_1U^n(U_x)^{n+1} - \lambda_2U^{n-1}(U_x)^{n+1}\right] - \mu\,(U_{xx})^{n+1}\\ = & U^n - \mu\,(U_{xx})^n - \theta_2\Delta t\,(U_x)^n + \Delta t\varepsilon(4\lambda_1^2\theta_1 - \theta_2)U^n(U_x)^n - 2\lambda_2\theta_1\Delta t\varepsilon(U^{n-1}\,(U_x)^n\\ & + (U_x)^{n-1}U^n) - \theta_3\Delta t\,(U_x)^{n-1} + \Delta t\varepsilon(\lambda_2\theta_1 - \theta_3)U^{n-1}\,(U_x)^{n-1}.\end{aligned} \tag{1.20}$$

If $\theta_1 = \theta_2 = 1/2, \theta_3 = 0$, $\lambda_1 = 1/2$ and $\lambda_2 = 0$, the proposed time discretization method (CN) is of order 2 and then if $\theta_1 = 5/12$, $\theta_2 = 2/3$, $\theta_3 = -1/12$ and $\lambda_1 = \lambda_2 = 1$, the proposed time discretization method (AM) is of order 3. Applying Galerkin method to Eq. (1.20) with weight function $W(x)$ and then integrating by parts lead to the equation:

$$\begin{aligned}& \int_a^b W(x)\left[U^{n+1} + \theta_1\Delta t\,(U_x)^{n+1} + \theta_1\Delta t\varepsilon\left(2\lambda_1(U_x)^nU^{n+1} - \lambda_2(U_x)^{n-1}U^{n+1}\right.\right.\\ & \left.\left.+2\lambda_1U^n(U_x)^{n+1} - \lambda_2U^{n-1}(U_x)^{n+1}\right)\right]dx + \int_a^b \mu W_x(x)\,(U_x)^{n+1}\,dx\\ = & \mu\int_a^b W_x(x)\,(U_x)^n\,dx + \int_a^b W(x)\left[U^n - \theta_2\Delta t\,(U_x)^n + \Delta t\varepsilon(4\lambda_1^2\theta_1 - \theta_2)U^n(U_x)^n\right.\\ & - 2\lambda_2\theta_1\Delta t\varepsilon \times (U^{n-1}\,(U_x)^n + (U_x)^{n-1}U^n) - \theta_3\Delta t\,(U_x)^{n-1}\\ & \left.+\Delta t\varepsilon(\lambda_2\theta_1 - \theta_3)U^{n-1}\,(U_x)^{n-1}\right]dx.\end{aligned} \tag{1.21}$$

Now, identifying the weight function $W(x)$ with the quadratic trigonometric B-spline T_m and using the expression (1.12) in Eq. (1.21), a fully discrete approximation is obtained over the element $[x_m, x_{m+1}]$ as

$$\begin{aligned}
&\sum_{j=m-1}^{m+1}\Bigg\{\left(\int_{x_m}^{x_{m+1}} T_iT_jdx\right)\delta_j^{n+1}+\mu\left(\int_{x_m}^{x_{m+1}} T_i'T_j'dx\right)\delta_j^{n+1}\\
&+\theta_1\Delta t\left(\int_{x_m}^{x_{m+1}} T_iT_j'dx\right)\delta_j^{n+1}+\theta_1\Delta t\varepsilon\Bigg[2\lambda_1\sum_{k=m-1}^{m+1}\left(\int_{x_m}^{x_{m+1}} T_iT_k'\left(\delta_k^n\right)T_jdx\right)\delta_j^{n+1}\\
&-\lambda_2\sum_{k=m-1}^{m+1}\left(\int_{x_m}^{x_{m+1}} T_iT_k'\left(\delta_k^{n-1}\right)T_jdx\right)\delta_j^{n+1}\\
&+2\lambda_1\sum_{k=m-1}^{m+1}\left(\int_{x_m}^{x_{m+1}} T_iT_k\left(\delta_k^n\right)T_j'dx\right)\delta_j^{n+1}\\
&-\lambda_2\sum_{k=m-1}^{m+1}\left(\int_{x_m}^{x_{m+1}} T_iT_k\left(\delta_k^{n-1}\right)T_j'dx\right)\delta_j^{n+1}\Bigg]\Bigg\}-\sum_{j=m-1}^{m+1}\Bigg\{\left(\int_{x_m}^{x_{m+1}} T_iT_jdx\right)\delta_j^n\\
&+\mu\left(\int_{x_m}^{x_{m+1}} T_i'T_j'dx\right)\delta_j^n-\theta_2\Delta t\left(\int_{x_m}^{x_{m+1}} T_iT_j'dx\right)\delta_j^n+\Delta t\varepsilon(4\lambda_1^2\theta_1-\theta_2)\\
&\times\sum_{k=m-1}^{m+1}\left(\int_{x_m}^{x_{m+1}} T_iT_k\left(\delta_k^n\right)T_j'dx\right)\delta_j^n-2\lambda_2\theta_1\Delta t\varepsilon\sum_{k=m-1}^{m+1}\left(\int_{x_m}^{x_{m+1}} T_iT_k\left(\delta_k^{n-1}\right)T_j'dx\right.\\
&\left.+\int_{x_m}^{x_{m+1}} T_iT_k'\left(\delta_k^{n-1}\right)T_jdx\right)\delta_j^n-\theta_3\Delta t\left(\int_{x_m}^{x_{m+1}} T_iT_j'dx\right)\delta_j^{n-1}\\
&+\Delta t\varepsilon(\lambda_2\theta_1-\theta_3)\sum_{k=m-1}^{m+1}\left(\int_{x_m}^{x_{m+1}} T_iT_k\left(\delta_k^{n-1}\right)T_j'dx\right)\delta_j^{n-1}\Bigg\},
\end{aligned}\tag{1.22}$$

where i, j and k take only the values $m-1$, m, $m+1$ for this typical element $[x_m, x_{m+1}]$. (1.22) can be written in the matrices form as

$$\begin{aligned}
&\Big[\boldsymbol{A}^e+\mu\boldsymbol{D}^e+\theta_1\Delta t\boldsymbol{B}^e+\theta_1\Delta t\varepsilon\Big(2\lambda_1\boldsymbol{E}^e\left((\boldsymbol{\delta}^e)^n\right)-\lambda_2\boldsymbol{E}^e\left((\boldsymbol{\delta}^e)^{n-1}\right)+2\lambda_1\boldsymbol{C}^e\left((\boldsymbol{\delta}^e)^n\right)\\
&-\lambda_2\boldsymbol{C}^e\left((\boldsymbol{\delta}^e)^{n-1}\right)\Big)\Big](\boldsymbol{\delta}^e)^{n+1}-\Big[\boldsymbol{A}^e+\mu\boldsymbol{D}^e-\theta_2\Delta t\boldsymbol{B}^e+\Delta t\varepsilon(4\lambda_1^2\theta_1-\theta_2)\boldsymbol{C}^e\left((\boldsymbol{\delta}^e)^n\right)\\
&-2\lambda_2\theta_1\Delta t\varepsilon\Big(\boldsymbol{C}^e\left((\boldsymbol{\delta}^e)^{n-1}\right)+\boldsymbol{E}^e\left((\boldsymbol{\delta}^e)^{n-1}\right)\Big)\Big](\boldsymbol{\delta}^e)^n\\
&-\Big[-\theta_3\Delta t\boldsymbol{B}^e+\Delta t\varepsilon(\lambda_2\theta_1-\theta_3)\boldsymbol{C}^e\left((\boldsymbol{\delta}^e)^{n-1}\right)\Big](\boldsymbol{\delta}^e)^{n-1},
\end{aligned}\tag{1.23}$$

where the element matrices and element parameters are

$$\begin{aligned}
&A_{ij}^e=\int_{x_m}^{x_{m+1}} T_iT_jdx,\quad B_{ij}^e=\int_{x_m}^{x_{m+1}} T_iT_j'dx,\quad C_{ij}^e\left((\boldsymbol{\delta}^e)^n\right)=\int_{x_m}^{x_{m+1}} T_iT_k\left(\boldsymbol{\delta}_k^n\right)T_j'dx,\\
&D_{ij}^e=\int_{x_m}^{x_{m+1}} T_i'T_j'dx,\quad E_{ij}^e\left((\boldsymbol{\delta}^e)^n\right)=\int_{x_m}^{x_{m+1}} T_iT_k'\left(\boldsymbol{\delta}_k^n\right)T_jdx,\\
&(\boldsymbol{\delta}^e)^n=(\delta_{m-1}^n,\delta_m^n,\delta_{m+1}^n)^T.
\end{aligned}$$

Assembling together contributions from all elements leads to the matrix equation

$$\left[\mathbf{A}+\mu\mathbf{D}+\theta_1\Delta t\mathbf{B}+\theta_1\Delta t\varepsilon\left(2\lambda_1\mathbf{E}\left(\boldsymbol{\delta}^n\right)-\lambda_2\mathbf{E}\left(\boldsymbol{\delta}^{n-1}\right)+2\lambda_1\mathbf{C}\left(\boldsymbol{\delta}^n\right)-\lambda_{\mathbf{2}}\mathbf{C}\left(\boldsymbol{\delta}^{n-1}\right)\right)\right]\boldsymbol{\delta}^{n+1}$$

$$= \left[\mathbf{A}+\mu\mathbf{D}-\theta_2\Delta t\mathbf{B}+\Delta t\varepsilon(4\lambda_1^2\theta_1-\theta_2)\mathbf{C}\left(\boldsymbol{\delta}^n\right)-2\lambda_2\theta_1\Delta t\varepsilon(\mathbf{C}\left(\boldsymbol{\delta}^{n-1}\right)+\mathbf{E}\left(\boldsymbol{\delta}^{n-1}\right))\right]\boldsymbol{\delta}^n$$
$$+\left[-\theta_3\Delta t\mathbf{B}+\Delta t\varepsilon(\lambda_2\theta_1-\theta_3)\mathbf{C}\left(\boldsymbol{\delta}^{n-1}\right)\right]\boldsymbol{\delta}^{n-1}, \tag{1.24}$$

where global element parameters

$$\boldsymbol{\delta}=(\delta_{-1},\delta_0,\ldots,\delta_{N-1},\delta_N)^T$$

and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$ are calculated from the corresponding element matrices $\boldsymbol{A}^e$, $\boldsymbol{B}^e$, $\boldsymbol{D}^e$, $\boldsymbol{C}^e$ and $\boldsymbol{E}^e$.

The pentadiagonal system of equations (1.24) consists of $(N+2)$ equations of $(N+2)$ unknown parameters $(\delta_{-1},\delta_0,\ldots,\delta_{N-1},\delta_N)$. After the first and last equations are deleted in the system (1.24), imposition of the boundary conditions $U(a,x)=U(b,x)=0$ at the both ends of the region yields to eliminate δ_{-1}^{n+1} and δ_N^{n+1} from the system (1.24). Therefore the solution of the pentadiagonal matrix equations with the dimensions $N\times N$ is obtained by way of Thomas algorithms. After initial vector $\mathbf{d}^0=(\delta_{-1}^0,\ldots,\delta_{N-1}^0,\delta_N^0)$ is found with the help of the boundary and initial conditions, $\mathbf{d}^1=(\delta_{-1}^1,\ldots,\delta_{N-1}^1,\delta_N^1)$ unknown vector is obtained using Crank-Nicolson method (CN2). Therefore $\mathbf{d}^{n+1},(n=1,2,\ldots)$ unknown vectors can be found repeatedly by solving the recurrence relation (1.24) using two precious $\mathbf{d}^n$ and $\mathbf{d}^{n-1}$ unknown vectors (AM2).

2. Numerical Experiments

Since an accurate numerical scheme must keep the conservation properties of the RLW equation, we will monitor the three invariants of numerical solution for the equation corresponding to conservation of mass, momentum and energy given by the following integrals [13]:

$$\begin{aligned} I_1 &= \int_{-\infty}^{\infty} u\,dx \approx \int_a^b U\,dx, \\ I_2 &= \int_{-\infty}^{\infty} (u^2+\mu(u_x)^2)dx \approx \int_a^b (U^2+\mu(U_x)^2)dx, \\ I_3 &= \int_{-\infty}^{\infty} \left(u^3+3u^2\right)dx \approx \int_a^b (U^3+3U^2)dx. \end{aligned} \tag{2.1}$$

Integrals for the conservation invariants are computed approximately with the trapezoidal rule for the space interval at all time steps. For the first test problem, accuracy of the proposed algorithms is worked out by measuring error norm L_∞

$$L_\infty = \max_m |u_m - U_m|, \tag{2.2}$$

and the order of convergence is computed with fixed space step by the formula

$$\text{order}=\frac{\log\left|\dfrac{u-U_{\Delta t_m}}{u-U_{\Delta t_{m+1}}}\right|}{\log\left|\dfrac{\Delta t_m}{\Delta t_{m+1}}\right|}, \tag{2.3}$$

where u is the exact solution and $U_{\Delta t_m}$ is the numerical solution with time step Δt_m.

2.1. Motion of Single Solitary Wave for the RLW Equation

The solitary wave theoretical solution of the RLW equation is

$$u(x,t) = 3c\mathrm{sech}^2(k[x - \tilde{x}_0 - vt]), \tag{2.4}$$

and the initial condition of the equation is

$$u(x,0) = 3c\mathrm{sech}^2(k[x - \tilde{x}_0]), \tag{2.5}$$

where $v = 1+\varepsilon c$ is the wave velocity, $3c$ is amplitude of the solitary wave, $\tilde{x}_0$ is peak position of the initially centered wave and $k = \sqrt{\dfrac{\varepsilon c}{4\mu v}}$. This solution corresponds to a solitary wave of magnitude $3c$, initially centered on the position $\tilde{x}_0$ propagating towards the right across the interval $[a, b]$ up to the time T without change of shape at a steady velocity v.

In this test problem, firstly single solitary wave simulation is carried out over the solution domain $-80 \le x \le 100$ in the time period $0 \le t \le 20$ with the parameters $\varepsilon = \mu = 1$, $\tilde{x}_0 = 0$ and the amplitude $3c = 0.3$. Using these parameters and $h = \Delta t = 0.01$, the initial and numerical solutions are drawn in Fig. 1 for visual view of the the solution up to time $t = 20$ for AM2. According to the figure we can say that numerical solution profile keeps its initial form.

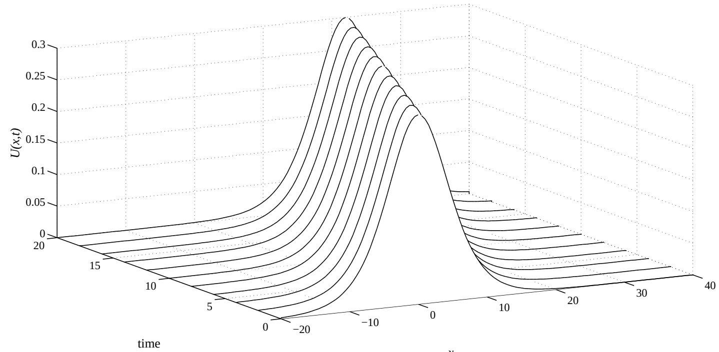

Figure 1. $U(x,t)$ at various time with $h = \Delta t = 0.01$ for AM2.

Three invariants (2.1) for the RLW equation using the initial condition (2.5) can be determined analytically as

$$I_1 = \frac{6c}{k}, \quad I_2 = \frac{12c^2}{k} + \frac{48kc^2\mu}{5}, \quad I_3 = \frac{36c^2}{k}\left(1 + \frac{4c}{5}\right).$$

After the program is run up to time $t = 20$ with fixed space and various time steps error norm L_∞, invariants, Cpu time and Order of convergence for the proposed algorithms are presented in Table 1. According to Table 1, we observe that while $h = 0.01$ and Δt are decreased from 1 to 0.01, the error norms L_∞ are also decreased for the all algorithms. When we compare the performance of the algorithms by their order of convergence, it can be seen that AM1 and AM2 have a cubic order of convergence whereas CN1 and CN2 have a quadratic order of convergence. Comparing the Cpu time of the all proposed algorithms, CN2 and AM2 require up

Table 1. Error norms L_∞ and invariants for a single solitary wave at time $t = 20$ with $c = 0.1$, $-80 \leq x \leq 100$ $h = 0.01$ and various time step.

	Δt	L_∞	I_1	I_2	I_3	Cpu	Order
CN1	1	8.41×10^{-3}	3.9799497	0.8105456	2.5791817	7.769	1.95
	0.5	2.17×10^{-3}	3.9799497	0.8104679	2.5790192	11.20	1.99
	0.2	3.51×10^{-4}	3.9799497	0.8104624	2.5790077	22.31	2.00
	0.1	8.79×10^{-5}	3.9799497	0.8104623	2.5790075	43.63	2.00
	0.05	2.20×10^{-5}	3.9799497	0.8104623	2.5790074	70.61	2.00
	0.02	3.52×10^{-6}	3.9799497	0.8104623	2.5790074	172.0	2.00
	0.01	8.79×10^{-7}	3.9799497	0.8104623	2.5790074	266.5	
CN2	Δt	L_∞	I_1	I_2	I_3	Cpu	Order
	1	8.56×10^{-3}	3.9799497	0.8104906	2.5789980	2.636	1.98
	0.5	2.19×10^{-3}	3.9799497	0.8104660	2.5790129	3.713	1.99
	0.2	3.51×10^{-4}	3.9799497	0.8104624	2.5790077	6.873	2.00
	0.1	8.79×10^{-5}	3.9799497	0.8104623	2.5790075	12.68	2.00
	0.05	2.20×10^{-5}	3.9799497	0.8104623	2.5790074	23.98	2.00
	0.02	3.52×10^{-6}	3.9799497	0.8104623	2.5790074	57.95	2.00
	0.01	8.79×10^{-7}	3.9799497	0.8104623	2.5790074	114.6	
AM1	Δt	L_∞	I_1	I_2	I_3	Cpu	Order
	1	2.05×10^{-3}	3.9799497	0.8129323	2.5871508	8.674	3.00
	0.5	2.56×10^{-4}	3.9799497	0.8107781	2.5800497	12.26	3.00
	0.2	1.64×10^{-5}	3.9799497	0.8104828	2.5790751	24.71	3.00
	0.1	2.05×10^{-6}	3.9799497	0.8104648	2.5790159	48.47	3.00
	0.05	2.57×10^{-7}	3.9799497	0.8104626	2.5790085	76.75	3.00
	0.02	1.64×10^{-8}	3.9799497	0.8104623	2.5790074	194.1	3.02
	0.01	2.03×10^{-9}	3.9799497	0.8104623	2.5790074	286.1	
AM2	Δt	L_∞	I_1	I_2	I_3	Cpu	Order
	1	2.10×10^{-3}	3.9799497	0.8128778	2.5869703	2.434	3.00
	0.5	2.62×10^{-4}	3.9799497	0.8107766	2.5800445	3.744	3.00
	0.2	1.68×10^{-5}	3.9799497	0.8104828	2.5790750	6.880	3.00
	0.1	2.09×10^{-6}	3.9799497	0.8104648	2.5790159	13.20	3.00
	0.05	2.62×10^{-7}	3.9799497	0.8104626	2.5790085	24.38	2.99
	0.02	1.69×10^{-8}	3.9799497	0.8104623	2.5790074	58.30	2.96
	0.01	2.17×10^{-9}	3.9799497	0.8104623	2.5790074	115.6	

to 50% less Cpu time than CN1 and AM1. Finally according to the table, we can see that three invariants I_1, I_2 and I_3 remain almost constants during the computer run for all algorithms so that propagation of the solitary wave represented faithfully. Absolute error (analytical–numerical) distributions for all of the proposed methods are drawn at time $t = 20$ in Fig. 2. As seen from all of the figures, the maximum errors occurred at the middle of the interval and are compatible with Table 1.

Comparisons are made with several previous works listed in Table 2 and Table 3. The present methods especially AM1 and AM2 provide smaller errors than the results of papers [3–5,17,22] for solitary wave amplitudes 0.3 and 0.09. In the table it can also be seen the effect of changing the range of x from $-40 \leq x \leq 60$ to $-80 \leq x \leq 100$.

Finally when ε and μ are much smaller than 1, error norms L_∞.are presented in Table 4 at time $t = 20$. As seen from the table, the proposed all of the methods produced almost the same results while ε and μ are reducing from 1 to 1/256.

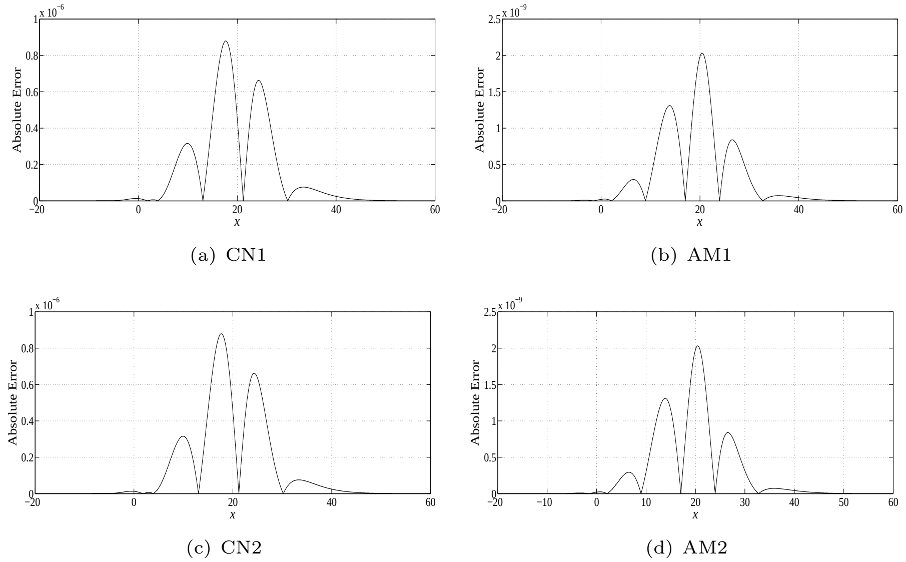

Figure 2. Absolute errors with $h = \Delta t = 0.01$.

Table 2. Solitary wave amplitude= 0.3 at $t = 20$ with $h = 0.125, \Delta t = 0.1$.

	$L_\infty \times 10^5$	I_1	I_2	I_3
CN1$(-80 \leq x \leq 100)$	8.78896	3.9799498	0.8104273	2.5790075
CN2$(-80 \leq x \leq 100)$	8.79019	3.9799498	0.8104273	2.5790075
AM1$(-40 \leq x \leq 60)$	1.26845	3.9798823	0.8104298	2.5790159
AM1$(-80 \leq x \leq 100)$	0.20615	3.9799498	0.8104625	2.5790074
AM2$(-40 \leq x \leq 60)$	1.26845	3.9798822	0.8104298	2.5790159
AM2$(-80 \leq x \leq 100)$	0.21010	3.9799498	0.8104298	2.5790159
[22]$(-40 \leq x \leq 60)$	25.398	3.97989	0.80925	2.57501
[22]$(-80 \leq x \leq 120, h = 0.2)$	14.240	3.97829	0.80983	2.57692
[5]$(-40 \leq x \leq 60)$	8.6	3.97988	0.810465	2.57901
[4]QBGM1$(-40 \leq x \leq 60)$	7.3	3.97988	0.81046	2.57900
[4]QBGM2$(-40 \leq x \leq 60)$	12.8	3.97988	0.81046	2.57900
[17]$(-40 \leq x \leq 60)$	7.344	3.9798879	0.8104622	2.5790063
[3]$(-40 \leq x \leq 60)$	19.8	3.98206	0.811164	2.58133
Exact		3.9799497	0.8104625	2.5790074

2.2. Interaction of Two Solitary Waves for RLW Equation

We consider interaction of two solitary waves using the following initial condition

$$u(x,0) = 3c_1\mathrm{sech}^2(k_1[x-\tilde{x}_1]) + 3c_2\mathrm{sech}^2(k_2[x-\tilde{x}_2]), \tag{2.6}$$

where $k_i = \sqrt{\dfrac{\varepsilon c_i}{4\mu\,(1+\varepsilon c_i)}}, i = 1, 2.$

To ensure an interaction of two solitary waves and for the purpose of comparing with the earlier works, all of the computations are done for the parameters $\varepsilon = \mu = 1$, $c_1 = 0.2$, $c_2 = 0.1$, $\tilde{x}_1 = -177$ and $\tilde{x}_2 = -147$ over the region $-200 \leq x \leq 600$. These parameters provide two well separated solitary waves of magnitudes 0.6 and

Table 3. Solitary wave amplitude= 0.09 at $t = 20$ with $h = 0.125, \Delta t = 0.1$.

	$L_\infty \times 10^4$	I_1	I_2	I_3
CN1($-40 \le x \le 60$)	4.31512	2.1045731	0.1273043	0.3888024
CN1($-100 \le x \le 120$)	0.04381	2.1094073	0.1272998	0.3888060
CN2($-40 \le x \le 60$)	4.31512	2.1045708	0.1273043	0.3888024
CN2($-100 \le x \le 120$)	0.04380	2.1094073	0.1272998	0.3888060
AM1($-40 \le x \le 60$)	4.31512	2.1045735	0.1273044	0.3888024
AM1($-100 \le x \le 120$)	0.00070	2.1094073	0.1272999	0.3888061
AM2($-40 \le x \le 60$)	4.31512	2.1045667	0.1273044	0.3888024
AM2($-100 \le x \le 120$)	0.00070	2.1094073	0.1272999	0.3888061
[22]($-40 \le x \le 60$)	1.2464	2.10641	0.12723	0.38859
[22]($-80 \le x \le 120, h = 0.1$)	0.1193	2.10938	0.38879	0.02746
[4]QBGM1($-40 \le x \le 60$)	2.05	2.10460	0.12730	0.38880
[4]QBGM2($-40 \le x \le 60$)	2.07	2.10457	0.12730	0.38880
[17]($-40 \le x \le 60$)	1.9806	2.1047075	0.1273006	0.3888025
[3]($-40 \le x \le 60$)	1.98	2.10906	0.127305	0.388815
Exact		2.1094075	0.1273017	0.3888060

Table 4. Error norms L_∞ for a single solitary wave at time $t = 20$ with $h = 0.125$, $\Delta t = 0.1$.

	$c = 0.03, -100 \le x \le 120$				$c = 0.1, -80 \le x \le 100$			
$\varepsilon = \mu$	CN1$\times 10^6$	CN2$\times 10^6$	AM1$\times 10^8$	AM2$\times 10^8$	CN1$\times 10^5$	CN2$\times 10^5$	AM1$\times 10^6$	AM2$\times 10^6$
1/256	3.98	3.99	7.01	7.01	8.10	8.10	2.53	2.53
1/128	3.99	3.99	7.01	7.01	8.13	8.13	2.53	2.53
1/64	3.99	3.99	7.01	7.01	8.18	8.18	2.53	2.53
1/32	4.00	4.00	7.02	7.02	8.27	8.27	2.53	2.53
1/16	4.02	4.02	7.04	7.03	8.43	8.43	2.51	2.51
1/8	4.06	4.06	7.05	7.05	8.63	8.63	2.45	2.46
1/4	4.12	4.12	7.07	7.06	8.79	8.79	2.34	2.35
1/2	4.23	4.23	7.06	7.06	8.83	8.83	2.19	2.22
1	4.38	4.38	7.01	7.03	8.79	8.79	2.06	2.10

0.3 and peak positions of them are located at $x = -177$ and -147. The analytical invariants can be found as

$$I_1 = \frac{6c_1}{k_1} + \frac{6c_2}{k_2}, \; I_2 = \frac{12c_1^2}{k_1} + \frac{48k_1c_1^2\mu}{5} + \frac{12c_2^2}{k_2} + \frac{48k_2c_2^2\mu}{5},$$
$$I_3 = \frac{36c_1^2}{k_1}\left(1 + \frac{4c_1}{5}\right) + \frac{36c_2^2}{k_2}\left(1 + \frac{4c_2}{5}\right).$$

The program is run until $t = 400$ with $h = 0.12$, $\Delta t = 0.1$ and numerical solutions of $U(x, t)$ at several times are drawn for visual views of the solutions in Fig. 3 for the algorithm AM2. The interaction process can be observed clearly from the figure. The nonlinear interaction takes place about time 200. Then, two solitary waves regain their original shapes after the interaction. At $t = 400$, the amplitude of the larger wave is 0.5999903 at the point $x = 311.56$, whereas the amplitude of the smaller wave is 0.2999681 at the point $x = 281.56$. The absolute difference in amplitude is 0.0000319 for the smaller wave and 0.0000601 for the larger wave. Table 5 displays a comparison of the values of the invariants obtained by the proposed methods CN1, CN2, AM1, AM2 with those obtained in [3, 5] at some selected time t.

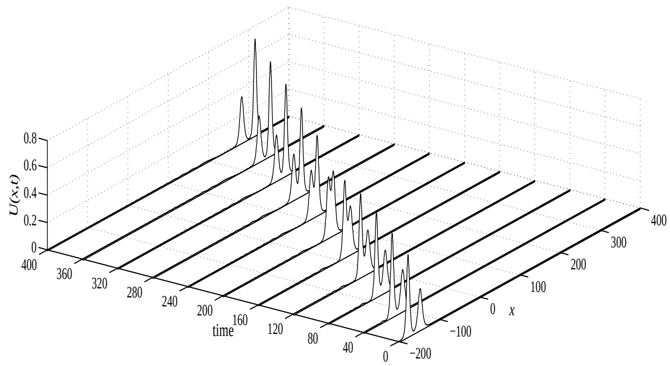

Figure 3. Interaction of two solitary waves.

Table 5. Invariants for interaction of two solitary waves with amplitude 0.6 and 0.3.

	[5]			[3]			CN1		
t	I_1	I_2	I_3	I_1	I_2	I_3			
0	9.8583	3.2448	10.7783	9.8586	3.2449	10.7788	9.8582	3.2446	10.7783
80	9.8583	3.2448	10.7783	9.8683	3.2475	10.7872	9.8582	3.2446	10.7783
160	9.8583	3.2449	10.7785	9.8751	3.2506	10.7979	9.8581	3.2448	10.7785
240	9.8583	3.2449	10.7785	9.8825	3.2544	10.8109	9.8580	3.2448	10.7785
320	9.8583	3.2448	10.7783	9.8883	3.2569	10.8197	9.8579	3.2446	10.7781
400	9.8583	3.2448	10.7782	9.8930	3.2585	10.8251	9.8577	3.2445	10.7779
	CN2			AM1			AM2		
0	9.8582	3.2446	10.7783	9.8582	3.2446	10.7783	9.8582	3.2446	10.7783
80	9.8582	3.2446	10.7784	9.8582	3.2448	10.7789	9.8582	3.2448	10.7789
160	9.8581	3.2448	10.7787	9.8581	3.2450	10.7793	9.8581	3.2450	10.7794
240	9.8580	3.2449	10.7787	9.8580	3.2450	10.7795	9.8581	3.2450	10.7796
320	9.8580	3.2446	10.7784	9.8579	3.2451	10.7799	9.8580	3.2451	10.7801
400	9.8579	3.2446	10.7783	9.8578	3.2452	10.7805	9.8579	3.2453	10.7807

3. Conclusion

In this study, four numerical algorithms for the numerical solution of the RLW equation have been presented using Galerkin method based on quadratic trigonometric B-splines as weight and trial functions for space discretization and one-two step Adams-Moulton method for time discretization. The proposed methods are tested on the propagation of single solitary wave and the interaction of two solitary waves. To see the accuracy of the methods error norms L_∞ for the first test problem and conservation quantities for both of the test problems are documented based on the obtained results. To compare all the proposed methods, AM2 gives accurate, reliable results and less Cpu time for the RLW equation. Also all of the methods have an advantage due to their small matrix operations. Therefore, the obtained results show that proposed algorithms, especially AM2, exhibit high accuracy and efficiency in both conservation of the invariants and error norm for the numerical solution of the RLW equation.

4. Acknowledgements

This work was supported by the Scientific Research Council of Eskişehir Osmangazi University under Project No. 2014-581.

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