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EXISTENCE OF GENERALIZED HOMOCLINIC SOLUTIONS OF A COUPLED KDV-TYPE BOUSSINESQ SYSTEM UNDER A SMALL PERTURBATION*

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Abstract This paper considers the coupled KdV-type Boussinesq system with a small perturbation $u_{xx} = 6cv - 6u - 6uv + \varepsilon f(\varepsilon, u, u_x, v, v_x)$, $v_{xx} = 6cu - 6v - 3u^2 + \varepsilon g(\varepsilon, u, u_x, v, v_x)$, where $c = 1 + \mu$, $\mu > 0$ and ε are small parameters. The linear operator has a pair of real eigenvalues and a pair of purely imaginary eigenvalues. We first change this system into an equivalent system with dimension 4, and then show that its dominant system has a homoclinic solution and the whole system has a periodic solution if the perturbation functions g and h satisfy some conditions. By using the contraction mapping theorem, the perturbation theorem, and the reversibility, we theoretically prove that this homoclinic solution, when higher order terms are added, will persist and exponentially approach to the obtained periodic solution (called generalized homoclinic solution) for small ε and $\mu > 0$.

Keywords Generalized homoclinic solution, coupled KdV-type Boussinesq system, reversibility.

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1. Introduction

Bona, Chen and Saut [5,6] derived a four-parameter family of Boussinesq systems

$$\begin{aligned} &v_t + u_\xi + (uv)_\xi + au_{\xi\xi\xi} - bv_{t\xi\xi} = 0, \\ &u_t + v_\xi + uu_\xi + cv_{\xi\xi\xi} - du_{t\xi\xi} = 0 \end{aligned} \tag{1.1}$$

to describe the two-way propagation of small amplitude gravity waves on the surface of water in a canal or near the shore with a flat bottom, where v represents the elevation from the equilibrium position and u represents the horizontal velocity in the flow at height βh (h is the undisturbed depth of the liquid and $\beta \in [0, 1]$). Here the parameters a, b, c and d are required to satisfy the following consistency conditions

$$a + b = \frac{1}{2}(\beta^2 - \frac{1}{3}), \quad c + d = \frac{1}{2}(1 - \beta^2) \geq 0,$$

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which implies $a + b + c + d = \frac{1}{3}$. This system has also be obtained in two space dimensions over a variable bottom for the generation and propagation of tsunami waves [23]. (1.1) has been studied mathematically and numerically in many literatures and numerous interesting results have been obtained (for example, see [1–4, 6, 9–12, 16, 18, 22, 25, 26]).

Bona and Chen [4] pointed out that if $b = d = \frac{1}{6}$, $a = c = 0$, the system (1.1) is a good candidate to model the two-dimensional surface water waves. When $b = d = 0$, the system (1.1) become a KdV-type Boussinesq system and [1] investigated its boundary stabilization posed on a bounded domain with the condition $a = c > 0$. If $a = c = \frac{1}{6}$ and $b = d = 0$, a purely coupled KdV-type Boussinesq system is given in [5] as

$$v_t + u_\xi + (uv)_\xi + \tfrac{1}{6}u_{\xi\xi\xi} = 0, \qquad u_t + v_\xi + uu_\xi + \tfrac{1}{6}v_{\xi\xi\xi} = 0, \tag{1.2}$$

which is free of the presumption of unidirectionality, while the classical coupled KdV equation assumes that the waves travel only in one direction. There are a lot of research papers devoted to the study of various properties of the system (1.2). For example, [24] discussed the exponential decay of the total energy; [6] considered its Hamiltonian structure and its initial-value problem; [7, 8] stressed that (1.2) does not possess classical solitary waves decaying to zero at infinity, and they investigated numerically its generalized solitary wave solutions—solitary wave solutions exponentially approaching to a periodic solution with a small amplitude at infinity. In this paper, we study the system (1.2) and theoretically prove the existence of generalized solitary wave solutions.

Let $x = \xi - ct$ where the constant c is the speed. Integrating the system (1.2) and taking the constant of integration equal to zero, we obtain the ordinary differential system

$$u_{xx} = 6cv - 6u - 6uv, \qquad v_{xx} = 6cu - 6v - 3u^2. \tag{1.3}$$

Since the real world always has some small noises or disturbances, here we are specially interested in the solutions of the system (1.3) under small perturbations, that is, we will focus on the following system

$$\begin{aligned} u_{xx} &= 6cv - 6u - 6uv + \varepsilon f(\varepsilon, u, u_x, v, v_x), \\ v_{xx} &= 6cu - 6v - 3u^2 + \varepsilon g(\varepsilon, u, u_x, v, v_x), \end{aligned} \tag{1.4}$$

where ε is a small parameter and the conditions for the smooth real functions f and g will be given as we need (see (1.5) or Section 2). The linearized operator of (1.4) at the origin has two pairs of purely imaginary eigenvalues for $c < 1$; a double eigenvalue 0 and a pair of purely imaginary eigenvalues for $c = 1$; and a positive eigenvalue, a negative eigenvalue and a pair of purely imaginary eigenvalues for $c > 1$. As we know, a bifurcation will appear if an eigenvalue transversely cross the purely imaginary axis. In this paper, we are interested in the travelling speed c close to 1. Using a dynamical system approach, we will rigorously prove the existence of a generalized homoclinic solution—homoclinic solutions exponentially tending to a periodic solution for $c > 1$ but close to 1, which corresponds to a generalized solitary wave solution of (1.2) under a small perturbation. We would like to mention that Lombardi [19–21] has obtained some good results about the generalized homoclinic solutions for the general system with the aid of the norm form theory and the complex analysis. Our method is basically similar to ones in [19–21]. The main theorem can be stated as follows.

Theorem 1.1. *Write $c = 1 + \mu$ ($\mu > 0$) and assume that*

$$\begin{aligned}&f(\varepsilon, 0, 0, 0, 0) = 0, \qquad g(\varepsilon, 0, 0, 0, 0) = 0,\\ &f(\varepsilon, u, -u_x, v, -v_x) = f(\varepsilon, u, u_x, v, v_x),\ \ g(\varepsilon, u, -u_x, v, -v_x) = g(\varepsilon, u, u_x, v, v_x).\end{aligned} \tag{1.5}$$

There exist constants μ_0 and $\gamma > 0$ such that for each $\mu \in (0, \mu_0)$, if we suppose that

$$\gamma = O(\mu^{17/12}), \qquad \varepsilon = O(\mu^{17/12+\alpha_2})$$

with a positive constant α_2, then the system (1.4) has an even generalized homoclinic solution

$$\begin{aligned}u(x) =& 2\mu \mathrm{sech}^2\left(\frac{\sqrt{6\mu}}{2}x\right) + \frac{\gamma}{\sqrt{3}}\varsigma(x)\cos\left(2\sqrt{3}(1+r_1)(x+\theta)\right)\\ &+ \mathcal{D}_1(x;\mu) + \mathcal{T}_1(x;\mu),\\ v(x) =& 2\mu \mathrm{sech}^2\left(\frac{\sqrt{6\mu}}{2}x\right) - \frac{\gamma}{\sqrt{3}}\varsigma(x)\cos\left(2\sqrt{3}(1+r_1)(x+\theta)\right)\\ &+ \mathcal{D}_2(x;\mu) + \mathcal{T}_2(x;\mu),\end{aligned}$$

where the phase shift θ is of order $O(\mu^{1/24})$, $\varsigma(x)$ is a smooth even cut-off function with $\varsigma(x) = 0$ for $|x| \le 1$ and $\varsigma(x) = 1$ for $|x| \ge 2$. Here $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{T}_1$ and $\mathcal{T}_2$ are smooth functions in their arguments, $\mathcal{T}_1$ and $\mathcal{T}_2$ are periodic with a period $\pi/(\sqrt{3}(1+r_1))$ for some constant $r_1 = O(\mu)$, and $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy uniformly with respect to the parameter μ that

$$|\mathcal{D}_1(x;\mu)| + |\mathcal{D}_2(x;\mu)| \le M\mu^{4/3}e^{-\nu|x|}, \qquad |\mathcal{T}_1(x;\mu)| + |\mathcal{T}_2(x;\mu)| \le M\mu^{11/6}$$

for $x \in \mathbf{R}$ and some fixed constant $\nu \in (\frac{\sqrt{6\mu}}{2}, \sqrt{6\mu})$, where M is a generic constant.

The paper is organized as follows. Section 2 changes the system (1.4) into a system with a dimension 4. A homoclinic solution of the dominant system is given. In Section 3, some lemmas are presented. The contraction mapping theorem and the perturbation method are applied to prove that the obtained homoclinic solution deforms to a generalized homoclinic solution exponentially approaching to a periodic solution for $x \in [0, \infty)$. In Section 4, using the reversibility and adjusting the phase shift, we extend this generalized homoclinic solution to $x \in (-\infty, \infty)$, which yields the proof of Theorem 1.1. Section 5 yields the proofs of two lemmas left in previous sections.

Throughout this paper, M denotes a positive constant and $B = O(C)$ means that $|B| \le M|C|$.

2. Homoclinic solutions

Let $u_1 = u_x$ and $v_1 = v_x$, which changes (1.4) into

$$\begin{aligned} u_x &= u_1, \\ u_{1x} &= -6u + 6v + 6\mu v - 6uv + \varepsilon f(\varepsilon, u, u_1, v, v_1), \\ v_x &= v_1, \\ v_{1x} &= 6u - 6v + 6\mu u - 3u^2 + \varepsilon g(\varepsilon, u, u_1, v, v_1), \end{aligned} \tag{2.1}$$

where $c = 1 + \mu$ is used and μ is a small parameter. Under the assumption (1.5), we know that the origin is an equilibrium and the system (2.1) is reversible with a reverser S defined by

$$S(u, u_1, v, v_1) = (u, -u_1, v, -v_1), \tag{2.2}$$

that is, $S(u, u_1, v, v_1)(-x)$ is also a solution whenever $(u, u_1, v, v_1)(x)$ is. If $S(u, u_1, v, v_1)(-x) = (u, u_1, v, v_1)(x)$, we call the solution (u, u_1, v, v_1) reversible. This means that $u(x)$ and $v(x)$ are even functions, and $u_1(x)$ and $v_1(x)$ are odd functions. The reversibility will play an important role in the existence of the generalized homoclinic solution.

For $\mu = 0$ and $\varepsilon = 0$, the linear operator of the system (2.1) has a double eigenvalue 0 and a pair of purely imaginary eigenvalues $\pm 2\sqrt{3}i$. The corresponding eigenvectors and the generalized eigenvectors are given by

$$\begin{aligned} &\eta_1 = (1, 0, 1, 0)^T, && \eta_2 = (0, 1, 0, 1)^T, \\ &\eta_3 = \left(\frac{i}{2\sqrt{3}}, -1, -\frac{i}{2\sqrt{3}}, 1\right)^T, && \eta_4 = \bar{\eta}_3 = \left(-\frac{i}{2\sqrt{3}}, -1, \frac{i}{2\sqrt{3}}, 1\right)^T, \end{aligned} \tag{2.3}$$

and they satisfy

$$S\eta_1 = \eta_1, \quad S\eta_2 = -\eta_2, \quad S\eta_3 = -\eta_4, \quad S\eta_4 = -\eta_3. \tag{2.4}$$

Note that the solution of (2.1) can be expressed in terms of the above vectors. Since the system (2.1) is real and η_3, η_4 are complex, we let $U = (u, u_1, v, v_1)^T$ and

$$U = A\eta_1 + B\eta_2 - iV_1(\eta_3 - \eta_4) + V_2(\eta_3 + \eta_4). \tag{2.5}$$

Then (2.1) is equivalent to the real system for (A, B, V_1, V_2), which is

$$\begin{aligned} A_x &= B, \\ B_x &= 6\mu A - \frac{9}{2}A^2 - \sqrt{3}AV_1 + \frac{1}{2}V_1^2 + \varepsilon \tilde{f}(\varepsilon, A, B, V_1, V_2), \\ V_{1x} &= -2\sqrt{3}V_2, \\ V_{2x} &= 2\sqrt{3}V_1 + \sqrt{3}\mu V_1 + \frac{3}{4}A^2 - \frac{\sqrt{3}}{2}AV_1 - \frac{3}{4}V_1^2 + \varepsilon \tilde{g}(\varepsilon, A, B, V_1, V_2), \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} \tilde{f}(\varepsilon, A, B, V_1, V_2) &= \frac{1}{2}\Big(f(\varepsilon, u, u_1, v, v_1) + g(\varepsilon, u, u_1, v, v_1)\Big), \\ \tilde{g}(\varepsilon, A, B, V_1, V_2) &= -\frac{1}{4}\Big(f(\varepsilon, u, u_1, v, v_1) - g(\varepsilon, u, u_1, v, v_1)\Big). \end{aligned} \tag{2.7}$$

By (2.2), (2.4) and (2.5), the reverser S is now given by

$$S(A, B, V_1, V_2) = (A, -B, V_1, -V_2). \tag{2.8}$$

The assumption (1.5) implies

$$\begin{aligned} \tilde{f}(\varepsilon, A, -B, V_1, -V_2) &= \tilde{f}(\varepsilon, A, B, V_1, V_2), \\ \tilde{g}(\varepsilon, A, -B, V_1, -V_2) &= \tilde{g}(\varepsilon, A, B, V_1, V_2). \end{aligned} \tag{2.9}$$

Thus, the system (2.6) is reversible with the reverser S. Symbolically, the system (2.6) can be written as

$$\frac{d\tilde{U}}{dx} = L\tilde{U} + N_1(\tilde{U}) + N_2(\tilde{U}) + \varepsilon R(\varepsilon, \tilde{U}), \tag{2.10}$$

where $\tilde{U} = (A, B, V_1, V_2)^T$, and

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 6\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\sqrt{3} \\ 0 & 0 & 2\sqrt{3} & 0 \end{pmatrix}, \qquad N_1(\tilde{U}) = \begin{pmatrix} 0 \\ -\frac{9}{2}A^2 \\ 0 \\ 0 \end{pmatrix},$$

$$N_2(\tilde{U}) = \begin{pmatrix} 0 \\ -\sqrt{3}AV_1 + \frac{1}{2}V_1^2 \\ 0 \\ \sqrt{3}\mu V_1 + \frac{3}{4}A^2 - \frac{\sqrt{3}}{2}AV_1 - \frac{3}{4}V_1^2 \end{pmatrix}, \; R(\varepsilon, \tilde{U}) = \begin{pmatrix} 0 \\ \tilde{f}(\varepsilon, A, B, V_1, V_2) \\ 0 \\ \tilde{g}(\varepsilon, A, B, V_1, V_2) \end{pmatrix}. \tag{2.11}$$

The dominant system of (2.6) is

$$\frac{d\tilde{U}}{dx} = L\tilde{U} + N_1(\tilde{U}), \tag{2.12}$$

which has a homoclinic solution

$$H(x) = (2\mu \mathrm{sech}^2 \frac{\sqrt{6\mu}}{2} x, -2\sqrt{6}\mu^{3/2} \mathrm{sech}^2 \frac{\sqrt{6\mu}}{2} x \tanh \frac{\sqrt{6\mu}}{2} x, 0, 0)^T \tag{2.13}$$

with

$$SH(-x) = H(x). \tag{2.14}$$

Moreover $H(x)$ satisfies the following inequality

$$|H(x)| \leq M\mu e^{-\sqrt{6\mu}|x|}, \qquad x \in (-\infty, \infty). \tag{2.15}$$

In Section 3 and Section 4, we will prove that this homoclinic solution deforms into a generalized homoclinic solution for the whole system (2.10).

3. Generalized homoclinic solution for $x \in [0, \infty)$

In this section, by using the contraction mapping theorem, we will demonstrate that for $x \in [0, \infty)$, (2.10) has a generalized homoclinic solution exponentially approaching to a periodic solution $X_{\mu,\varepsilon,\gamma}(x)$ as $x \to \infty$.

First we look for the periodic solution $X_{\mu,\varepsilon,\gamma}(x)$. Let $\tau = 2\sqrt{3}(1 + r_1)x$ for a constant r_1 to be determined. By the reversibility defined in (2.8), we can assume that the reversible periodic solution of (2.10) with a period 2π has the form

$$(A, B, V_1, V_2) = \left(\sum_{n=0}^{\infty} A_n \cos n\tau, \sum_{n=1}^{\infty} B_n \sin n\tau, \sum_{n=0}^{\infty} V_{1,n} \cos n\tau, \sum_{n=1}^{\infty} V_{2,n} \sin n\tau\right).$$

Choose $V_{1,1}$ as a small positive real parameter (We use γ to denote $V_{1,1}$ in the following). Plugging the above expressions into (2.10) and making the coefficient of each term in the Fourier series equal, together with the contraction mapping theorem, we can solve for r_1, A_n, B_n, $V_{1,n}$ $(n \neq 1)$ and $V_{2,n}$ as functions of $(x, \mu, \varepsilon, \gamma)$. The result is given by the following lemma.

Lemma 3.1. *There exists a positive constant μ_0 such that for each $\mu \in (0, \mu_0]$, if*

$$\varepsilon = O(\mu^{1+\alpha_1+\alpha_2}), \quad \gamma = O(\mu^{1+\alpha_1}) \tag{3.1}$$

with any positive constants α_1 and α_2, then the system (2.10) has a reversible smooth periodic solution

$$X_{\mu,\varepsilon,\gamma}(x) = (A_p(\mu, \varepsilon, \gamma), B_p(\mu, \varepsilon, \gamma), V_{1p}(\mu, \varepsilon, \gamma), V_{2p}(\mu, \varepsilon, \gamma))^T(x) \tag{3.2}$$

satisfying

$$\|X_{\mu,\varepsilon,\gamma}[j](x)\|_m \leq M\mu^{-1}\gamma^2, \quad \|X_{\mu,\varepsilon,\gamma}[k](x)\|_m \leq M\gamma, \quad j = 1, 2, \ k = 3, 4, \tag{3.3}$$

and r_1 is a smooth function of $(\mu, \varepsilon, \gamma)$ such that

$$|r_1| \leq M\mu. \tag{3.4}$$

Here $f[j]$ denotes the j-th component of f, and the norm $\|\cdot\|$ denotes $C_B^m(R)$-norm, which is a space of continuously differentiable functions up to order m with a supremum norm and any positive integer m.

The proof is given in Section 5.

Assume that the solution of the system (2.10) has the following form

$$\hat{U}(x) = H(x) + Z(x) + \varsigma(x)X_{\mu,\varepsilon,\gamma}(x + \theta), \tag{3.5}$$

where $H(x)$ and $X_{\mu,\varepsilon,\gamma}$ are defined in (2.15) and (3.2) respectively, the phase shift $\theta \in S^1 = [-\pi, \pi]$ is a constant, the cut-off function $\varsigma(x)$ is in $C^\infty(\mathrm{R}, \mathrm{R})$ satisfying $0 \leq \varsigma(x) \leq 1$ and

$$\varsigma(x) = \begin{cases} 1, & |x| \geq 2, \\ 0, & |x| \leq 1, \end{cases} \tag{3.6}$$

where $Z(x)$ is a perturbation term to be determined, which exponentially tends to 0 as $x \to \infty$, so that $\hat{U}(x)$ is a solution of (2.10) that approaches to the periodic solution $X_{\mu,\varepsilon,\gamma}(x+\theta)$ as $x \to \infty$. Plugging (3.5) into (2.10) yields

$$\frac{dZ}{dx} = \ell(x)Z + \hat{N}(x,Z) + \varepsilon\hat{R}(x,\varepsilon,Z), \tag{3.7}$$

where

$$\begin{aligned}
\ell(x) &= L + dN_1[H(x)] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 6\mu - 18\mu\mathrm{sech}^2\frac{\sqrt{6\mu}}{2}x & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\sqrt{3} \\ 0 & 0 & 2\sqrt{3} & 0 \end{pmatrix}, \\
\hat{N}(x,Z) &= N_1(H(x) + Z(x) + \varsigma(x)X_{\mu,\varepsilon,\gamma}(x+\theta)) - dN_1[H(x)]Z(x) - N_1(H(x)) \\
&\quad - \varsigma(x)N_1\big(X_{\mu,\varepsilon,\gamma}(x+\theta)\big) + N_2(H(x) + Z(x) + \varsigma(x)X_{\mu,\varepsilon,\gamma}(x+\theta)) \\
&\quad - \varsigma(x)N_2(X_{\mu,\varepsilon,\gamma}(x+\theta)), \\
\hat{R}(x,\varepsilon,Z) &= R(\varepsilon, H(x) + Z(x) + \varsigma(x)X_{\mu,\varepsilon,\gamma}(x+\theta)) - \varsigma(x)R(\varepsilon, X_{\mu,\varepsilon,\gamma}(x+\theta)) \\
&\quad - \frac{1}{\varepsilon}\varsigma'(x)X_{\mu,\varepsilon,\gamma}(x+\theta),
\end{aligned} \tag{3.8}$$

and d means taking the Fréchet derivative.

By (2.15) and (3.3), a direct calculation yields the following lemma.

Lemma 3.2. *If μ, ε and γ are small enough and $|Z| + |Z_1| + |Z_2| \le M_0$ for some positive constant M_0, then for $x > 0$, $\hat{N}(x,Z)$ and $\hat{R}(x,\varepsilon,Z)$ satisfy*

$$\begin{aligned}
&\hat{N}[1](x,Z) = \hat{N}[3](x,Z) = 0, \\
&|\hat{N}[2](x,Z)| \le M\left(\mu e^{-\sqrt{6\mu}x}|\tilde{v}| + \gamma(\mu e^{-\sqrt{6\mu}x} + |Z|) + e^{-x}\gamma^2 + |Z|^2\right), \\
&|\hat{N}[4](x,Z)| \le M\Big(\mu^2 e^{-2\sqrt{6\mu}x} + \mu|Z| + \gamma(\mu e^{-\sqrt{6\mu}x} + |Z|) + e^{-x}\gamma^2 + |Z|^2\Big), \\
&|\hat{N}[2](x,Z_1) - \hat{N}[2](x,Z_2)| \le M\Big(|Z_1| + |Z_2| + \mu + \gamma\Big)|Z_1 - Z_2|, \\
&|\hat{N}[4](x,Z_1) - \hat{N}[4](x,Z_2)| \le M\Big(\mu + \gamma + |Z_1| + |Z_2|\Big)|Z_1 - Z_2|, \\
&|\hat{R}[j](x,\varepsilon,Z)| \le M\Big(\mu e^{-\sqrt{6\mu}x} + |Z| + (\gamma + \frac{\gamma^2}{\mu|\varepsilon|})e^{-x}\Big), \\
&|\hat{R}[k](x,\varepsilon,Z)| \le M\Big(\mu e^{-\sqrt{6\mu}x} + |Z| + \frac{\gamma}{|\varepsilon|}e^{-x}\Big), \\
&|\hat{R}(x,\varepsilon,Z_1) - \hat{R}(x,\varepsilon,Z_2)| \le M|Z_1 - Z_2|,
\end{aligned} \tag{3.9}$$

where $j = 1,2$, $k = 3,4$, $Z(x) = (\tilde{u}, \tilde{u}_1, \tilde{v}, \tilde{v}_1)^T(x)$ and $f[j]$ denotes the j-th component of f.

It is easy to obtain that the solution $Z(x)$ of (3.7) exists if x is in a finite interval and an initial condition is given. In order to prove the existence of $Z(x)$ for $x \ge 0$ with decay to zero at infinity, we change (3.7) to an integral equation and then use the contraction mapping theorem to prove the existence of a fixed point of the integral equation.

First, consider the linear equation

$$\frac{dZ}{dx} = \ell(x)Z(x), \tag{3.10}$$

which has four linearly independent solutions

$$\begin{aligned}
s_1(x) = \mu^{3/2}\Big(& -2\sqrt{6}\operatorname{sech}^2(\sqrt{\frac{3\mu}{2}}x)\tanh(\sqrt{\frac{3\mu}{2}}x),\\
& 6\sqrt{\mu}(-2+\cosh(\sqrt{6\mu}x)\operatorname{sech}^4(\sqrt{\frac{3\mu}{2}}x), 0, 0\Big)^T,\\
s_2(x) = \frac{1}{96\mu^2}\Big(& -2(6+\cosh(\sqrt{6\mu}x)) - 15\operatorname{sech}^2(\sqrt{\frac{3\mu}{2}}x)\\
& \times(-2+\sqrt{6\mu}x\tanh(\sqrt{\frac{3\mu}{2}}x)), -\frac{\sqrt{\mu}}{8}\operatorname{sech}^4(\sqrt{\frac{3\mu}{2}}x)\\
& \times(720\sqrt{\mu}x - 360\sqrt{\mu}x\cosh(\sqrt{6\mu}x) + \sqrt{6}(185\sinh(\sqrt{6\mu}x)\\
& + 4\sinh(2\sqrt{6\mu}x) + \sinh(3\sqrt{6\mu}x))), 0, 0\Big)^T,\\
s_3(x) = \Big(& 0, 0, \cos(2\sqrt{3}x), \sin(2\sqrt{3}x)\Big)^T,\\
s_4(x) = \Big(& 0, 0, \sin(2\sqrt{3}x), -\cos(2\sqrt{3}x)\Big)^T.
\end{aligned} \tag{3.11}$$

Moreover,

$$|s_1(x)| \le M\mu^{\frac{3}{2}}e^{-\sqrt{6\mu}x}, \quad |s_2(x)| \le M\mu^{-2}e^{\sqrt{6\mu}x}, \quad |s_3(x)| + |s_4(x)| \le M \tag{3.12}$$

for $x \in [0, \infty)$, and

$$\begin{aligned}
& s_1(0) = (0, -6\mu^2, 0, 0)^T, && s_2(0) = \frac{1}{6\mu^2}(1, 0, 0, 0)^T,\\
& s_3(0) = (0, 0, 1, 0)^T, && s_4(0) = (0, 0, 0, -1)^T.
\end{aligned} \tag{3.13}$$

The adjoint equation of (3.10) has four linearly independent solutions given by

$$\begin{aligned}
s_1^*(x) =& -\frac{1}{96\mu^2}\Big(\frac{\sqrt{\mu}}{8}\operatorname{sech}^4(\sqrt{\frac{3\mu}{2}}x)(720\sqrt{\mu}x - 360\sqrt{\mu}x\cosh(\sqrt{6\mu}x)\\
& + \sqrt{6}(185\sinh(\sqrt{6\mu}x) + 4\sinh(2\sqrt{6\mu}x) + \sinh(3\sqrt{6\mu}x))),\\
& -2(6+\cosh(\sqrt{6\mu}x)) - 15\operatorname{sech}^2(\sqrt{\frac{3\mu}{2}}x)(-2+\sqrt{6\mu}x\tanh(\sqrt{\frac{3\mu}{2}}x)), 0, 0\Big)^T,\\
s_2^*(x) =& -\mu^{3/2}\Big(6\sqrt{\mu}(-2+\cosh(\sqrt{6\mu}x)\operatorname{sech}^4(\sqrt{\frac{3\mu}{2}}x), 2\sqrt{6}\operatorname{sech}^2(\sqrt{\frac{3\mu}{2}}x)\\
& \times\tanh(\sqrt{\frac{3\mu}{2}}x), 0, 0\Big)^T,\\
s_3^*(x) =& \Big(0, 0, \cos(2\sqrt{3}x), \sin(2\sqrt{3}x)\Big)^T,\\
s_4^*(x) =& \Big(0, 0, \sin(2\sqrt{3}x), -\cos(2\sqrt{3}x)\Big)^T.
\end{aligned} \tag{3.14}$$

Moreover,

$$|s_1^*(x)| \le M\mu^{-2}e^{\sqrt{6\mu}x}, \quad |s_2^*(x)| \le M\mu^{\frac{3}{2}}e^{-\sqrt{6\mu}x}, \quad |s_3^*(x)| + |s_4^*(x)| \le M \tag{3.15}$$

for $x \in [0,\infty)$, and

$$\begin{aligned} s_1^*(0) &= -\frac{1}{6\mu^2}(0,1,0,0)^T, \qquad & s_2^*(0) &= (6\mu^2,0,0,0)^T,\\ s_3^*(0) &= (0,0,1,0)^T, & s_4^*(0) &= (0,0,0,-1)^T. \end{aligned} \tag{3.16}$$

For any $x \in [0,\infty)$, we have

$$\langle s_i(x), s_j^*(x)\rangle = 0 \text{ for } i \ne j, \qquad \langle s_i(x), s_i^*(x)\rangle = 1, \quad i,j = 1,2,3,4, \tag{3.17}$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product on $\mathbf{R}^4$.

The solution of (3.7) that decays to zero at infinity can be expressed as

$$\begin{aligned} Z = \mathcal{F}(Z) &\triangleq \int_0^x \langle \hat{N}(t,Z) + \varepsilon\hat{R}(t,\varepsilon,Z), s_1^*(t)\rangle dt s_1(x)\\ &\quad - \sum_{j=2}^4 \int_x^\infty \langle \hat{N}(t,Z) + \varepsilon\hat{R}(t,\varepsilon,Z), s_j^*(t)\rangle dt s_j(x). \end{aligned} \tag{3.18}$$

Fix $\nu \in (\frac{\sqrt{6\mu}}{2}, \sqrt{6\mu})$ and consider (3.18) as a fixed point problem in a Banach space

$$\mathcal{E}_\nu = \{Z \in C([0,\infty)\times S^1)\Big| \sup_{x\in[0,\infty),\theta\in S^1}\{|Z(x,\theta)|e^{\nu x}\} < \infty\}$$

with the norm

$$\|Z\|_\nu = \sup\{|Z(x,\theta)|e^{\nu x}\Big| x\in[0,\infty), \theta\in S^1\}.$$

Then we have

Lemma 3.3. *Under the assumption (3.1), for $Z, Z_1, Z_2 \in \mathcal{E}_v$, the function $\mathcal{F}$ satisfies*

$$\begin{aligned} &\|\mathcal{F}[j](Z)\|_\nu \le M\mu^{-1}\Big(\mu^{5/2} + \mu\gamma + \frac{\gamma^2}{\sqrt{\mu}} + (\mu^{3/2}+\gamma)\|Z\|_\nu + \|Z\|_\nu^2\Big),\\ &\|\mathcal{F}[k](Z)\|_\nu \le M\mu^{-1/2}\Big(\mu^2 + \sqrt{\mu}\gamma + \mu\|Z\|_\nu + \|Z\|_\nu^2\Big),\\ &\|\mathcal{F}(Z_1) - F(Z_2)\|_\nu \le M\mu^{-1}(\mu^{3/2} + \gamma + \|Z_1\|_\nu + \|Z_2\|_\nu)\|Z_1 - Z_2\|_\nu \end{aligned} \tag{3.19}$$

for $j = 1,2$, and $k = 3,4$ where $f[j]$ means the $j-th$ component of f.

Proof. By (3.12), (3.15), (3.18) and Lemma 3.2, it is obtained that for $x \geq 0$

$$
\begin{aligned}
&\Big|\int_0^x \langle \hat{N}(t,Z)+\varepsilon\hat{R}(t,\varepsilon,Z), s_1^*(t)\rangle dt s_1(x)\Big| e^{\nu x}\\
\leq& M\mu^{-2}\int_0^x \Big(\mu e^{-\sqrt{6\mu}t}|\tilde{v}| + \gamma(\mu e^{-\sqrt{6\mu}t}+|Z|) + e^{-t}\gamma^2 + |Z|^2\\
&+|\varepsilon|(\mu e^{-\sqrt{6\mu}t}+|Z|+(\gamma+\frac{\gamma^2}{\mu|\varepsilon|})e^{-t})\Big)e^{\sqrt{6\mu}t}dt\mu^{\frac{3}{2}}e^{-\sqrt{6\mu}x}e^{\nu x}\\
\leq& M\mu^{-2}\int_0^x \Big(\mu\|\tilde{v}\|_\nu e^{-\sqrt{6\mu}t-\nu t} + \gamma(\mu+\|Z\|_\nu)e^{-\nu t} + e^{-t}\gamma^2 + \|Z\|_\nu^2 e^{-2\nu t}\\
&+|\varepsilon((\mu+\|Z\|_\nu)|e^{-\nu t}+(\gamma+\frac{\gamma^2}{\mu|\varepsilon|})e^{-t})\Big)e^{\sqrt{6\mu}t}dt\mu^{\frac{3}{2}}e^{-\sqrt{6\mu}x}e^{\nu x}\\
\leq& M\mu^{-1}\Big(\mu\|\tilde{v}\|_\nu + \gamma(\mu+\|Z\|_\nu) + \sqrt{\mu}\gamma^2 + \|Z\|_\nu^2 + |\varepsilon|(\mu+\|Z\|_\nu+\sqrt{\mu}\gamma+\frac{\gamma^2}{\sqrt{\mu}|\varepsilon|})\Big)\\
\leq& M\mu^{-1}\Big(\mu\gamma+\frac{\gamma^2}{\sqrt{\mu}}+\mu\|\tilde{v}\|_\nu+\gamma\|Z\|_\nu+\|Z\|_\nu^2\Big),
\end{aligned}
\tag{3.20}
$$

$$
\begin{aligned}
&\Big|\int_x^\infty \langle \hat{N}(t,Z)+\varepsilon\hat{R}(t,\varepsilon,Z), s_2^*(t)\rangle dt s_2(x)\Big| e^{\nu x}\\
\leq& M\mu^{\frac{3}{2}}\int_x^\infty \Big(\mu e^{-\sqrt{6\mu}t}|\tilde{v}| + \gamma(\mu e^{-\sqrt{6\mu}t}+|Z|) + e^{-t}\gamma^2 + |Z|^2\\
&+|\varepsilon|(\mu e^{-\sqrt{6\mu}t}+|Z|+(\gamma+\frac{\gamma^2}{\mu|\varepsilon|})e^{-t})\Big)e^{-\sqrt{6\mu}t}dt\mu^{-2}e^{\sqrt{6\mu}x}e^{\nu x}\\
\leq& M\mu^{\frac{3}{2}}\int_x^\infty \Big(\mu\|\tilde{v}\|_\nu e^{-\sqrt{6\mu}t-\nu t} + \gamma(\mu+\|Z\|_\nu)e^{-\nu t} + e^{-t}\gamma^2 + \|Z\|_\nu^2 e^{-2\nu t}\\
&+|\varepsilon((\mu+\|Z\|_\nu)|e^{-\nu t}+(\gamma+\frac{\gamma^2}{\mu|\varepsilon|})e^{-t})\Big)e^{-\sqrt{6\mu}t}dt\mu^{-2}e^{\sqrt{6\mu}x}e^{\nu x}\\
\leq& M\mu^{-1}\Big(\mu\|\tilde{v}\|_\nu + \gamma(\mu+\|Z\|_\nu) + \sqrt{\mu}\gamma^2 + \|Z\|_\nu^2 + |\varepsilon|(\mu+\|Z\|_\nu+\sqrt{\mu}\gamma+\frac{\gamma^2}{\sqrt{\mu}|\varepsilon|})\Big)\\
\leq& M\mu^{-1}\Big(\mu\gamma+\frac{\gamma^2}{\sqrt{\mu}}+\mu\|\tilde{v}\|_\nu+\gamma\|Z\|_\nu+\|Z\|_\nu^2\Big),
\end{aligned}
\tag{3.21}
$$

$$
\begin{aligned}
&\Big|\int_x^\infty \langle \hat{N}(t,Z)+\varepsilon\hat{R}(t,\varepsilon,Z), s_j^*(t)\rangle dt s_j(x)\Big| e^{\nu x}\\
\leq& M\int_x^\infty \Big(\mu^2 e^{-2\sqrt{6\mu}t} + \mu|Z| + \gamma(\mu e^{-\sqrt{6\mu}t}+|Z|)\\
&+e^{-t}\gamma^2+|Z|^2+|\varepsilon|(\mu e^{-\sqrt{6\mu}t}+|Z|+\frac{\gamma}{|\varepsilon|}e^{-t})\Big)dte^{\nu x}\\
\leq& M\int_x^\infty \Big(\mu^2 e^{-2\sqrt{6\mu}t} + \mu\|Z\|_\nu e^{-\nu t} + \gamma(\mu+\|Z\|_\nu)e^{-\nu t}\\
&+e^{-t}\gamma^2+\|Z\|_\nu^2 e^{-2\nu t}+|\varepsilon|((\mu+\|Z\|_\nu)e^{-\nu t}+\frac{\gamma}{|\varepsilon|}e^{-t})\Big)dte^{\nu x}\\
\leq& M\mu^{-1/2}\Big(\mu^2+\mu\|Z\|_\nu+\gamma(\mu+\|Z\|_\nu)+\sqrt{\mu}\gamma^2+\|Z\|_\nu^2+|\varepsilon|(\mu+\|Z\|_\nu+\frac{\sqrt{\mu}\gamma}{|\varepsilon|})\Big)
\end{aligned}
$$

$$\leq M\mu^{-1/2}\Big(\mu^2+\sqrt{\mu}\gamma+\mu\|Z\|_\nu+\|Z\|_\nu^2\Big) \tag{3.22}$$

for $j=3,4$. Note that (3.22) corresponds to the estimate of $\|\tilde{v}\|_\nu$, which yields the second inequality of (3.19). Replace $\|\tilde{v}\|_\nu$ in (3.20) and (3.21) with (3.22), and we obtain the first inequality of (3.19). The remaining estimates can be similarly obtained. The proof is completed. □

Now we are ready to use the contraction mapping theorem to look for a fixed point of the mapping $\mathcal{F}$. For simplicity, we take

$$\alpha_1 \in \left(\frac{1}{3},\frac{1}{2}\right). \tag{3.23}$$

Let $\bar{B}_r(0)\subset\mathcal{E}_\nu$ be a closed ball in $\mathcal{E}_\nu$ with radius $r=O(\mu^{1+\alpha_1/2})$. It is easy to see that $\mathcal{F}$ is a contraction mapping on $\tilde{B}_r(0)\in\mathcal{E}_v$ for small $\mu>0$ under the assumption (3.1). This yields that (3.18) has a unique solution $Z(x;\mu,\varepsilon,\gamma)$ satisfying

$$|Z(x;\mu,\varepsilon,\gamma)|\leq M\mu^{1+\alpha_1/2},\quad x\in[0,\infty). \tag{3.24}$$

With a more subtle estimate, we can get for $x\in[0,\infty)$

$$|Z[j](x;\mu,\varepsilon,\gamma)|\leq M\mu^{-3/2}\gamma^2,\ |Z[k](x;\mu,\varepsilon,\gamma)|\leq M\gamma,\ j=1,2,\ k=3,4. \tag{3.25}$$

Using the same argument as that for (3.25) and an extension of a contraction mapping principle [27], we can show that $Z(x;\mu,\varepsilon,\gamma)$ is smooth in its arguments. Note that the solution Z of (3.18) exists if x belongs to any finite interval and the initial condition is given so that $\hat{U}(x;\theta,\mu,\varepsilon,\gamma)$ defined in (3.5) exists for $x\geq\tilde{x}_0$ with any fixed $\tilde{x}_0\in(-\infty,\infty)$.

4. Reversible homoclinic solution for $x\in(-\infty,\infty)$

In this section, we show that the system (2.10) has a reversible generalized homoclinic solution for $x\in(-\infty,\infty)$. Due to the reversibility of (2.10), this problem is equivalent to solve the following equation

$$(I-S)\hat{U}(0;\theta,\mu,\varepsilon,\gamma)=0 \tag{4.1}$$

for $\theta\in S^1$. Let $Z(x)=(\tilde{u},\tilde{u}_1,\tilde{v},\tilde{v}_1)^T(x)$. By (2.14) and the definition of $\varsigma(x)$ in (3.6), it is easy to check that (4.1) is equivalent to

$$\tilde{u}_1(0)=0, \tag{4.2}$$
$$\tilde{v}_1(0)=0. \tag{4.3}$$

Using (3.13) and (3.18), we know that (4.2) holds automatically. Thus, we only need to study (4.3).

Lemma 4.1. *Under the conditions (3.1) and (3.23), if we take $\alpha_1=\frac{5}{12}$, then the equation (4.3) can be transformed into*

$$\theta=\mu^{\frac{1}{24}}\varphi(\theta,\mu,\varepsilon,\gamma), \tag{4.4}$$

where φ is differentiable with respect to its arguments, and φ and its derivative with respect to θ are uniformly bounded for bounded θ and small $\mu>0$.

The proof of Lemma 4.1 is given in Section 5.

Again using the contraction mapping theorem and Lemma 4.1, we can solve (4.4) for θ as a smooth function of μ, ε and γ, which gives that the equation (4.3) is valid.

Notice that both $\hat{U}(x;\theta,\mu,\varepsilon,\gamma)$ for $x \geq 0$ and $S\hat{U}(-x;\theta,\mu,\varepsilon,\gamma)$ for $x \leq 0$ are solutions of (2.10) by the reversibility. Hence we may define a solution of (2.10) as follows

$$\mathcal{U}(x) = \begin{cases} \hat{U}(x;\theta,\mu,\varepsilon,\gamma) & \text{for } x \geq 0, \\ S\hat{U}(-x;\theta,\mu,\varepsilon,\gamma) & \text{for } x \leq 0. \end{cases} \tag{4.5}$$

Then $S\mathcal{U}(-x) = \mathcal{U}(x)$. Since the equation (4.1) is true, we know that

$$S\hat{U}(0;\theta,\mu,\varepsilon,\gamma) = \hat{U}(0;\theta,\mu,\varepsilon,\gamma).$$

The uniqueness of the solution for an initial value problem implies that the system (2.10) has a reversible generalized homoclinic solution $\mathcal{U}(x;\theta,\varepsilon,\gamma)$ for $x \in (-\infty,\infty)$.

By (1.4), (2.1), (2.10) and (3.5), we get the proof of Theorem 1.1 in Section 1.

5. Appendix

In this section, we will give the proofs of Lemma 3.1 and Lemma 4.1.

5.1. Proof of Lemma 3.1

The general theory for the existence of periodic solutions of a reversible system can be found in the book [17]. Here we need the exact estimates for some terms which will play an important role in the construction of the generalized homoclinic solution. We just sketch the proof. More details can be seen in [13, 14]. The main method for this purpose is to use the Fourier series expansion technique and the contraction mapping theorem.

Let

$$\tau = 2\sqrt{3}(1+r_1)x, \tag{5.1}$$

where r_1 is a small real constant to be determined later. Then system (2.10) is changed into

$$\begin{aligned} A_\tau &= \frac{1}{2\sqrt{3}(1+r_1)}B, & B_\tau &= \frac{\sqrt{3}\mu}{1+r_1}A + h_1(\mu,\varepsilon,A,B,V_1,V_2), \\ V_{1\tau} &= -\frac{1}{1+r_1}V_2, & V_{2\tau} &= \frac{1}{1+r_1}V_1 + h_2(\mu,\varepsilon,A,B,V_1,V_2), \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} h_1(\mu,\varepsilon,A,B,V_1,V_2) &= \frac{1}{2\sqrt{3}(1+r_1)}\Big(-\frac{9}{2}A^2 - \sqrt{3}AV_1 + \frac{1}{2}V_1^2 \\ &\quad + \varepsilon\tilde{f}(\varepsilon,A,B,V_1,V_2)\Big), \\ h_2(\mu,\varepsilon,A,B,V_1,V_2) &= \frac{1}{2\sqrt{3}(1+r_1)}\Big(\sqrt{3}\mu V_1 + \frac{3}{4}A^2 - \frac{\sqrt{3}}{2}AV_1 - \frac{3}{4}V_1^2 \\ &\quad + \varepsilon\tilde{g}(\varepsilon,A,B,V_1,V_2)\Big). \end{aligned} \tag{5.3}$$

Since we look for a reversible periodic solution, we assume that

$$\begin{aligned} A &= \sum_{n=0}^{\infty} A_n \cos n\tau, & B &= \sum_{n=1}^{\infty} B_n \sin n\tau, \\ V_1 &= \sum_{n=0}^{\infty} V_{1,n} \cos n\tau, & V_2 &= \sum_{n=1}^{\infty} V_{2,n} \sin n\tau. \end{aligned} \tag{5.4}$$

Plugging (5.4) into (5.2) and making the coefficient of each term in the Fourier series equal yields

$$\begin{aligned} A_n &= \frac{-(1+r_1)}{2\sqrt{3}n^2(1+r_1)^2+\sqrt{3}\mu}\Big[h_1(\mu,\varepsilon,A,B,V_1,V_2)\Big]_n, \quad n \geq 1, \\ A_0 &= \frac{-(1+r_1)}{\sqrt{3}\mu}\Big[h_1(\mu,\varepsilon,A,B,V_1,V_2)\Big]_0, \\ B_n &= \frac{2n(1+r_1)^2}{2n^2(1+r_1)^2+\mu}\Big[h_1(\mu,\varepsilon,A,B,V_1,V_2)\Big]_n, \qquad n \geq 1, \\ V_{1,n} &= \frac{1+r_1}{n^2(1+r_1)^2-1}\Big[h_2(\mu,\varepsilon,A,B,V_1,V_2)\Big]_n, \qquad n \neq 1, \\ V_{2,n} &= \frac{n(1+r_1)^2}{n^2(1+r_1)^2-1}\Big[h_2(\mu,\varepsilon,A,B,V_1,V_2)\Big]_n, \qquad n \geq 2, \\ V_{2,1} &= (1+r_1)V_{1,1}, \end{aligned} \tag{5.5}$$

where $[f]_k$ denotes the k-th Fourier coefficient of f. For $n = 1$,

$$V_{1,1}(r_1^2+2r_1) = (1+r_1)\Big[h_2(\mu,\varepsilon,A,B,V_1,V_2)\Big]_1. \tag{5.6}$$

Now we activate $V_{1,1}$ and choose $V_{1,1} > 0$ (Remember that we use γ to denote $V_{1,1}$ for simplicity). We first solve (5.5) for $A_n, B_n, V_{1,n}(n \neq 1), V_{2,n}(n \geq 2)$ and then solve (5.6) for r_1.

Let $H^m(0,2\pi)$ be a space of periodic functions of τ with a period 2π such that their derivatives up to order m are in $L^2(0,2\pi)$, and the norm is denoted by $\|\cdot\|_m$. Fix γ and define spaces

$$\begin{aligned} H_1^1(0,2\pi) &= \{f(\tau) = \sum_{n=0}^{\infty} f_n \cos n\tau \in H^1(0,2\pi)\}, \\ H_2^1(0,2\pi) &= \{f(\tau) = \sum_{n=1}^{\infty} f_n \sin n\tau \in H^1(0,2\pi)\}, \\ H_3^1(0,2\pi) &= \{f(\tau) = \sum_{n=0}^{\infty} f_n \cos n\tau \in H^1(0,2\pi) \big| f_1 = 0\}, \\ H_4^1(0,2\pi) &= \{f(\tau) = \sum_{n=2}^{\infty} f_n \sin n\tau \in H^1(0,2\pi) \big| f_1 = 0\}. \end{aligned}$$

Using (5.5), we define a mapping $\Theta(E,F,G,P;\tilde{w})$ from $\mathcal{H}^1 = H_1^1(0,2\pi) \times H_2^1(0,2\pi)$

$\times H^1_3(0,2\pi)\times H^1_4(0,2\pi)$ to itself by

$$\Theta(E,F,G,P;\tilde{w})=\begin{pmatrix}\sum_{n=1}^{\infty}\frac{-(1+r_1)}{2\sqrt{3}n^2(1+r_1)^2+\sqrt{3}\mu}[h_1(\mu,\varepsilon,E,F,\tilde{G},\tilde{P})]_n\cos n\tau+\tilde{A}_0\\ \sum_{n=1}^{\infty}\frac{2n(1+r_1)^2}{2n^2(1+r_1)^2+\mu}[h_1(\mu,\varepsilon,E,F,\tilde{G},\tilde{P})]_n\sin n\tau\\ \sum_{n=0,n\neq 1}^{\infty}\frac{1+r_1}{n^2(1+r_1)^2-1}[h_2(\mu,\varepsilon,E,F,\tilde{G},\tilde{P})]_n\cos n\tau\\ \sum_{n=2}^{\infty}\frac{n(1+r_1)^2}{n^2(1+r_1)^2-1}[h_2(\mu,\varepsilon,E,F,\tilde{G},\tilde{P})]_n\sin n\tau\end{pmatrix},\tag{5.7}$$

where

$$\begin{aligned}&\tilde{w}=(\mu,\varepsilon,\gamma,r_1),\quad \tilde{A}_0=\frac{-(1+r_1)}{\sqrt{3}\mu}\Big[h_1(\mu,\varepsilon,E,F,\tilde{G},\tilde{P})\Big]_0,\\ &\tilde{G}=G+\gamma\cos\tau,\quad \tilde{P}=P+(1+r_1)\gamma\sin\tau.\end{aligned}\tag{5.8}$$

Assume that $\bar{B}_{\tilde{r}}(0)$ is a closed ball with a radius $\tilde{r}$ in the space $\mathcal{H}^1$. We have the following lemma.

Lemma 5.1. *For (E,F,G,P), (E_1,F_1,G_1,P_1), $(E_2,F_2,G_2,P_2)\in\bar{B}_{\tilde{r}}(0)$ and any small bounded $\tilde{w}$ and $\tilde{r}$, Θ is smooth in its arguments and satisfies*

$$\begin{aligned}&\|\Theta(E,F,G,P;\tilde{w})\|_1\\ \le& M\mu^{-1}\Big(\gamma^2+(\mu^2+\gamma+|\varepsilon|)(\|E\|_1+\|F\|_1+\|G\|_1+\|P\|_1)+\|E\|_1^2+\|F\|_1^2+\|G\|_1^2\\ &+\|P\|_1^2\Big),\\ &\|\Theta(E_2,F_2,G_2,P_2;\tilde{w})-\Theta(E_1,F_1,G_1,P_1;\tilde{w})\|_1\\ \le& M\mu^{-1}\Big(\mu^2+\gamma+|\varepsilon|+\|E_1\|_1+\|E_2\|_1+\|F_1\|_1+\|F_2\|_1+\|G_1\|_1+\|G_2\|_1+\|P_1\|_1\\ &+\|P_2\|_1\Big)\Big(\|E_1-E_2\|_1+\|F_1-F_2\|_1+\|G_1-G_2\|_1+\|P_1-P_2\|_1\Big).\end{aligned}\tag{5.9}$$

The factor μ^{-1} appears since the denominator of $\tilde{A}_0$ in (5.8) has μ. Under the assumption (3.1), if we take $\tilde{r}=\hat{r}\gamma$ for a positive constant $\hat{r}$, we can show from Lemma 5.1 that Θ is a contraction mapping on $\bar{B}_{\tilde{r}}(0)$ for small $\mu>0$. Thus Θ has a unique fixed point which is a smooth function in its arguments. Write this fixed point as

$$(A_p^0,B_p^0,V_{1p}^0,V_{2p}^0)(\mu,\varepsilon,\gamma,r_1)(\tau),\tag{5.10}$$

which satisfies with a more subtle estimate

$$\|A_p^0\|_1+\|B_p^0\|_1\le M\mu^{-1}\gamma^2,\qquad \|V_{1p}^0\|_1+\|V_{2p}^0\|_1\le M\mu\gamma.\tag{5.11}$$

Using the same argument we can show that (5.10) is in $H^m(0,2\pi)$ and satisfies (5.11) with $H^m(0,2\pi)$-norm for any integer $m>0$. We use $(A_p,B_p,V_{1p},V_{2p})(\tau)$ to denote

$$(A_p^0(\tau),B_p^0(\tau),V_{1p}^0+\gamma\cos\tau,V_{2p}^0+(1+r_1)\gamma\sin\tau).\tag{5.12}$$

Now we solve (5.6) for r_1. Substituting (5.12) into (5.6), we obtain

$$2\gamma r_1 = \tilde{g}(\mu, \varepsilon, \gamma, r_1),$$

where

$$\tilde{g}(\mu, \varepsilon, \gamma, r_1) = (1 + r_1)[h_2(\mu, \varepsilon, A_p, B_p, V_{1p}, V_{2p})]_1 - r_1^2\gamma \tag{5.13}$$

is smooth. Let

$$\tilde{g}_1(\mu, \varepsilon, \gamma, r_1) = \frac{(1 + r_1)[h_2(\mu, \varepsilon, A_p, B_p, V_{1p}, V_{2p})]_1}{2\gamma} - \frac{1}{2}r_1^2.$$

Then we have $r_1 = \tilde{g}_1(\mu, \varepsilon, \gamma, r_1)$. Under the assumption (1.5), we know that $(0, 0, 0, 0)^T$ is a trivial periodic solution of (5.2), which corresponds to $\gamma = 0$. This means that $[h_2(\mu, \varepsilon, A_p, B_p, V_{1p}, V_{2p})]_1$ has a factor γ. Hence $\tilde{g}_1$ is smooth in its arguments. Similarly we can prove that $\tilde{g}_1$ is a contraction mapping satisfying $|\tilde{g}_1| \le M(\mu + \gamma + |\varepsilon|)$. Thus, $\tilde{g}_1$ has a unique fixed point

$$r_1 = r_1(\mu, \varepsilon, \gamma)$$

as a smooth function for small $(\mu, \varepsilon, \gamma)$, which satisfies

$$|r_1| \le M(\mu + \gamma + |\varepsilon|) \le M\mu.$$

Therefore, (2.10) has a periodic solution

$$\big(A_p(\mu, \varepsilon, \gamma), B_p(\mu, \varepsilon, \gamma), V_{1p}(\mu, \varepsilon, \gamma), V_{2p}(\mu, \varepsilon, \gamma)\big)(\tau) \tag{5.14}$$

in $H^m(0, 2\pi)$. By the relation $\tau = 2\sqrt{3}(1 + r_1)x$, we write the periodic solution (5.14) as

$$X_{\mu,\varepsilon,\gamma}(x) = (A_p(\mu, \varepsilon, \gamma), B_p(\mu, \varepsilon, \gamma), V_{1p}(\mu, \varepsilon, \gamma), V_{2p}(\mu, \varepsilon, \gamma))^T(x),$$

which is smooth for x and small μ, γ, ε. Then $X_{\mu,\varepsilon,\gamma}(x)$ is a reversible periodic solution of (2.10) which satisfies for any integer $m > 0$

$$\|X_{\mu,\varepsilon,\gamma}(x)\|_m \le M\gamma. \tag{5.15}$$

The Sobolev embedding theorem gives that (5.15) holds also in $C_B^m(\mathrm{R})$-norm, which is a space of continuously differentiable function up to order m with a supremum norm. The proof of Lemma 3.1 is finished.

□

5.2. Proof of Lemma 4.1

First we estimate V_{1p} and V_{2p}. Let

$$C_p(\tau) = V_{1p}(\tau) + iV_{2p}(\tau), \tag{5.16}$$

which yields

$$V_{1p} = \frac{C_p + \bar{C}_p}{2}, \qquad V_{2p} = i\frac{\bar{C}_p - C_p}{2}. \tag{5.17}$$

Thus, by (5.2), we know that $\left(A_p, B_p, C_p, \bar{C}_p\right)^T(\tau)$ is a periodic solution of the following system

$$\begin{aligned} A_{p\tau} &= \frac{1}{2\sqrt{3}(1+r_1)} B_p, \\ B_{p\tau} &= \frac{\sqrt{3}\mu}{1+r_1} A_p + \tilde{h}_1(\mu, \varepsilon, A_p, B_p, C_p, \bar{C}_p), \\ C_{p\tau} &= \frac{i}{1+r_1} C_p + \tilde{h}_2(\mu, \varepsilon, A_p, B_p, C_p, \bar{C}_p), \\ \bar{C}_{p\tau} &= \frac{-i}{1+r_1} \bar{C}_p - \tilde{h}_2(\mu, \varepsilon, A_p, B_p, C_p, \bar{C}_p), \end{aligned} \tag{5.18}$$

where

$$\begin{aligned} &\tilde{h}_1(\mu, \varepsilon, A_p, B_p, C_p, \bar{C}_p) \\ =&\frac{1}{2\sqrt{3}(1+r_1)}\Big(- \frac{9}{2} A_p^2 - \frac{\sqrt{3}(C_p + \bar{C}_p)}{2} A_p \\ &+ \frac{(C_p + \bar{C}_p)^2}{8} + \varepsilon \tilde{f}(\varepsilon, A_p, B_p, \frac{C_p + \bar{C}_p}{2}, i\frac{C_p - \bar{C}_p}{2})\Big), \\ &\tilde{h}_2(\mu, \varepsilon, A_p, B_p, C_p, \bar{C}_p) \\ =&\frac{i}{2\sqrt{3}(1+r_1)}\Big(\sqrt{3}\mu\frac{C_p + \bar{C}_p}{2} + \frac{3}{4} A_p^2 - \frac{\sqrt{3}(C_p + \bar{C}_p)}{4} A_p \\ &- \frac{3(C_p + \bar{C}_p)^2}{16} + \varepsilon \tilde{g}(\varepsilon, A_p, B_p, \frac{C_p + \bar{C}_p}{2}, i\frac{C_p - \bar{C}_p}{2})\Big). \end{aligned} \tag{5.19}$$

We can express $C_p(\tau)$ as

$$C_p(\tau) = C_p(0) e^{\frac{i}{1+r_1}\tau} + w(\tau), \tag{5.20}$$

where

$$w(\tau) = \int_0^\tau e^{\frac{i(\tau - s)}{1+r_1}} \tilde{h}_2(\mu, \varepsilon, A_p, B_p, C_p, \bar{C}_p) ds. \tag{5.21}$$

Note that the coefficient of $e^{i\tau}$ in $C_p(\tau)$ is γ. Thus,

$$\begin{aligned} \gamma &= \frac{1}{2\pi}\int_0^{2\pi} C_p(s) e^{-is} ds \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-is}\Big(C_p(0) e^{\frac{i}{1+r_1}s} + w(s)\Big) ds \\ &= (1 + K(r_1)) C_p(0) + \frac{1}{2\pi}\int_0^{2\pi} e^{-is} w(s) ds, \end{aligned} \tag{5.22}$$

where $K(r_1) = \frac{1+r_1}{2\pi r_1 i}\big(1 - e^{-\frac{2\pi r_1 i}{1+r_1}}\big) - 1 = O(r_1)$ and $K(0) = 0$, which yields

$$C_p(0) = \frac{1}{1 + K(r_1)}\left(\gamma - \frac{1}{2\pi}\int_0^{2\pi} e^{-is} w(s) ds\right), \tag{5.23}$$

so

$$C_p(\tau) = \frac{e^{\frac{i}{1+r_1}\tau}}{1 + K(r_1)}\left(\gamma - \frac{1}{2\pi}\int_0^{2\pi} e^{-is} w(s) ds\right) + w(\tau), \tag{5.24}$$

or

$$C_p(x) = \frac{e^{2\sqrt{3}(1+r_1)ix}}{1+K(r_1)}\Big(\gamma - \frac{1}{2\pi}\int_0^{2\pi} e^{-is}w(s)ds\Big) + w\big(2\sqrt{3}(1+r_1)x\big). \tag{5.25}$$

From (3.1), (3.3) and (3.23), we can get $w(x) = O(\mu\gamma + |\varepsilon|\gamma + \gamma^2) = O(\mu\gamma)$ so that $C_p(x) = O(\gamma)$. Then we obtain by (3.4) and (5.17)

$$\begin{aligned} V_{1p}(x) &= \operatorname{Re} C_p(x) = \gamma\cos(2\sqrt{3}x) + \tilde{P}(x,\mu,\varepsilon,\gamma), \\ V_{2p}(x) &= \operatorname{Im} C_p(x) = \gamma\sin(2\sqrt{3}x) + \tilde{Q}(x,\mu,\varepsilon,\gamma), \end{aligned} \tag{5.26}$$

where $\tilde{P}(x,\mu,\varepsilon,\gamma) = O(\mu\gamma)$ and $\tilde{Q}(x,\mu,\varepsilon,\gamma) = O(\mu\gamma)$.

By (3.13) and (3.18), we can transformed (4.3) to

$$0 = -\int_0^\infty < \hat{N}(t,Z) + \varepsilon\hat{R}(t,\varepsilon,Z), s_4^*(t) > dt, \tag{5.27}$$

which is equivalent to the following equation by (3.8), (3.9) and (3.14)

$$\begin{aligned} 0 = \int_0^\infty \bigg(&\Big(-\varsigma'(t)V_{1p}(t+\theta) + \varepsilon\mathcal{R}[3](t)\Big)\sin(2\sqrt{3}t) \\ &- \Big(-\varsigma'(t)V_{2p}(t+\theta) + \hat{N}[4](t,Z) + \varepsilon\mathcal{R}[4](t)\Big)\cos(2\sqrt{3}t)\bigg)dt, \end{aligned} \tag{5.28}$$

where

$$\mathcal{R}(x) = R(\varepsilon, H(x) + Z(x) + \varsigma(x)X_{\mu,\varepsilon,\gamma}(x+\theta)) - \varsigma(x)R(\varepsilon, X_{\mu,\varepsilon,\gamma}(x+\theta)).$$

Using (3.1), (3.23), (3.25) and Lemma 3.2, we obtain the following estimate

$$\begin{aligned} \Big|\int_0^\infty \hat{N}[4](t,Z)\cos(2\sqrt{3}t)dt\Big| &\le M\mu^{-1/2}\Big(\mu^2 + \mu\|Z\|_\nu + \gamma(\mu + \|Z\|_\nu) \\ &\quad + \sqrt{\mu}\gamma^2 + \|Z\|_\nu^2\Big) \le M\mu^{3/2}, \end{aligned} \tag{5.29}$$

and

$$\begin{aligned} &\Big|\int_0^\infty \varepsilon\mathcal{R}[3](t)\sin(2\sqrt{3}t)dt\Big| + \Big|\varepsilon\mathcal{R}[4](t)\cos(2\sqrt{3}t)dt\Big| \\ \le& M|\varepsilon|\int_0^\infty \Big(\mu e^{-\sqrt{6\mu}t} + |Z| + \gamma^2 e^{-t}\Big)dt \\ \le& M|\varepsilon|\mu^{-1/2}\Big(\mu + \|Z\|_\nu + \sqrt{\mu}\gamma^2\Big) \\ \le& M|\varepsilon|\mu^{-1/2}\Big(\mu + \mu^{-3/2}\gamma^2\Big) \le M\sqrt{\mu}|\varepsilon|. \end{aligned} \tag{5.30}$$

By (5.26), we transform the equation (5.28) into

$$\begin{aligned} 0 =& \int_0^\infty \Big(-\varsigma'(t)\gamma\cos(2\sqrt{3}(t+\theta))\sin(2\sqrt{3}t) \\ &+ \varsigma'(t)\gamma\sin(2\sqrt{3}(t+\theta))\cos(2\sqrt{3}t)\Big)dt + \mathcal{P}(\theta,\mu,\varepsilon,\gamma) \\ =& \gamma\sin(2\sqrt{3}\theta)\int_1^2 \varsigma'(t)dt + \mathcal{P}(\theta,\mu,\varepsilon,\gamma) = \gamma\sin(2\sqrt{3}\theta) + \mathcal{P}(\theta,\mu,\varepsilon,\gamma), \end{aligned} \tag{5.31}$$

where $\mathcal{P}$ is smooth in its arguments and

$$|\mathcal{P}| \le M(\mu^{3/2} + \sqrt{\mu}|\varepsilon|) \le M\mu^{3/2}.$$

Thus,

$$\theta = \mu^{1/24}\tilde{\mathcal{P}}(\theta, \mu, \varepsilon, \gamma), \tag{5.32}$$

where

$$\tilde{\mathcal{P}}(\theta, \mu, \varepsilon, \gamma) = -\frac{\mu^{-1/24}}{2\sqrt{3}} \arcsin(\mathcal{P}/\gamma).$$

For computational simplicity, we take $\alpha_1 = \frac{5}{12}$ in (3.1), which implies that (3.23) holds and $\gamma = O(\mu^{17/12})$ and $\varepsilon = O(\mu^{17/12+\alpha_2})$. Thus, for small $\mu > 0$

$$|\tilde{\mathcal{P}}(\theta, \mu, \varepsilon, \gamma)| \le M.$$

Using a similar argument as above, we may show that $\tilde{\mathcal{P}}$ is differentiable with respect to its arguments and its derivative with respect to θ is also uniformly bounded for bounded θ and small $\mu > 0$, which gives the equation (4.4). The lemma is proved. $\square$

References

[1] P. Ademir and R. Lionel, *Stabilization of a Boussinesq system of KdV-KdV type*, Systems Control Lett., 2008, 57(8), 595–601.

[2] C. J. Amick, *Regularity and uniqueness of solutions to the Boussinesq system of equations*, J. Diff. Eqs., 1983, 54(2), 231–247.

[3] D. C. Antonopoulos and V. A. Dougalis, *Error estimates for Galerkin approximations of the "classical" Boussinesq system*, Math. Comp., 2013, 82(282), 689–717.

[4] J. Bona and M. Chen, *A Boussinesq system for two-way propagation of nonlinear dispersive waves*, Phys. D, 1998, 116(1), 191–224.

[5] J. Bona, M. Chen and J. Saut, *Boussinesq equations and other sytems for small-amplitude long waves in nonlinear dispersive media. I: Derivation and linear theory*, J. Nonlinear Sci., 2002, 12(4), 283–318.

[6] J. Bona, M. Chen and J. Saut, *Boussinesq equations and other sysytems for small-amplitude long waves in nonlinear dispersive media. II: Nonlinear theory*, Nonlinearity, 2004, 17(3), 925–952.

[7] J. Bona, A. Dougalis and D. Mitsotakis, *Numerical solution of KdV-KdV systems of Boussinesq equations: I. The numerical scheme and existence of generalized solitary waves*, Math. Comput. Simulat., 2007, 74(2-3), 214–228.

[8] J. Bona, A. Dougalis and D. Mitsotakis, *Numerical solution of Boussinesq systems of KdV-KdV type: II. Evolution of radiating solitary waves*, Nonlinearity, 2008, 21(12), 2825–2848.

[9] J. Bona, T. Colin and C. Guillopé, *Propagation of long-crested water waves*, Discrete Contin. Dyn. Syst., 2013, 33(2), 599–628.

[10] J. Bona, Z. Grujić and H. Kalisch, *A KdV-type Boussinesq system: from the energy level to analytic spaces*, Discrete Contin. Dyn. Syst., 2010, 26(4), 1121–113.

[11] T. J. Bridges and E. G. Fan, *Solitary waves, periodic waves, and a stability analysis for Zufiria's higher-order Boussinesq model for shallow water waves*, Phy. Lett., 2004, 326A(5-6), 381–390.

[12] P. Daripa and R. K. Dash, *A class of model equations for bi-directional propagation of capillary-gravity waves*, Internat. J. Engrg. Sci., 2003, 41(2), 201–218.

[13] S. Deng and S. Sun, *Three-dimensional gravity-capillary waves on water–small surface tension case*, Phys. D, 2009, 238(18), 1735–1751.

[14] S. Deng and S. Sun, *Exact theory of three-dimensional water waves at the critical speed*, SIAM J. Math. Anal., 2010, 42(6), 2721–2761.

[15] R. Hirota and J. Satsuma, *Soliton-solutions of a coupled korteveg-devries*, Phys. Lett. A., 1981, 85(8-9), 407–408.

[16] H. Kalisch andM. Bjøkavåg, *Energy budget in a dispersive model for undular bores*, Proc. Estonian Acad. Sci., 2010, 59(2), 172–181.

[17] H. Kielhöfer, *Bifurcation theory: an introduction with applications to PDEs*, Springer, New York, 2003.

[18] Y. S. Li, *Some water wave equations and integrability*, J. Nonlinear Math. Phy., 2005, 12(Supplement 1), 466–481.

[19] E. Lombardi, *Homoclinic orbits to small periodic orbits for a class of reversible systems*, Proc. Roy. Soc. Edinburgh Sect. A, 1996, 126(5), 1035–1054.

[20] E. Lombardi, *Orbits homoclinic to exponentially small periodic orbits for a class of reversible systems. Application to water waves*, Arch. Rational Mech. Anal., 1997, 137(3), 227–304.

[21] E. Lombardi, *Oscillatory integrals and phenomena beyond all algebraic orders*, Lecture Notes in Mathematics, Vol. 1741, Springer, New York, 2000.

[22] Y. Mammeri, *Unique continuation property for Boussinesq-type systems*, Comm. Math. Anal., 2010, 9(1), 121–127.

[23] D. Mitsotakis, *Boussinesq systems in two space dimensions over a variable bottom for the generation and propagation of tsunami waves*, Math. Comput. Simulat., 2009, 80(4), 860–873.

[24] A. F. Pazoto and L. Rosier, *Stabilization of a Boussinesq system of KdV-KdV type*, Systems Control Lett., 2008, 57(8), 595–601.

[25] M. E. Schonbek, *Existence of solutions for the Boussinesq system of equations*, J. Diff. Eqs., 1981, 42(3), 325–352.

[26] M. Sorin, R. Lionel and Z. Bing, *Control and stabilization of a family of Boussinesq systems*, Discrete. Contin. Dyn. Syst., 2009, 24(2), 273–313.

[27] W. Walter, *Gewöhnliche Differential gleichungen*, Springer, New York, 1972.