SIZE-STRUCTURED DYNAMICS IN A JUVENILE-ADULT POPULATION WITH FIXED SEX-RATIO*

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Abstract We present and analyze a nonlinear size-structured juvenile-adult model with a fixed sex-ratio, in which juveniles are structured by size, while adults by age. Global or local stability results are investigated by the method of characteristics and prior estimations, which show that, if the net reproduction rate is less than one, the population will be extinct for any initial distributions; On the other hand, the population may be extinct or increase in a manner faster than exponential style if the net reproduction rate is greater than one. By using Laplace transform methods, asymptotic behavior of solutions is analyzed too.

Keywords Juvenile-adult model, size-structured, fixed sex-ratio, stability, asymptotic behavior.

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1. Introduction

For many animal species (e.g. amphibians, insects and crustaceans), the individuals take on different morphological shapes between juvenile stage and adult stage. In resent decades, many discrete and continuous juvenile-adult models have been developed and studied in the literature(see [1–7, 11, 13, 15, 18, 19] and the references therein). We briefly comment on a few of them. In [11], J. M. Cushing discussed a general discrete juvenile-adult model with periodically varying coefficients by using bifurcation theory. The authors in [18] studied two lumped age-structured models of juvenile against adult competition. They discussed existence and stability of positive equilibria and existence of oscillatory solutions, and investigated the effects of the competitive interactions between juveniles and adults on the dynamics of the population too. In [19], the authors studied the evolution of a spatially structured population with two age classes using spatial moment equations. In their model, adults can either help juveniles by increasing their survival, or adopt a cannibalistic behaviour and consume juveniles. In [13], J. Z. Farkas and T. Hagen developed a nonlinear size-structured juvenile-adult population model, and analyzed the linearized dynamical behavior of stationary solutions by using semigroup theory. In [15], X. Fu and D. Zhu discussed the asymptotic behavior of a size-structured juvenile-adult population equation with resource-dependent and delayed birth process. In [1–7], A. S. Ackleh and his coauthors studied an amphibian population using discrete and continuous juvenile-adult models. In these articles, they divided

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individuals into juveniles (tadpoles) and adults (frogs). Since juveniles and adults depend on different resources, and no resource competition takes place between them. In [1], they developed a non-autonomous nonlinear continuous amphibian juvenile-adult model, and obtained the existence-uniqueness results and long-time behavior via a comparison principle. In [7], they established a second-order explicit high-resolution scheme to approximate the solution of the urban amphibian model. Therein, convergence of the finite difference approximation to the unique weak solution with bounded total variation is proved.

On the other hand, since R. A. Fisher developed the famous argument that parental expenditure on sons and daughters should be equal in [14], sex-ratio population models and relevant models have been widely investigated(see [9, 10, 17, 20] and the references therein). À. Calsina and J. Ripoll considered an origin (age and sex)-structured model for a sequential hermaphrodite population with general parameters accounting for the birth, transition and death processes in [9]. The most particular feature of the model is the transition from one sex to the other which is described by a given probability law. Then, in [19] they analyzed the long-term evolution of sex-reversal, through the long-term evolution of the probability law of the age at sex-reversal as a function-valued trait. In [20], T. Schmickl and I. Karsai investigated a population model with consideration of the factors sex-ratio, male success and gender-specific density-independent mortality. They analyzed how these factors affect population dynamics. In [17], M. Iannelli and J. Ripoll considered a two sex population with a fixed sex ratio. Therein, existence and stability of steady states are established, the population ergodicity is studied, survival thresholds are discussed and the effect of intraspecific competition is analysed too.

Taking both stage factor and sex-ratio factor into account, we present a juvenile-adult population model with a fixed sex-ratio for adults. In juvenile stage we consider size difference while age in adult stage. Our model is composed of two first-order partial differential equations with initial and boundary conditions. The remainder of the paper is organized as follows. In Section 2, we introduce the basic model and deduce the equilibria solution of model. In Section 3, we obtain some conditions for stability/instability of the equilibria. By applying Laplace transform methods, asymptotic behavior of solutions is analyzed in Section 4. The final Section contains some concluding remarks.

2. The basic model and its equilibria

We propose the following model to describe the dynamics of a juvenile-adult population with a fixed sex-ratio:

$$J_t(s,t) + (g(s)J(s,t))_s + \mu(s)J(s,t) = 0 \ , \ s_1 \leq s \leq s_2, \ t > 0, \tag{2.1}$$

$$g(s_1)J(s_1,t) = \Phi(M(t)) \int_0^{a_\dagger} \beta(a)A(a,t)\mathrm{d}a \ , \ t > 0, \tag{2.2}$$

$$J(s,0) = J_0(s) \ , \ s_1 \leq s \leq s_2, \tag{2.3}$$

$$A_t(a,t) + A_a(a,t) + v(a)A(a,t) = 0 \ , \ 0 \leq a < a_\dagger, \ t > 0, \tag{2.4}$$

$$A(0,t) = g(s_2)J(s_2,t) \ , \ t > 0 \ , \tag{2.5}$$

$$A(a,0) = A_0(a) \ , \ 0 \leq a < a_\dagger, \tag{2.6}$$

$$M(t) = \int_0^{a_\dagger} \gamma(a)A(a,t)\mathrm{d}a \ , \ t > 0, \tag{2.7}$$

where $J(s,t)$ and $A(a,t)$ denote the densities of juveniles of size s and adults of age a, respectively, at time t. The parameters μ and g are the mortality and growth rates of juveniles, respectively. s_1 and s_2 denote the minimum and maximum size of juveniles, respectively. The functions β and v are the fertility and mortality of adults, respectively. $a_\dagger$ denotes the maximum age of adults. Equation (2.5) says that s_2 is the size at which juveniles mature into adults. γ denotes the proportion of males in adults. The function Φ is relative to the number of males of adults that a female of adults meets in the time unit, and has the same properties as described in [17]. That is, the function Φ is increasing and takes all the values between 0 to 1, see [17] for a detailed description.

Throughout this paper the following assumptions hold:

(A1): $\mu \in L^1_{loc,+}(s_1, s_2)$, $\beta \in L^\infty_+(0, a_\dagger)$;

(A2): $g \in C^1_+(s_1, s_2)$, $\int_{s_1}^{s_2} \frac{1}{g(x)}\mathrm{d}x < +\infty$, this condition will guarantee that juveniles reach the maximum size in a finite time;

(A3): $\gamma \in L^\infty_+(0, a_\dagger)$, $v \in L^1_{loc,+}(0, a_\dagger)$, $\int_0^{a_\dagger} v(a)\mathrm{d}a = +\infty$;

(A4): $J_0 \in L^1_+(s_1, s_2)$, $A_0 \in L^1_+(0, a_\dagger)$;

(A5): All of variables and parameters are nonnegative in their domains and are extended by zero outside their domains.

We first introduce the definition of the solution of problem (2.1)-(2.7) via the method of characteristics.

Definition 2.1. A couple of integrable nonnegative functions $(J(s,t), A(s,t))$ is said to be a solution of system (2.1)-(2.7) if it satisfies the following equations:

$$DJ(s,t) = -[\mu(s) + \frac{\partial g(s)}{\partial s}]J(s,t),$$
$$DA(a,t) = -v(a)A(a,t),$$

with

$$DJ(s,t) = \lim_{h\to 0} \frac{J\left(\tau^{-1}(\tau(s) + h), t + h\right) - J(s,t)}{h},$$
$$DA(a,t) = \lim_{h\to 0} \frac{A(a + h, t + h) - A(a,t)}{h},$$

where $\tau(s) = \int_{s_1}^s \frac{1}{g(x)}\mathrm{d}x$, which denotes the time required for a newborn with the minimum size s_1 to reach a size s.

Any equilibrium $(J^*(s), A^*(a))$ of the system (2.1)-(2.7) satisfies the equations:

$$(g(s)J^*(s))_s + \mu(s)J^*(s) = 0, \tag{2.8}$$

$$g(s_1)J^*(s_1) = \Phi(M^*) \int_0^{a_\dagger} \beta(a)A^*(a)\mathrm{d}a, \tag{2.9}$$

$$A_a^*(a) + v(a)A^*(a) = 0, \tag{2.10}$$

$$A^*(0) = g(s_2)J^*(s_2), \tag{2.11}$$

$$M^* = \int_0^{a_\dagger} \gamma(a)A^*(a)\mathrm{d}a. \tag{2.12}$$

The general solution of Eq. (2.8) and Eq. (2.10) are found as

$$J^*(s) = J^*(s_1)\Pi_1(s), \tag{2.13}$$

and

$$A^*(a) = A^*(0)\Pi_0(a), \tag{2.14}$$

respectively. Hereafter,

$$\Pi_0(a) = \exp\left\{-\int_0^a v(x)\mathrm{d}x\right\}, \ \Pi_1(s) = \exp\left\{-\int_{s_1}^s \frac{\mu(x) + g_x(x)}{g(x)}\mathrm{d}x\right\}.$$

From Eq. (2.12) and Eq. (2.14) we obtain

$$A^*(a) = \frac{M^*\Pi_0(a)}{\int_0^{a_\dagger} \gamma(a)\Pi_0(a)\mathrm{d}a}. \tag{2.15}$$

Setting $s = s_2$ in Eq. (2.13), we have $J^*(s_2) = J^*(s_1)\Pi_1(s_2)$, together with Eq. (2.11) and Eqs. (2.13-2.15), it yields that

$$J^*(s) = \frac{M^*\Pi_1(s)}{g(s_2)\Pi_1(s_2)\int_0^{a_\dagger} \gamma(a)\Pi_0(a)\mathrm{d}a}. \tag{2.16}$$

Substituting Eq.(2.13) and Eqs.(2.15)-(2.16) into Eq.(2.9) we see that M^* must satisfy the following nonlinear equation:

$$R\Phi(M^*) = 1, \tag{2.17}$$

where

$$R = \frac{g(s_2)}{g(s_1)}\Pi_1(s_2)\int_0^{a_\dagger} \beta(a)\Pi_0(a)\mathrm{d}a. \tag{2.18}$$

The term $\frac{g(s_2)}{g(s_1)}\Pi_1(s_2)$ represents the probability that a juvenile survives and reaches the maturation size s_2(see [12]). $\Pi_0(a)$ denotes the survival probability for an adult to reach age a, together with the adult fertility $\beta(a)$ give the so-called maternity function for age-structured models(see [16]). So that the parameter R in (2.18), which can be called the net reproduction rate, gives the expected number of offspring produced by an individual during his lifespan in the case of a complete availability of males considering both stages juvenile and adult.

So, we have shown that, to each positive solution M^* of Eq. (2.17), there belongs a uniquely determined equilibrium $(J^*(s), A^*(a))$. According to the properties of Φ, it is not difficult to see that there is no non-trivial equilibrium if $R \leq 1$ whereas there exists a unique non-trivial equilibrium if $R > 1$.

3. Stability of equilibria

In this section we study the stability of the equilibria of system (2.1)-(2.7). Some sufficient conditions for stability/instability of equilibria will be given here.

Utilizing the method of curve technique, the solution of adult system (2.4)-(2.6) can be given as:

$$A(a,t) = \begin{cases} A_0(a-t)\frac{\Pi_0(a)}{\Pi_0(a-t)}, & t \leq a, \\ g(s_2)J(s_2, t-a)\Pi_0(a), & t > a, \end{cases} \tag{3.1}$$

Substituting Eq. (3.1) into Eq. (2.2), we obtain

$$J_t(s,t) + g(s)J_s(s,t) = -[\mu(s) + g_s(s)]J(s,t) \ , \ s_1 \leq s \leq s_2, \ t > 0, \tag{3.2}$$

$$g(s_1)J(s_1,t) = \begin{cases} \Phi(M(t))\int_t^{a_\dagger} \beta(a)A_0(a-t)\frac{\Pi_0(a)}{\Pi_0(a-t)}\mathrm{d}a \\ +\Phi(M(t))\int_0^t \beta(a)g(s_2)J(s_2, t-a)\Pi_0(a)\mathrm{d}a \ , \ 0 < t \leq a_\dagger, \\ \Phi(M(t))\int_0^{a_\dagger} \beta(a)g(s_2)J(s_2, t-a)\Pi_0(a)\mathrm{d}a \ , \ t > a_\dagger, \end{cases} \tag{3.3}$$

$$J(s,0) = J_0(s) \ , \ s_1 \leq s \leq s_2. \tag{3.4}$$

Then, a standard calculation for system (3.2)-(3.4) leads to:

$$J(s,t) = \begin{cases} J_0(\tau^{-1}(\tau(s)-t))\frac{\Pi_1(s)}{\Pi_1(\tau^{-1}(\tau(s)-t))}, & 0 < t \leq \tau(s), \\ J(s_1, t-\tau(s))\Pi_1(s), & t > \tau(s). \end{cases} \tag{3.5}$$

Noting that $\tau(s_2) < +\infty$, and denoting $B(t) = J(s_2, t)$, we have, for $t > \tau(s_2)$

$$J(s_2, t) = J(s_1, t-\tau(s_2))\Pi_1(s_2),$$

hence, from Eqs. (3.1), (3.3) and (3.5), and introducing the notations,

$$\Pi_0(a) = \exp\left\{-\int_0^a v(x)\mathrm{d}x\right\}, \ \Pi_1(s) = \exp\left\{-\int_{s_1}^s \frac{\mu(x) + g_x(x)}{g(x)}\mathrm{d}x\right\},$$

$$F_0(t) = J_0(\tau^{-1}(\tau(s_2)-t))\frac{\Pi_1(s_2)}{\Pi_1(\tau^{-1}(\tau(s_2)-t))},$$

$$K_0(a) = \beta(a)\Pi_0(a), \quad H_0(a) = \gamma(a)\Pi_0(a),$$

$$K(a) = \frac{g(s_2)}{g(s_1)}\Pi_1(s_2)K_0(a), \quad H(a) = g(s_2)H_0(a),$$

$$F(t) = \frac{\Pi_1(s_2)}{g(s_1)}\int_0^{a_\dagger-t} K_0(a+t)\frac{A_0(a)}{\Pi_0(a)}\mathrm{d}a,$$

$$G(t) = \int_0^{a_\dagger-t} H_0(a+t)\frac{A_0(a)}{\Pi_0(a)}\mathrm{d}a,$$

with all the functions extended by zero outside the interval $[0, a_\dagger]$, we obtain the following integral system

$$\begin{cases} B(t) = F_0(t) + \Phi(M(t-\tau(s_2)))\left(\int_0^{t-\tau(s_2)} K(a)B(t-\tau(s_2)-a)\mathrm{d}a + F(t-\tau(s_2))\right), \\ M(t) = \int_0^t H(a)B(t-a)\mathrm{d}a + G(t), \end{cases}$$

$$\tag{3.6}$$

which is equivalent to the juvenile system (3.2)-(3.4). From the non-negative solution of (3.6) we can get the solution of the basic system as

$$
A(a,t) = \begin{cases} A_0(a-t)\dfrac{\Pi_0(a)}{\Pi_0(a-t)}, & t \leq a, \\[2mm] B(t-a)g(s_2)\Pi_0(a), & t > a, \end{cases} \tag{3.7}
$$

$$
J(s,t) = \begin{cases} J_0(\tau^{-1}(\tau(s)-t))\dfrac{\Pi_1(s)}{\Pi_1(\tau^{-1}(\tau(s)-t))}, & 0 < t \leq \tau(s), \\[3mm] \dfrac{\Pi_1(s)}{g(s_1)}\Phi(M(t-\tau(s)))\Big\{ \int_0^{a_\dagger -t+\tau(s)} \dfrac{K_0(a+t-\tau(s))A_0(a)}{\Pi_0(a)}\mathrm{d}a \\[3mm] \qquad + \int_0^{t-\tau(s)} g(s_2)K_0(a)B(t-\tau(s)-a)\mathrm{d}a \Big\}, & \tau(s) < t \leq \tau(s)+a_\dagger, \\[3mm] \dfrac{\Pi_1(s)}{g(s_1)}\Phi(M(t-\tau(s)))\int_0^{a_\dagger} g(s_2)K_0(a)B(t-\tau(s)-a)\mathrm{d}a, & t > \tau(s)+a_\dagger. \end{cases} \tag{3.8}
$$

Based on these formulations, we first have a global result concerning the trivial equilibrium.

Theorem 3.1. *If $R < 1$, that is the net reproduction rate is less than one, then the trivial equilibrium is globally asymptotically stable.*

Proof. From the first equation in (3.6) we have

$$
\max_{x\in[0,t]} B(x) \leq R\max_{x\in[0,t]} B(x) + \max_{x\in[\tau(s_2),\tau(s_2)+a_\dagger]} F(x) + \max_{x\in[0,\tau(s_2)]} F_0(x)
$$

consequently

$$
\sup_{t\geq 0} B(t) \leq \frac{1}{1-R}\left(\max_{x\in[\tau(s_2),\tau(s_2)+a_\dagger]} F(x) + \max_{x\in[0,\tau(s_2)]} F_0(x)\right)
$$

$$
\leq \frac{1}{1-R}\left(\frac{\Pi_1(s_2)}{g(s_1)}\|\beta\|_{L^\infty(0,a_\dagger)}\|A_0(\cdot)\|_{L^1(0,a_\dagger)} + \|J_0(\cdot)\|_{L^1(s_1,s_2)}\right),
$$

which means that $B(t)$ is bounded.

Then, using this estimate and the form (3.7)-(3.8) of the solution, a straightforward calculation leads to

$$
\|A(\cdot,t)\|_{L^1(0,a_\dagger)} + \|J(\cdot,t)\|_{L^1(s_1,s_2)}
$$
$$
\leq \|A_0(\cdot)\|_{L^1(0,a_\dagger)} + g(s_2)a_\dagger\sup_{t\geq 0} B(t)
$$
$$
+ \|J_0(\cdot)\|_{L^1(s_1,s_2)} + \frac{\|\beta\|_{L^\infty(0,a_\dagger)}}{g(s_1)}\|A_0(\cdot)\|_{L^1(0,a_\dagger)} + \frac{2g(s_2)}{g(s_1)}a_\dagger\|\beta\|_{L^\infty(0,a_\dagger)}\sup_{t\geq 0} B(t)
$$
$$
\leq \eta_1\|A_0(\cdot)\|_{L^1(0,a_\dagger)} + \eta_2\|J_0(\cdot)\|_{L^1(s_1,s_2)},
$$

where

$$
\eta_1 = 1 + \frac{g(s_2)\Pi_1(s_2)a_\dagger\|\beta\|_{L^\infty(0,a_\dagger)}}{(1-R)g(s_1)} + \frac{\|\beta\|_{L^\infty(0,a_\dagger)}}{g(s_1)} + \frac{2g(s_2)\Pi_1(s_2)a_\dagger\|\beta\|^2_{L^\infty(0,a_\dagger)}}{(1-R)g^2(s_1)},
$$
$$
\eta_2 = 1 + \frac{g(s_2)a_\dagger}{1-R} + \frac{2g(s_2)a_\dagger\|\beta\|_{L^\infty(0,a_\dagger)}}{(1-R)g(s_1)},
$$

which shows that the solution is stable. For $t > \tau(s_2) + a_\dagger$, the first equation in (3.6) implies that

$$B(t) = \Phi(M(t - \tau(s_2))) \int_0^{t-\tau(s_2)} K(a)B(t - \tau(s_2) - a)\mathrm{d}a,$$

hence,

$$\limsup_{t\to\infty} B(t) \le R \limsup_{t\to\infty} B(t),$$

since $R < 1$ and $B(t)$ is bounded, it follows that

$$\lim_{t\to\infty} B(t) = 0.$$

By (3.7) and (3.8) we finally obtain

$$\lim_{t\to\infty} \|A(\cdot,t)\|_{L^1(0,a_\dagger)} = 0, \qquad \lim_{t\to\infty} \|J(\cdot,t)\|_{L^1(s_1,s_2)} = 0,$$

which yield what we desire. $\square$

Now we discuss the stability of equilibria in the case of $R > 1$. The following two Theorems show that the population will be extinct or increase in suitable conditions.

Theorem 3.2. *If $R > 1$, and suppose that the following additional assumptions are satisfied:*

(A6): $A_0(a) \le \delta_1 A^*(a)$, a.e. in $[0, a_\dagger]$, $\delta_1 < 1$;

(A7): $J_0(s) \le \delta_2 \frac{\Pi_1(s_2)}{\Pi_1(s)} J^*(s)$, a.e. in $[s_1, s_2]$, $\delta_2 < 1$.

Then we have

$$\lim_{t\to\infty} \|A(\cdot,t)\|_{L^1(0,a_\dagger)} = 0, \qquad \lim_{t\to\infty} \|J(\cdot,t)\|_{L^1(s_1,s_2)} = 0.$$

Proof. From the second equation in (3.6) we have

$$M(0) = G(0) = \int_0^\infty \gamma(a)A_0(a)\mathrm{d}a \le \int_0^\infty \gamma(a)\delta_1 A^*(a)\mathrm{d}a = \delta_1 M^*.$$

Then we define

$$t_0 = \sup\left\{ T | M(t) \le \tilde{\delta}M^*, \quad t \in [0,T] \right\},$$

where $\tilde{\delta}$ is chosen such that

$$\delta_1 < \tilde{\delta} < 1, \text{ and } \tilde{\delta}(1 - R\Phi(\tilde{\delta}M^*)) > \delta_2. \tag{3.9}$$

If $t_0 < \infty$, a straightforward calculation from (3.6) yields

$$\max_{x\in[0,t_0]} B(x)$$

$$\le \Phi(\tilde{\delta}M^*)\left(\max_{x\in[0,t_0]} B(x) \int_0^t K(a)\mathrm{d}a + \frac{\delta_1 M^* \int_t^\infty K(a)\mathrm{d}a}{\int_0^{a_\dagger} H(a)\mathrm{d}a} \right) + \frac{\delta_2 M^*}{\int_0^{a_\dagger} H(a)\mathrm{d}a}$$

$$\le R\Phi(\tilde{\delta}M^*)\max\left\{ \max_{x\in[0,t_0]} B(x), \frac{\delta_1 M^*}{\int_0^{a_\dagger} H(a)\mathrm{d}a} \right\} + \frac{\delta_2 M^*}{\int_0^{a_\dagger} H(a)\mathrm{d}a}.$$

If $\max\limits_{x\in[0,t_0]} B(x) \le \frac{\delta_1 M^*}{\int_0^{a_\dagger} H(a)\mathrm{d}a}$, then putting this inequality into the second of (3.6) we have

$$\tilde{\delta} M^* = M(t_0) = \int_0^{t_0} H(a)B(t_0 - a)\mathrm{d}a + G(t_0) \le \delta_1 M^*,$$

which is a contradiction.

On the other hand, if $\max\limits_{x\in[0,t_0]} B(x) > \frac{\delta_1 M^*}{\int_0^{a_\dagger} H(a)\mathrm{d}a}$, and noting that $R\Phi(\tilde{\delta} M^*) < 1$, we have

$$\max\limits_{x\in[0,t_0]} B(x) \le \frac{\delta_2 M^*}{(1 - R\Phi(\tilde{\delta} M^*)) \int_0^{a_\dagger} H(a)\mathrm{d}a}.$$

Plugging the above relation into the second of (3.6), it follows that

$$\tilde{\delta} M^* = M(t_0) = \int_0^{t_0} H(a)B(t_0 - a)\mathrm{d}a + G(t_0)$$

$$\le \int_0^{t_0} H(a)\mathrm{d}a \frac{\delta_2 M^*}{(1 - R\Phi(\tilde{\delta} M^*)) \int_0^{a_\dagger} H(a)\mathrm{d}a} + \int_{t_0}^{a_\dagger} H(a)\mathrm{d}a \frac{\delta_1 M^*}{\int_0^{a_\dagger} H(a)\mathrm{d}a}$$

$$\le \max\left\{ \frac{\delta_2 M^*}{(1 - R\Phi(\tilde{\delta} M^*))}, \; \delta_1 M^* \right\},$$

which contradicts the condition (3.9).

Up to now, we have shown that $t_0 = \infty$. Hence, $\sup\limits_{t\ge 0} M(t) \le \tilde{\delta} M^*$, $\sup\limits_{t\ge 0} B(t) < +\infty$, and from the first equation in (3.6), we get

$$\lim\limits_{t\to\infty} \sup B(t) \le R\Phi(\tilde{\delta} M^*) \lim\limits_{t\to\infty} \sup B(t),$$

which implies $\lim\limits_{t\to\infty} B(t) = 0$ by $R\Phi(\tilde{\delta} M^*) < 1$.

By (3.7) and (3.8) we finally obtain

$$\lim\limits_{t\to\infty} \|A(\cdot,t)\|_{L^1(0,a_\dagger)} = 0, \qquad \lim\limits_{t\to\infty} \|J(\cdot,t)\|_{L^1(s_1,s_2)} = 0.$$

$\square$

Now we present the population increasing result.

Theorem 3.3. *If $R > 1$, and suppose the following additional assumption holds too:*

$(A8): A_0(a) \ge \delta_3 A^*(a), \; a.e. \; in \; [0,a_\dagger], \; \delta_3 > 1;$

Then

$$\lim\limits_{t\to\infty} \|A(\cdot,t)\|_{L^1(0,a_\dagger)} = +\infty, \qquad \lim\limits_{t\to\infty} \|J(\cdot,t)\|_{L^1(s_1,s_2)} = +\infty.$$

Proof. Let $t_0 = \sup\left\{ T | M(t) \ge \tilde{\delta} M^*, \; t \in [0,T] \right\}$ with $1 < \tilde{\delta} < \delta_3$. If $t_0 < \infty$, from the equation (3.6), we have

$$\min\limits_{x\in[0,t_0]} B(x) \ge R\Phi(\tilde{\delta} M^*) \min\left\{ \min\limits_{x\in[0,t_0]} B(x), \; \frac{\delta_3 M^*}{\int_0^{a_\dagger} H(a)\mathrm{d}a} \right\}.$$

If we could prove that $B(t) > 0$ for all t, together with the fact that $R\Phi(\tilde{\delta}M^*) > 1$, it leads to

$$\min_{x \in [0,t_0]} B(x) \geq \frac{\delta_3 M^*}{\int_0^{a_\dagger} H(a)\mathrm{d}a},$$

then from the second equation in (3.6) we have $\tilde{\delta}M^* = M(t_0) \geq \delta_3 M^*$, which is a contradiction. That is $t_0 = \infty$, from this and the above inequality we draw

$$\inf_{t \geq 0} M(t) \geq \tilde{\delta}M^*, \ \inf_{t \geq 0} B(t) > 0,$$

considering the first equation of (3.6) again, we obtain

$$\lim_{t \to \infty} \inf B(t) \geq R\Phi(\tilde{\delta}M^*) \lim_{t \to \infty} \inf B(t),$$

which yields the conclusion we desire by (3.7) and (3.8).

In what follows we show that $B(t) > 0$ for all t. For $t < \tau(s_2)$, we get $B(t) = F_0(t)$ from Eq. (3.6). Since $F_0(t) > 0$ by the assumption (A4), we have $B(t) > 0$ for $t < \tau(s_2)$.

Applying the assumption (A8), it is not hard to show that $M(0) \geq \delta_3 M^* > 0$, and consequently we have

$$B(\tau(s_2)) = \Phi(M(0))F(0) + F_0(\tau(s_2)) > R\Phi(M(0))\frac{\delta_3 M^*}{\int_0^{a_\dagger} H(a)\mathrm{d}a} > 0.$$

If there exists $t_0 \in [\tau(s_2), \infty)$ such that

$$B(t_0) = 0, \quad B(t) > 0 \text{ for } t \in [\tau(s_2), \infty),$$

we have

$$0 = B(t_0) \geq \int_0^{t_0 - \tau(s_2)} K(a + \tau(s_2) - t_0)B(a)\mathrm{d}a$$
$$+ \frac{\Pi_1(s_2)\delta_3 M^*}{g(s_1)\int_0^{a_\dagger} H(a)\mathrm{d}a} \int_{t_0 - \tau(s_2)}^{a_\dagger} K(a)\mathrm{d}a,$$

which implies the wrong conclusion $K(a) = 0$ for $a \in [0, a_\dagger]$. Hence, we have $B(t) > 0$ for all t, and the proof is complete. $\qquad\square$

Therefore, we have the following corollary.

Corollary 3.1. *If $R > 1$, then the trivial equilibrium is locally asymptotically stable and the non-trivial one is unstable.*

The following theorem shows that the population will increase in a manner faster than exponential style, if the net reproduction rate is greater than one.

Theorem 3.4. *If $R > 1$ and the assumption (A8) are satisfied, then we have*

$$B(t) > Ce^{\omega t}, \ \|A(\cdot, t)\|_{L^1(0,a_\dagger)} > Ce^{\omega t},$$
$$M(t) > Ce^{\omega t}, \ \|J(\cdot, t)\|_{L^1(s_1,s_2)} > Ce^{\omega t},$$

where C and ω are suitable positive constants.

Proof.   From the proof of Theorem 3.3 we have, for $t \geq 0$,

$$B(t) > 0, \ M(t) \geq \tilde{\delta} M^*.$$

Setting

$$B_n = \min_{t \in [n(\tau(s_2)+a_\dagger),(n+1)(\tau(s_2)+a_\dagger)]} B(t), \ n = 0, 1, 2, \cdots.$$

By the first equation of (3.6) and the assumption (A8), we obtain

$$B_n \geq R\Phi(\tilde{\delta} M^*) \min\{B_n, B_{n-1}\}, \ n = 1, 2, \cdots.$$

Since $B_n > 0$ and $R\Phi(\tilde{\delta} M^*) > 1$, we have $B_n \geq R\Phi(\tilde{\delta} M^*) B_{n-1}$. By iteration it yields $B_n \geq (R\Phi(\tilde{\delta} M^*))^n B_0$. Then, if $t \in [n(\tau(s_2)+a_\dagger), (n+1)(\tau(s_2)+a_\dagger)]$, that is $B(t) > (R\Phi(\tilde{\delta} M^*))^{-1} B_0 (R\Phi(\tilde{\delta} M^*))^{\frac{t}{\tau(s_2)+a_\dagger}}$. Therefore, we have $B(t) > Ce^{\omega t}$ with suitable positive C and ω. Finally the conclusion follows because, using (3.6)-(3.8), for $t > \tau(s_2) + a_\dagger$ we have

$$\|A(\cdot, t)\|_{L^1(0,a_\dagger)} = \int_0^{a_\dagger} B(t-a)g(s_2)\Pi_0(a)\mathrm{d}a > Ce^{\omega t},$$

$$\|J(\cdot, t)\|_{L^1(s_1,s_2)} = \int_{s_1}^{s_2} \Phi(M(t-\tau(s))) \int_0^t \frac{\Pi_1(s)g(s_2)K_0(a)}{g(s_1)} B(t-\tau(s)-a)\mathrm{d}a\mathrm{d}s,$$
$$> Ce^{\omega t},$$

$$M(t) = \int_0^t H(a)B(t-a)\mathrm{d}a + G(t) > Ce^{\omega t},$$

with a suitable constant C. $\square$

4. Asymptotic behavior of solutions

In this section, we will investigate the asymptotic behavior of solutions of system (2.1)-(2.7) by virtue of Laplace transform methods (see [8, 16]). To this end, we start by discussing the asymptotic behavior of $B(t)$. By (3.6) we may infer via Bellman's lemma that

$$B(t) \leq \left(\|J_0\|_{L^1(s_1,s_2)} + \frac{\|\beta\|_{L^\infty(0,a_\dagger)}\|A_0\|_{L^1(0,a_\dagger)}}{g(s_1)} \right) e^{t\frac{g(s_2)}{g(s_1)}\|\beta\|_{L^\infty(0,a_\dagger)}}, \ \forall t \in [0, \infty)$$

and this implies that $B(t)$ is absolutely Laplace transformable. Now rewriting the first equation of (3.6) as

$$B(t) = F_0(t) + F(t-\tau(s_2)) - h(t-\tau(s_2)) + \int_0^{t-\tau(s_2)} K(a)B(t-\tau(s_2)-a)\mathrm{d}a,$$

where

$$h(t) = \left(1 - \Phi(M(t))\right) \left(\int_0^t K(a)B(t-a)\mathrm{d}a + F(t) \right),$$

then we obtain in the same manner that $|h(t)| \leq Ce^{\omega t}$ with some positive constants C and ω. Thus $h(t)$ is also Laplace transformable.

Setting $\mathcal{L}(B)(\lambda)$ to denote the Laplace transform of $B(t)$, we have

$$\mathcal{L}(B)(\lambda)$$
$$= \int_0^{\tau(s_2)} e^{-\lambda t} F_0(t) \mathrm{d}t + \int_{\tau(s_2)}^{\tau(s_2)+a_\dagger} e^{-\lambda t} F(t - \tau(s_2)) \mathrm{d}t$$
$$- \int_{\tau(s_2)}^{\infty} e^{-\lambda t} h(t - \tau(s_2)) \mathrm{d}t + \int_{\tau(s_2)}^{\infty} e^{-\lambda t} \int_0^{t-\tau(s_2)} K(a) B(t - \tau(s_2) - a) \mathrm{d}a \mathrm{d}t$$
$$= \mathcal{L}(F_0)(\lambda) + e^{-\lambda \tau(s_2)} (\mathcal{L}(F)(\lambda) - \mathcal{L}(h)(\lambda)) + e^{-\lambda \tau(s_2)} \mathcal{L}(K)(\lambda) \mathcal{L}(B)(\lambda),$$

and in conclusion

$$\mathcal{L}(B)(\lambda) = \frac{\mathcal{L}(F_0)(\lambda) + e^{-\lambda \tau(s_2)} (\mathcal{L}(F)(\lambda) - \mathcal{L}(h)(\lambda))}{1 - e^{-\lambda \tau(s_2)} \mathcal{L}(K)(\lambda)}. \tag{4.1}$$

It is well known that the asymptotic behavior of $B(t)$ is related to the singularities of $\mathcal{L}(B)(\lambda)$. By (4.1) we conclude that $\mathcal{L}(B)(\lambda)$ can have poles only among the roots of the equation

$$1 - e^{-\lambda \tau(s_2)} \mathcal{L}(K)(\lambda) = 0. \tag{4.2}$$

With respect to this latter equation we have:

Theorem 4.1. *Equation (4.2) has a unique real solution α^*, which is a simple root. This solution is negative if and only if $R < 1$. Any other solution α of (4.2) satisfies $Re\alpha < \alpha^*$ and within any strip $\sigma_1 < Re\lambda < \sigma_2$ there is at most a finite number of roots.*

Proof. Let $\varphi(\lambda) = e^{-\lambda \tau(s_2)} \mathcal{L}(K)(\lambda)$, then it is strictly decreasing on $\mathbf{R}$ because

$$\varphi'(\lambda) = -(t + \tau(s_2)) \int_0^{\infty} e^{-\lambda(t+\tau(s_2))} K(t) \mathrm{d}t < 0$$

and satisfies

$$\lim_{\lambda \to -\infty} \varphi(\lambda) = +\infty, \quad \lim_{\lambda \to +\infty} \varphi(\lambda) = 0.$$

It follows that there exists a unique $\alpha^* \in \mathbf{R}$ such that $\varphi(\alpha^*) = 1$, and since $\varphi'(\alpha^*) < 0$ we get that α^* is simple. Comparing relation (2.18) with the notation $K(a)$ leads to $R = \int_0^{a_\dagger} K(a) \mathrm{d}a$. Therefore, it is obvious that $\alpha^* < 0$ if and only if $\varphi(0) = \int_0^{\infty} K(a) \mathrm{d}a < 1$.

Let α be another solution of (4.2), then

$$\int_0^{\infty} e^{-\alpha^*(t+\tau(s_2))} K(t) \mathrm{d}t = 1 = Re\left(\int_0^{\infty} e^{-\alpha(t+\tau(s_2))} K(t) \mathrm{d}t \right)$$
$$= \int_0^{\infty} e^{-(Re\alpha)(t+\tau(s_2))} \cos((Im\alpha)(t + \tau(s_2))) K(t) \mathrm{d}t$$
$$< \int_0^{\infty} e^{-(Re\alpha)(t+\tau(s_2))} K(t) \mathrm{d}t$$

and as a consequence we obtain that $Re\alpha < \alpha^*$.

Finally, since $\varphi(\lambda) \to 0$ as $|\lambda| \to +\infty$, all the roots within the strip $\sigma_1 < Re\lambda <$

σ_2 must lie in some bounded subset and be finite in number because otherwise $\varphi(\lambda)$ would vanish identically. $\square$

Now we are ready to formulate the result concerning the asymptotic behavior of $B(t)$.

Theorem 4.2. *Let $R > 1$ so that $\alpha^* > 0$. Suppose the assumption (A8) are satisfied. Then the solution $B(t)$ of (3.6) satisfies*

$$B(t) = b_0 e^{\alpha^* t}(1 + \Omega(t)),$$

where $b_0 > 0$ and $\lim_{t \to +\infty} \Omega(t) = 0$.

Proof.   To simplify the notation, we first set

$$\mathcal{L}(\zeta)(\lambda) = \mathcal{L}(F_0)(\lambda) + e^{-\lambda \tau(s_2)} \mathcal{L}(F)(\lambda),$$
$$\mathcal{L}(\xi)(\lambda) = e^{-\lambda \tau(s_2)} \mathcal{L}(h)(\lambda),$$
$$\mathcal{L}(\eta)(\lambda) = e^{-\lambda \tau(s_2)} \mathcal{L}(K)(\lambda).$$

Then the relation (4.1) can be rewritten as

$$\mathcal{L}(B)(\lambda) = \frac{\mathcal{L}(\zeta)(\lambda) - \mathcal{L}(\xi)(\lambda)}{1 - \mathcal{L}(\eta)(\lambda)}. \tag{4.3}$$

Since, by a classical result we get that

$$\lim_{|\lambda| \to +\infty, Re\lambda > \alpha} \mathcal{L}(\zeta)(\lambda) = \lim_{|\lambda| \to +\infty, Re\lambda > \alpha} \mathcal{L}(\xi)(\lambda) = \lim_{|\lambda| \to +\infty, Re\lambda > \alpha} \mathcal{L}(\eta)(\lambda) = 0, \forall \alpha \in \mathbf{R}. \tag{4.4}$$

We conclude that

$$\lim_{|\lambda| \to +\infty, Re\lambda > \alpha} \frac{\mathcal{L}(\zeta)(\lambda) - \mathcal{L}(\xi)(\lambda)}{1 - \mathcal{L}(\eta)(\lambda)} = 0.$$

On the other hand

$$\int_{-\infty}^{+\infty} \left| \frac{\mathcal{L}(\zeta)(x + iy) - \mathcal{L}(\xi)(x + iy)}{1 - \mathcal{L}(\eta)(x + iy)} \right| dy < +\infty, \tag{4.5}$$

where $x \in \mathbf{R}$ is such that $\mathcal{L}(\eta)(x + iy) \neq 1$, $\forall y \in \mathbf{R}$. Indeed, relation (4.4) implies that $m(x) = \inf_{y \in \mathbf{R}} |1 - \mathcal{L}(\eta)(x + iy)| > 0$.

Define the functions

$$\zeta_x(t) = \begin{cases} e^{-xt}\zeta(t), \ t \geq 0, \\ 0, \ t < 0, \end{cases} \qquad \xi_x(t) = \begin{cases} e^{-xt}\xi(t), \ t \geq 0, \\ 0, \ t < 0. \end{cases}$$

It is evident that

$$\widehat{\zeta_x}(y) = \mathcal{L}(\zeta)(x + iy), \quad \widehat{\xi_x}(y) = \mathcal{L}(\xi)(x + iy),$$

with $\widehat{\zeta_x}$ and $\widehat{\xi_x}$ are Fourier transforms of ζ_x and ξ_x, respectively.

Thus

$$\left|\frac{\mathcal{L}(\zeta)(x+iy) - \mathcal{L}(\xi)(x+iy)}{1 - \mathcal{L}(\eta)(x+iy)}\right| \le \frac{|\widehat{\zeta}_x(y) - \widehat{\xi}_x(y)|}{m(x)},$$

and consequently

$$\int_{-\infty}^{+\infty} \left|\frac{\mathcal{L}(\zeta)(x+iy) - \mathcal{L}(\xi)(x+iy)}{1 - \mathcal{L}(\eta)(x+iy)}\right| \le \frac{\|\widehat{\zeta}_x - \widehat{\xi}_x\|_{L^1(\mathbf{R})}}{m(x)},$$

which shows that (4.5) is satisfied.

For any fixed $x > \alpha^*$ we consider the Laplace transform inversion formula

$$g(t) := \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{\lambda t} \frac{\mathcal{L}(\zeta)(\lambda) - \mathcal{L}(\xi)(\lambda)}{1 - \mathcal{L}(\eta)(\lambda)} \mathrm{d}\lambda, \qquad (4.6)$$

which has the Laplace transform

$$\mathcal{L}(g)(t) = \frac{\mathcal{L}(\zeta)(\lambda) - \mathcal{L}(\xi)(\lambda)}{1 - \mathcal{L}(\eta)(\lambda)}.$$

Using (4.3) we may infer that $g(t) = B(t)$, $t \ge 0$.

Finally, chosen $x_1 < \alpha^*$, such that any root α of (4.2), other than α^*, satisfies $Re\alpha < x_1$. By (4.4) and (4.5) we shift the integration in (4.6), from x to x_1. Then, applying the theorem of residuals, it follows that

$$g(t) = e^{\alpha^* t}(b_0 + \omega(t)),$$

where

$$b_0 = Res\left(\frac{\mathcal{L}(\zeta)(\lambda) - \mathcal{L}(\xi)(\lambda)}{1 - \mathcal{L}(\eta)(\lambda)}\right)(\alpha^*) = \frac{\mathcal{L}(\zeta)(\alpha^*) - \mathcal{L}(\xi)(\alpha^*)}{\int_0^\infty (t + \tau(s_2)) e^{-\alpha^*(t+\tau(s_2))} K(t)\mathrm{d}t} \ne 0,$$

$$|\omega(t)| = \frac{e^{-\alpha^* t}}{2\pi}\left|\int_{x_1-i\infty}^{x_1+i\infty} e^{\lambda t} \frac{\mathcal{L}(\zeta)(\lambda) - \mathcal{L}(\xi)(\lambda)}{1 - \mathcal{L}(\eta)(\lambda)} \mathrm{d}\lambda\right| \le \frac{e^{-(\alpha^*-x_1)t}}{m(x_1)}\|\widehat{\zeta}_x - \widehat{\xi}_x\|_{L^1(\mathbf{R})}.$$

Let $\Omega(t) = \frac{\omega(t)}{b_0}$, then $B(t) = b_0 e^{\alpha^* t}(1 + \Omega(t))$. Noting that $\lim_{t\to\infty} \omega(t) = 0$ and $B(t) > 0$, and as a consequence we get $\lim_{t\to\infty} \Omega(t) = 0$ and $b_0 > 0$. □

Bearing in mind expressions (3.5) and (3.7), together with the relation $J(s_2, t) = J(s_1, t - \tau(s_2))\Pi_1(s_2)$ valid for $t > \tau(s_2)$, the distributions of juveniles and adults can be described in terms of $B(t)$ as

$$J(s, t) = \frac{\Pi_1(s)}{\Pi_1(s_2)} B(t + \tau(s_2) - \tau(s)), \ t > \tau(s_2),$$

$$A(a, t) = B(t - a)g(s_2)\Pi_0(a), \ t > a_\dagger.$$

Then, by using Theorem 4.2, we establish the asymptotic behavior of solutions of system (2.1)-(2.7) as follows.

Theorem 4.3. *The asymptotic behavior of the distributions of juveniles and adults is given by*

$$\lim_{t\to+\infty} e^{-\alpha^* t} J(s, t) = \frac{b_0 \Pi_1(s)}{\Pi_1(s_2)} e^{\alpha^*(\tau(s_2) - \tau(s))},$$

$$\lim_{t\to+\infty} e^{-\alpha^* t} A(a, t) = b_0 g(s_2)\Pi_0(a) e^{-\alpha^* a}.$$

These results show an exponential growth of the distributions of juveniles and adults, determined by the constant α^*, the unique real solution to Eq. (4.2).

5. Concluding remarks

In this paper, we study a nonlinear system of hyperbolic partial differential equations with boundary and initial conditions. The system represents a model of a fixed sex-ratio population based on a size-structured juvenile-adult model. The juvenile stage is structured by size, while the adult stage simply by age. By using the characteristics method and prior estimations, we discuss the existence and stability of steady states. The asymptotic behavior of the solutions of juveniles and adults are analyzed via Laplace transform methods.

Compared with the existing works in the literature, our research is three-featured. First, to our best knowledge, this paper is the first attempt to consider both stage factor and sex-ratio factor in one population model. Second, we derive a set of new stability conditions rather than applying some existing results. Third, we reduce the basic model to a delayed renewal equation with constant delay for the distribution of juveniles at the maturation size. More challenging behavioral analysis problems concerning the realistic indexes and complex population dynamics, such as a non-autonomous system or a system with diffusion process, will be explored in our future research.

Acknowledgements

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