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WEAK CONVERGENCE OF A SPLITTING ALGORITHM IN HILBERT SPACES*

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Abstract In this paper, we present a splitting algorithm with computational errors for solving common solutions of zero point, fixed point and equilibrium problems. Weak convergence theorems of common solutions are established in the framework of real Hilbert spaces.

Keywords Equilibrium problem, splitting algorithm, nonexpansive mapping, variational inequality, zero point.

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1. Introduction and preliminaries

In this paper, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and let $Proj_C$ be the metric projection from H onto C.

Let $A : C \to H$ be a mapping. Recall that A is said to be monotone iff

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

A is said to be strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C.$$

For such a case, we also call A is an α-strongly monotone mapping. A is said to be inverse-strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, we also call A is an α-inverse-strongly monotone mapping. It is clear that A is inverse-strongly monotone if and only if A^{-1} is strongly monotone. We also remark here that every α-inverse-strongly monotone mapping is strongly monotone and $\frac{1}{\alpha}$-Lipschitz continuous.

Recall that a set-valued mapping $T : H \to 2^H$ is said to be monotone iff for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal iff the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal iff, for any $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for all $(y, g) \in G(T)$ implies

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$f \in Tx$. Let A be a monotone mapping of C into H and $N_C v$ the normal cone to C at $v \in C$, *i.e.*,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \quad \forall u \in C\}$$

and define a mapping T on C by

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ iff $\langle Av, u - v\rangle \geq 0$ for all $u \in C$; see [20] and the references therein.

Let S be a mapping on C. $Fix(S)$ stands for the fixed point set of S. In what follows, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. S is said to be demiclosed at y iff $Sx_n \to y$ and $x_n \rightharpoonup x$, then x is a fixed point of S, that is, $Sx = y$.

Recall that S is said to be firmly nonexpansive iff

$$\|Sx - Sy\|^2 \leq \langle Sx - Sy, x - y\rangle, \quad \forall x, y \in C.$$

S is said to be nonexpansive iff

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

If C is closed, convex and bounded, then $Fix(S)$ is not empty; see [2] and the references therein.

Let I denote the identity operator on H and $T : H \to 2^H$ be a maximal monotone operator. Then we can define, for each $r > 0$, a nonexpansive single valued mapping $J_r^B : H \to H$ by $J_r^B = (I + rT)^{-1}$. It is called the resolvent of T. We know that $T^{-1}0 = Fix(J_r^B)$ for all $r > 0$ and J_r^B is firmly nonexpansive.

One of classical methods of studying the problem $0 \in Tx$, where T is a maximal monotone operator, is the proximal point algorithm (PPA) which was initiated by Martinet [13, 14] and further developed by Rockafellar [20, 21] . The PPA and its dual version in the context of convex programming, the method of multipliers of Hesteness and Powell, have been extensively studied and are known to yield as special cases decomposition methods such as the method of partial inverses [23], the Douglas-Rachford splitting method, and the alternating direction method of multipliers [8,9]. In the case of $T = A+B$, where A and B are monotone mappings, the splitting method $x_{n+1} = (I + r_n B)^{-1}(I - r_n A)x_n, n = 0, 1, \dots,$ where $r_n > 0$, was proposed by Lions and Mercier [12] and by Passty [15].

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. We consider the following equilibrium problem in the terminology of Blum and Oettli [1], which is also known as the Ky Fan inequality [10].

$$\text{Find } x \in C \text{ such that } F(x, y) \geq 0, \forall y \in C. \tag{1.1}$$

In this paper, the set of such an $x \in C$ is denoted by $EP(F)$, *i.e.*, $EP(F) = \{x \in C : F(x, y) \geq 0, \ \forall y \in C\}$.

To study equilibrium problem 1.1, we assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y)+F(y,x)\leq 0$ for all $x,y\in C$;

(A3) for each $x,y,z\in C$, $\limsup_{t\downarrow 0} F(tz+(1-t)x,y)\leq F(x,y)$;

(A4) for each $x\in C$, $y\mapsto F(x,y)$ is convex and lower semi-continuous.

There are many nonlinear problems arising in engineering areas needing more than one constraint. Solving such problems, we have to obtain some solution which is simultaneously the solution of two or more subproblems or the solution of one subproblem on the solution set of another subproblem; see [3–5, 7, 11, 16–19, 25–30] and the references therein.

The aim of this paper is to investigate a splitting algorithm with computational errors for solving common solutions of zero point, fixed point and equilibrium problems. The organization of this paper is as follows. In Section 1, we provide some necessary preliminaries. In Section 2, a splitting algorithm with computational errors is introduced and investigated. A weak convergence theorem is established. In Section 4, Applications of the main results are discussed.

In order to prove our main results, we need the following lemmas.

Lemma 1.1 ([1]). *Let C be a nonempty closed convex subset of H and let $F: C\times C\to\mathbb{R}$ be a bifunction satisfying (A1)-(A4). Then, for any $r>0$ and $x\in H$, there exists $z\in C$ such that*

$$F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\ \forall y\in C.$$

Further, define

$$T_r x=\{z\in C: F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\ \forall y\in C\}$$

for all $r>0$ and $x\in H$. Then, the following hold:

(a) T_r is single-valued;

(b) T_r is firmly nonexpansive;

(c) $Fix(T_r)=EP(F)$;

(d) $EP(F)$ is closed and convex.

Lemma 1.2 ([6]). *Let C be a nonempty closed convex subset of H. Let $A: C\to H$ be a mapping and let $B: H\rightrightarrows H$ be a maximal monotone operator. Then $Fix(J_r(I-rA))=(A+B)^{-1}(0)$.*

Lemma 1.3 ([24]). *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative sequences satisfying the following relation:*

$$a_{n+1}\leq(1+b_n)a_n+c_n,\ \forall n\geq n_0,$$

where n_0 is some nonnegative integer, $\sum_{n=1}^{\infty} b_n<\infty$ and $\sum_{n=1}^{\infty} c_n<\infty$. Then the limit $\lim_{n\to\infty} a_n$ exists.

Lemma 1.4 ([22]). *Let $0<p\leq t_n\leq q<1$ for all $n\geq 1$. Suppose that $\{x_n\}$, and $\{y_n\}$ are sequences in H such that*

$$\limsup_{n\to\infty}\|x_n\|\leq d,\quad \limsup_{n\to\infty}\|y_n\|\leq d$$

and

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = d$$

hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 1.5 ([2]). *Let C be a nonempty closed and convex subset of H and $S : C \to C$ a nonexpansive mapping. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$, and $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$, then $x = Sx$.*

2. Main results

Theorem 2.1. *Let C be a nonempty closed convex subset of H and let F be a bifunction from $C \times C$ to R which satisfies (A1)-(A4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $B : H \rightrightarrows H$ be a maximal monotone mapping. Let $S : C \to C$ be a nonexpansive mapping. Assume that $Fix(S) \cap EP(F) \cap (A+B)^{-1}(0)$ is nonempty. Let $\{r_n\}$ and $\{s_n\}$ be positive real number sequences. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n = 1$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$ and*

$$\begin{cases} F(y_n, y) + \frac{1}{s_n}\langle y - y_n, y_n - J_{r_n}\big(x_n - r_n Ax_n + e_n\big)\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n Sx_n + \beta_n y_n + \gamma_n f_n, \quad n \geq 1, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$ and $\{f_n\}$ is bounded sequence in C. Assume that the control sequences satisfy the following restrictions:

(a) $0 < \beta \leq \beta_n \leq \beta' < 1$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$,

(b) $0 < \liminf_{n\to\infty} s_n$ *and* $0 < r \leq r_n \leq r' < 2\alpha$,

where β, β', r and r' are real constants. Then $\{x_n\}$ converges weakly to some point in $Fix(S) \cap EP(F) \cap (A+B)^{-1}(0)$.

Proof. First, we show that the mapping $I - r_n A$ is nonexpansive. For any $x, y \in C$, we see that

$$\begin{aligned} &\|(I - r_n A)x - (I - r_n A)y\|^2 \\ =&\|x-y\|^2 - 2r_n\langle x - y, Ax - Ay\rangle + r_n^2\|Ax - Ay\|^2 \\ \leq&\|x-y\|^2 - r_n(2\alpha - r_n)\|Ax - Ay\|^2. \end{aligned}$$

In view of restriction (b), we see that $\|(I - r_n A)x - (I - r_n A)y\| \leq \|x - y\|$. This proves that $I - r_n A$ is nonexpansive. Let $p \in Fix(S) \cap EP(F) \cap (A+B)^{-1}(0)$ be fixed arbitrarily. By use of Lemmas 1.1 and 1.2, we find that $y_n = T_{s_n} J_{r_n}\big(x_n - r_n Ax_n + e_n\big)$. It follows that

$$\begin{aligned} &\|x_{n+1} - p\| \\ \leq&\alpha_n\|Sx_n - p\| + \beta_n\|y_n - p\| + \gamma_n\|f_n - p\| \\ \leq&\alpha_n\|x_n - p\| + \beta_n\|J_{r_n}\big(x_n - r_n Ax_n + e_n\big) - J_{r_n}\big(p - r_n Ap\big)\| + \gamma_n\|f_n - p\| \\ \leq&\alpha_n\|x_n - p\| + \beta_n\|\big(x_n - r_n Ax_n + e_n\big) - \big(p - r_n Ap\big)\| + \gamma_n\|f_n - p\| \\ \leq&\|x_n - p\| + \lambda_n, \end{aligned}$$

where $\lambda_n = \gamma_n\|f_n - p\| + \|e_n\|$. From restriction (a), we find that $\sum_{n=1}^{\infty} \|\lambda_n\| < \infty$. This implies from Lemma 1.3 that the limit $\lim_{n\to\infty} \|x_n - p\|$ exists. Hence, we have $\{x_n\}$ is bounded, so is $\{y_n\}$. Since $\{x_n\}$ is bounded, we may assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to ξ. Put $z_n = J_{r_n}\big(x_n - r_nAx_n + e_n\big)$. Since A is inverse-strongly monotone, we find that

$$\begin{aligned}
\|z_n - p\|^2 &\le \|(x_n - r_nAx_n) - (p - r_nAp) + e_n\|^2 \\
&\le \|(x_n - p) - r_n(Ax_n - Ap)\|^2 + \|e_n\|(\|e_n\| + 2\|e_n\|\|x_n - p\|) \\
&\le \|x_n - p\|^2 - r_n(2\alpha - r_n)\|Ax_n - Ap\|^2 + \|e_n\|(\|e_n\| + 2\|e_n\|\|x_n - p\|).
\end{aligned} \tag{2.1}$$

Hence, we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
\le&\alpha_n\|x_n - p\|^2 + \beta_n\|y_n - p\|^2 + \gamma_n\|f_n - p\|^2 \\
\le&\|x_n - p\|^2 - r_n(2\alpha - r_n)\beta_n\|Ax_n - Ap\|^2 + \|e_n\|(\|e_n\| + 2\|e_n\|\|x_n - p\|) \\
&+ \gamma_n\|f_n - p\|^2.
\end{aligned}$$

That is,

$$\begin{aligned}
r_n(2\alpha - r_n)\beta_n\|Ax_n - Ap\|^2 \le{}& \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&+ \|e_n\|(\|e_n\| + 2\|e_n\|\|x_n - p\|) + \gamma_n\|f_n - p\|^2.
\end{aligned}$$

By use of restrictions (a) and (b), we find that

$$\lim_{n\to\infty} \|Ax_n - Ap\| = 0. \tag{2.2}$$

Since J_{r_n} is firmly nonexpansive, we find that

$$\begin{aligned}
\|z_n - p\|^2 &= \|J_{r_n}\big(x_n - r_nAx_n + e_n\big) - J_{r_n}\big(p - r_nAp)\|^2 \\
&\le \langle\big(x_n - r_nAx_n + e_n\big) - \big(p - r_nAp\big), z_n - p\rangle \\
&= \frac{1}{2}\big(\|\big(x_n - r_nAx_n + e_n\big) - \big(p - r_nAp\big)\|^2 + \|z_n - p\|^2 \\
&\quad - \|\big((x_n - r_nAx_n + e_n\big) - \big(p - r_nAp\big)\big) - (z_n - p)\|^2 \\
&\le \frac{1}{2}\big(\|x_n - p\|^2 + \|e_n\|(\|e_n\| + 2\|x_n - p\|) + \|z_n - p\|^2 \\
&\quad - \|x_n - z_n - r_n(Ax_n - Ap) + e_n\|^2\big) \\
&\le \frac{1}{2}\big(\|x_n - p\|^2 + \|e_n\|(\|e_n\| + 2\|x_n - p\|) + \|z_n - p\|^2 - \|x_n - z_n\|^2 \\
&\quad - \|r_n(Ax_n - Ap) - e_n\|^2 + 2\|x_n - z_n\|\|r_n(Ax_n - Ap) - e_n\|\big).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|z_n - p\|^2 \le{}& \|x_n - p\|^2 + \|e_n\|(\|e_n\| + 2\|x_n - p\|) - \|x_n - z_n\|^2 \\
&+ 2r_n\|x_n - z_n\|\|Ax_n - Ap\| + 2\|x_n - z_n\|\|e_n\|.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \alpha_n\|x_n - p\|^2 + \beta_n\|y_n - p\|^2 + \gamma_n\|f_n - p\|^2 \\
&\le \|x_n - p\|^2 + \|e_n\|(\|e_n\| + 2\|x_n - p\|) - \beta_n\|x_n - z_n\|^2 \\
&\quad + 2r_n\|x_n - z_n\|\|Ax_n - Ap\| + 2\|x_n - z_n\|\|e_n\| + \gamma_n\|f_n - p\|^2.
\end{aligned}$$

It follows that

$$\beta_n\|x_n - z_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \|e_n\|(\|e_n\| + 2\|x_n - p\|) + 2r_n\|x_n - z_n\|\|Ax_n - Ap\| + 2\|x_n - z_n\|\|e_n\| + \gamma_n\|f_n - p\|^2.$$

By use of restrictions (a) and (b), we find from (2.2) that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{2.3}$$

It follows that the subsequence $\{z_{n_i}\}$ of $\{z_n\}$ converges weakly to ξ. Notice that

$$\frac{x_n - z_n + e_n}{r_n} - Ax_n \in Bz_n.$$

Let $\mu \in B\nu$. Since B is monotone, we find that

$$\left\langle \frac{x_n - z_n + e_n}{r_n} - Ax_n - \mu, z_n - \nu \right\rangle \ge 0.$$

It follows from (2.2) that $\langle -A\xi - \mu, \xi - \nu\rangle \ge 0$. This implies that $-A\xi \in B\bar{x}$, that is, $\xi \in (A+B)^{-1}(0)$.

Next, we show that $\xi \in EP(F)$. Since T_{s_n} is firmly nonexpansive, we find that

$$\begin{aligned}\|y_n - p\|^2 &\le \langle z_n - p, y_n - p\rangle \\ &= \frac{1}{2}\big(\|z_n - p\|^2 + \|y_n - p\|^2 - \|y_n - z_n\|^2\big).\end{aligned}$$

It follows from (2.1) that

$$\begin{aligned}\|y_n - p\|^2 &\le \|z_n - p\|^2 - \|y_n - z_n\|^2 \\ &\le \|x_n - p\|^2 + \|e_n\|(\|e_n\| + 2\|x_n - p\|) - \|y_n - z_n\|^2.\end{aligned}$$

This implies that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \alpha_n\|x_n - p\|^2 + \beta_n\|y_n - p\|^2 + \gamma_n\|f_n - p\|^2 \\ &\le \|x_n - p\|^2 + \|e_n\|(\|e_n\| + 2\|x_n - p\|) - \beta_n\|y_n - z_n\|^2 + \gamma_n\|f_n - p\|^2.\end{aligned}$$

Hence, we have

$$\beta_n\|y_n - z_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \|e_n\|(\|e_n\| + 2\|x_n - p\|) + \gamma_n\|f_n - p\|^2.$$

By use of restriction (a), we find that

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{2.4}$$

Notice that

$$F(y_n, y) + \frac{1}{s_n}\langle y - y_n, y_n - z_n\rangle \ge 0, \quad \forall y \in C.$$

By use of condition (A2), we see that $\frac{1}{s_n}\langle y - y_n, y_n - z_n\rangle \ge F(y, y_n)$, $\forall y \in C$. Replacing n by n_i, we arrive at

$$\langle y - y_{n_i}, \frac{y_{n_i} - z_{n_i}}{s_{n_i}}\rangle \ge F(y, y_{n_i}), \quad \forall y \in C.$$

By use of restriction (b) and (2.4), we find that $\{y_{n_i}\}$ converges weakly to ξ. It follows that $0 \geq F(y,\xi)$. For t with $0 < t \leq 1$, and $y \in C$, let $y_t = ty + (1-t)\xi$. Since $y \in C$, and $\xi \in C$, we have $y_t \in C$. It follows that $F(y_t, \xi) \leq 0$. Notice that

$$0 = F(y_t, y_t) \leq tF(y_t, z) + (1-t)F(y_t, \xi) \leq tF(y_t, y),$$

which yields that $F(y_t, y) \geq 0$, $\forall y \in C$. Letting $t \downarrow 0$, we obtain from restriction (A3) that $F(\xi, y) \geq 0$, $\forall y \in C$. This implies that $\xi \in EP(F)$. This proves that $\xi \in EP(F)$.

Now, we are in a position to show that $\xi \in Fix(S)$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists, we put $\lim_{n\to\infty} \|x_n - p\| = d > 0$. It follows that

$$\lim_{n\to\infty} \|(1-\alpha_n)\big((y_n - p) + \gamma_n(f_n - y_n)\big) + \alpha_n\big((Sx_n - p) + \gamma_n(f_n - y_n)\big)\| = d.$$

Notice that

$$\|(Sx_n - p) + \gamma_n(f_n - y_n)\| \leq \|x_n - p\| + \gamma_n\|f_n - x_n\|.$$

This shows that $\limsup_{n\to\infty} \|(Sx_n - p) + \gamma_n(f_n - y_n)\| \leq d$. By use of (2.1), we find that

$$\begin{aligned} &\|(y_n - p) + \gamma_n(f_n - y_n)\| \\ \leq&\|y_n - p\| + \gamma_n\|f_n - x_n\| \\ \leq&\|x_n - p\|^2 + \|e_n\|(\|e_n\| + 2\|e_n\|\|x_n - p\|) + \gamma_n\|f_n - x_n\|. \end{aligned}$$

It follows that $\limsup_{n\to\infty} \|(y_n - p) + \gamma_n(f_n - y_n)\| \leq d$. It follows from Lemma 1.4 that

$$\lim_{n\to\infty} \|Sx_n - y_n\| = 0. \tag{2.5}$$

Since S is nonexpansive, we find from (2.3) and (2.4) that $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$. This implies from Lemma 1.5 that $\xi \in Fix(S)$. Finally, we show that the whole sequence $\{x_n\}$ weakly converges to ξ. Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ converging weakly to ξ', where $\xi' \neq \xi$. In the same way, we can show that $\xi' \in (A+B)^{-1}(0) \cap EP(F)$. Since H has the Opial's condition, we, therefore, obtain that

$$\begin{aligned} d = \liminf_{i\to\infty} \|x_{n_i} - \xi\| &< \liminf_{i\to\infty} \|x_{n_i} - \xi'\| \\ &= \liminf_{j\to\infty} \|x_j - \xi'\| < \liminf_{j\to\infty} \|x_j - \xi\| = d. \end{aligned}$$

This is a contradiction. Hence $\xi = \xi'$. This proves that $\{x_n\}$ converges weakly to $\xi \in Fix(S) \cap EP(F) \cap (A+B)^{-1}(0)$. This completes the proof. □

From Theorem 2.1, the following results are not hard to derive.

Corollary 2.1. *Let C be a nonempty closed convex subset of H and let F be a bifunction from $C \times C$ to R which satisfies (A1)-(A4). Let $S : C \to C$ be a nonexpansive mapping with a nonempty fixed point set. Let $\{s_n\}$ be a positive real number sequence. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$ and*

$$\begin{cases} F(y_n, y) + \frac{1}{s_n}\langle y - y_n, y_n - x_n - e_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n Sx_n + \beta_n y_n + \gamma_n f_n, \quad n \geq 1, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in H such that $\sum_{n=1}^{\infty}\|e_n\| < \infty$ and $\{f_n\}$ is bounded sequence in C. Assume that the control sequences satisfy the following restrictions:

(a) $0 < \beta \le \beta_n \le \beta' < 1$ *and* $\sum_{n=1}^{\infty}\gamma_n < \infty$,

(b) $0 < \liminf_{n\to\infty} s_n$,

where β are β' are real constants. Then $\{x_n\}$ converges weakly to some point in $Fix(S) \cap EP(F)$.

Corollary 2.2. *Let C be a nonempty closed convex subset of H and Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $B : H \rightrightarrows H$ be a maximal monotone mapping. Let $S : C \to C$ be a nonexpansive mapping. Assume that $Fix(S)\cap(A+B)^{-1}(0)$ is nonempty. Let $\{r_n\}$ be a positive real number sequence. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0,1)$ such that $\alpha_n+\beta_n+\gamma_n = 1$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$ and*

$$x_{n+1} = \alpha_n S x_n + \beta_n J_{r_n}\big(x_n - r_n A x_n + e_n\big) + \gamma_n f_n, \quad n \ge 1,$$

where $\{e_n\}$ is a bounded sequence in H such that $\sum_{n=1}^{\infty}\|e_n\| < \infty$ and $\{f_n\}$ is bounded sequence in C. Assume that the control sequences satisfy the following restrictions:

(a) $0 < \beta \le \beta_n \le \beta' < 1$ *and* $\sum_{n=1}^{\infty}\gamma_n < \infty$,

(b) $0 < r \le r_n \le r' < 2\alpha$,

where β, β', r and r' are real constants. Then $\{x_n\}$ converges weakly to some point in $Fix(S) \cap (A+B)^{-1}(0)$.

3. Applications

In this section, we give some results on solutions variational inequalities and minimizers of convex functions.

Let C be a nonempty closed and convex subset of H and $A : C \to H$ be a mapping. Recall that the classical variational inequality is to find an $x \in C$ such that

$$\langle Ax, y - x\rangle \ge 0, \quad \forall y \in C. \tag{3.1}$$

The solution set of (3.1) is denoted by $VI(C, A)$. Projection methods have been recently investigated for solving variational inequality (3.1). It is known that x is a solution to (3.1) iff x is a fixed point of the mapping $Proj_C(I - rA)$, where I denotes the identity on H. If A is inverse-strongly monotone, then $Proj_C(I - rA)$ is nonexpansive. Moreover, if C is bounded, closed and convex, then the existence of solutions of the variational inequality is guaranteed by the nonexpansivity of the mapping $Proj_C(I - rA)$. Let i_C be a function defined by

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

It is easy to see that i_C is a proper lower and semicontinuous convex function on H, and the subdifferential ∂i_C of i_C is maximal monotone. Define the resolvent

$J_r := (I + r\partial i_C)^{-1}$ of the subdifferential operator ∂i_C. Letting $x = J_r y$, we find that

$$\begin{aligned} y \in x + r\partial i_C x &\Longleftrightarrow y \in x + rN_C x \\ &\Longleftrightarrow \langle y - x, v - x\rangle \leq 0, \forall v \in C \\ &\Longleftrightarrow x = Proj_C y, \end{aligned}$$

where $N_C x := \{e \in H : \langle e, v - x\rangle, \forall v \in C\}$. Putting $B = \partial i_C$ in Theorems 2.1, we find the following results immediately.

Theorem 3.1. *Let C be a nonempty closed convex subset of H and let F be a bifunction from $C \times C$ to R which satisfies (A1)-(A4). Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $S : C \to C$ be a nonexpansive mapping. Assume that $Fix(S) \cap EP(F) \cap VI(C, A)$ is nonempty. Let $\{r_n\}$ and $\{s_n\}$ be positive real number sequences. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$ and*

$$\begin{cases} F(y_n, y) + \frac{1}{s_n}\langle y - y_n, y_n - Proj_C\big(x_n - r_n Ax_n + e_n\big)\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n Sx_n + \beta_n y_n + \gamma_n f_n, \quad n \geq 1, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$ and $\{f_n\}$ is bounded sequence in C. Assume that the control sequences satisfy the following restrictions:

(a) $0 < \beta \leq \beta_n \leq \beta' < 1$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$,

(b) $0 < \liminf_{n\to\infty} s_n$ *and* $0 < r \leq r_n \leq r' < 2\alpha$,

where β, β', r and r' are real constants. Then $\{x_n\}$ converges weakly to some point in $Fix(S) \cap EP(F) \cap VI(C, A)$.

Putting $F(x, y) = 0$ for any $x, y \in C$ and $s_n = 1$ in Theorem 3.1, we have the following result.

Corollary 3.1. *Let C be a nonempty closed convex subset of H . Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $S : C \to C$ be a nonexpansive mapping. Assume that $Fix(S) \cap VI(C, A)$ is nonempty. Let $\{r_n\}$ be a positive real number sequence. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$ and*

$$x_{n+1} = \alpha_n Sx_n + \beta_n Proj_C\big(x_n - r_n Ax_n + e_n\big) + \gamma_n f_n, \quad n \geq 1,$$

where $\{e_n\}$ is a bounded sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$ and $\{f_n\}$ is bounded sequence in C. Assume that the control sequences satisfy the following restrictions:

(a) $0 < \beta \leq \beta_n \leq \beta' < 1$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$,

(b) $0 < r \leq r_n \leq r' < 2\alpha$,

where β, β', r and r' are real constants. Then $\{x_n\}$ converges weakly to some point in $Fix(S) \cap VI(C, A)$.

Now, we are in a position to consider the problem of finding minimizers of proper lower semicontinuous convex functions. For a proper lower semicontinuous convex function $g : H \to (-\infty, \infty]$, the subdifferential mapping ∂g of g is defined by $\partial g(x) = \{x^* \in H : g(x) + \langle y - x, x^* \rangle \leq g(y), \forall y \in H\}$, $\forall x \in H$. Rockafellar [21] proved that ∂g is a maximal monotone operator. It is easy to verify that $0 \in \partial g(v)$ if and only if $g(v) = \min_{x \in H} g(x)$.

Theorem 3.2. *Let $g : H \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. Let $\{r_n\}$ be a positive real number sequence. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real number sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{x_n\}$ be a sequence generated in the following process: $x_1 \in C$ and*

$$\begin{cases} y_n = \arg\min_{z \in H}\{g(z) + \frac{\|z - x_n + e_n\|^2}{2r_n}\}, \\ x_{n+1} = \alpha_n x_n + \beta_n y_n + \gamma_n f_n, \quad n \geq 1, \end{cases}$$

where $\{e_n\}$ is a bounded sequence in H such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$ and $\{f_n\}$ is bounded sequence in C. Assume that the control sequences satisfy restrictions: $0 < \beta \leq \beta_n \leq \beta' < 1$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, and $0 < r \leq r_n \leq r' < 2\alpha$, where β, β', r and r' are real constants. Then $\{x_n\}$ converges weakly to some point in $(\partial g)^{-1}(0)$.

Proof. Since $g : H \to (-\infty, \infty]$ is a proper convex and lower semicontinuous function, we see that subdifferential ∂g of g is maximal monotone. Put $F(x, y) = 0$ for any $x, y \in C$, $s_n = 1$, and $A = 0$. Then $y_n = J_{r_n}(x_n + e_n)$. It follows that $y_n = \arg\min_{z \in H}\{g(z) + \frac{\|z - x_n - e_n\|^2}{2r_n}\}$ is equivalent to $0 \in \partial g(y_n) + \frac{1}{r_n}(y_n - x_n - e_n)$. It follows that $x_n + e_n \in y_n + r_n \partial g(y_n)$. By use of Theorem 2.1, we find the desired conclusion immediately. □

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