

INVERSE PROBLEMS FOR THE STURM-LIOUVILLE EQUATION WITH THE DISCONTINUOUS COEFFICIENT*

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Abstract In this study we derive the Gelfand-Levitan-Marchenko type main integral equation of the inverse problem for the boundary value problem L and prove the uniquely solvability of the main integral equation. Further, we give the solution of the inverse problem by the spectral data and by two spectrum.

Keywords Sturm-Liouville equation, boundary value problems, spectral analysis of ordinary differential operators, transformation operator, integral representation, asymptotic formulas for eigenvalues, expansion formula.

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1. Introduction

We consider the boundary value problem (L) of the form

$$\begin{aligned} -y'' + q(x)y &= \lambda^2 \rho(x)y, \quad 0 < x < \pi, \\ y'(0) - hy(0) &= 0, \quad y'(\pi) + h_1y(\pi) = 0, \end{aligned}$$

where $q(x)$ is a real valued function in $L_2(0, \pi)$, λ is the spectral parameter, h, h_1 are real numbers, and $\rho(x)$ is the following piecewise-constant function with discontinuity at a point $a \in (0, \pi)$:

$$\rho(x) = \begin{cases} 1, & 0 \leq x \leq a \\ \alpha^2, & a \leq x \leq \pi \end{cases}, \quad 0 < \alpha \neq 1.$$

The goal of this article is to determine the potential $q(x)$ and the boundary constants h, h_1 by the spectral characteristics of the problem (L). Problems of this type are called inverse spectral problems. Inverse problems of spectral analysis consist in the recovery of operators from their spectral characteristics. The theory of inverse problems for differential operators occupies an important position in the current developments of the spectral theory of linear operators. The already existing literature on the theory of inverse problems of spectral analysis is abundant. The most comprehensive account of the current state of this theory and its applications

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can be found in the monographs of Marchenko, Levitan, Beals-Deift-Tomei and Yurko (see [15, 20, 21, 25–27, 31, 38, 42] and the references therein).

Boundary value problems for the Sturm-Liouville equation with discontinuous leading coefficients arise in geophysics, electromagnetics, elasticity and other fields of engineering and physics; for example, modelling toroidal vibrations and free vibrations of the earth, reconstructing the discontinuous material properties of a nonabsorbing media, as a rule leads to direct and inverse problems for the Sturm-Liouville equation with discontinuous coefficients (see [7, 18, 22, 23, 39, 40]).

The presence of discontinuities generates important qualitative modifications in the investigation of the boundary value problems. Direct and inverse problems for discontinuous Sturm-Liouville boundary-value problems in various formulations have been studied in [3, 4, 8, 10, 11, 14, 19, 32, 33, 39, 41, 43, 44] and other works. The discontinuous inverse scattering problem for the equation of type L on the half line $[0, +\infty)$ was investigated in [13, 16]. In these papers, the inverse problem on the half-line $[0, +\infty)$ has been reduced to two inverse problems on the intervals $[0, a]$ and $[a, +\infty)$ for the Sturm-Liouville operator without discontinuity, although the complete solution of this problem was given in [17] where the new integral representation, similar to transformation operators, was obtained for the Jost solution of the discontinuous Sturm-Liouville equation.

Note that, direct and inverse scattering problems on the half-line for the equation of type L with various boundary conditions also has been investigated in [28–30].

After the half-line inverse scattering problem for the equation of type L was successfully solved by using the integral representation of the Jost solutions, it is naturally to think about the inverse spectral problems in a finite interval. In this aspect, the direct and inverse spectral problem for the equation of the problem L with Dirichlet boundary conditions on the interval $(0, \pi)$ recently has been investigated in [1, 2] where the new integral representations for solutions have been also constructed. The boundary-value problem for the discontinuous Sturm-Liouville problem L recently has been investigated in [35]. Some other results related to the inverse spectral and inverse scattering problems for the discontinuous Sturm-Liouville equation recently have been investigated in [6, 34, 36, 37].

In the present paper we derive the Gelfand-Levitan-Marchenko type main integral equation of the inverse problem for the boundary value problem L and prove the uniquely solvability of the main integral equation. Further, we give the solution of the inverse problem by the spectral data and by two spectrum.

2. The main equation of the inverse problem

Let $s(x, \lambda)$, $c(x, \lambda)$ be solutions of the equation of the problem L with the initial conditions

$$s(0, \lambda) = c'(0, \lambda) = 0, s'(0, \lambda) = c(0, \lambda) = 1.$$

Then $\omega_1(x, \lambda) = hs(x, \lambda) + c(x, \lambda)$ is the solution with initial conditions

$$\omega_1(0, \lambda) = 1, \omega_1'(0, \lambda) = h$$

respectively. We know that (see [6])

$$w_1(x, \lambda) = c_0(x, \lambda) + \int_0^{\mu^+(x)} W(x, t) \cos \lambda t dt, \quad (2.1)$$

where

$$c_0(x, \lambda) = r^+(x) \cos \lambda \mu^+(x) + r^-(x) \cos \lambda \mu^-(x),$$

$$r^\pm(x) = \frac{1}{2} \left(1 \pm \frac{1}{\sqrt{\rho(x)}} \right), \quad \mu^\pm(x) = \pm x \sqrt{\rho(x)} + a(1 \mp \sqrt{\rho(x)}),$$

$W_1(x, \cdot) \in L_1(0; \mu^+(x))$ for all $x \in [0, \pi]$. Note that the function $W_1(x, t)$ also satisfies the conditions (see [35]).

$$\frac{d}{dx} W_1(x, \mu^+(x)) = \frac{1}{4\sqrt{\rho(x)}} \left(1 + \frac{1}{\sqrt{\rho(x)}} \right) q(x), \quad (2.2)$$

$$\frac{d}{dx} \{W_1(x, \mu^-(x) + 0) - W_1(x, \mu^-(x) - 0)\} = \frac{1}{4\sqrt{\rho(x)}} \left(1 - \frac{1}{\sqrt{\rho(x)}} \right) q(x). \quad (2.3)$$

It is also known that ([35]) the boundary value problem (L) has a countable set of eigenvalues $\{\lambda_n^2\}_{n \geq 1}$ and for sufficiently large values of n the asymptotic formula

$$\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + \frac{k_n}{n}$$

is held. Here $\lambda_n^0 = \frac{n\pi}{\mu^+(\pi)} + \theta_n$, $\sup_n |\theta_n| = \theta < +\infty$,

$$d_n = \frac{h^+ \cos \lambda_n^0 \mu^+(\pi) + h^- \cos \lambda_n^0 \mu^-(\pi)}{\frac{1}{2}(\alpha + 1) \mu^+(\pi) \cos \lambda_n^0 \mu^+(\pi) - \frac{1}{2}(\alpha - 1) \mu^-(\pi) \cos \lambda_n^0 \mu^-(\pi)}$$

is a bounded sequence and $k_n \in \ell_2$. Moreover,

$$\alpha_n = \int_0^\pi \rho(x) \omega_1^2(x, \lambda_n) dx, \quad n = 1, 2, \dots$$

are the normalized numbers of the problem L. We also define

$$\alpha_n^0 = \int_0^\pi \rho(x) c_0^2(x, \lambda_n^0) dx.$$

Using Eq. (2.1) it is easy to obtain

$$\Phi_N(x, t) = \sum_{k=1}^4 I_{N_k}(x, t),$$

where

$$\Phi_N(x, t) = \sum_{n=0}^N \left(\frac{w_1(x, \lambda_n) w_1(t, \lambda_n)}{\alpha_n} - \frac{c_0(x, \lambda_n^0) c_0(t, \lambda_n^0)}{\alpha_n^0} \right),$$

$$I_{N_1}(x, t) = \sum_{n=0}^N \left(\frac{c_0(x, \lambda_n) c_0(t, \lambda_n)}{\alpha_n} - \frac{c_0(x, \lambda_n^0) c_0(t, \lambda_n^0)}{\alpha_n^0} \right),$$

$$I_{N_2}(x, t) = \int_0^{\mu^+(x)} W(x, \xi) \sum_{n=0}^N \frac{c_0(t, \lambda_n^0) \cos \lambda_n^0 \xi}{\alpha_n^0} d\xi,$$

$$I_{N_3}(x, t) = \int_0^{\mu^+(x)} W(x, \xi) \sum_{n=0}^N \left(\frac{c_0(t, \lambda_n) \cos \lambda_n \xi}{\alpha_n} - \frac{c_0(t, \lambda_n^0) \cos \lambda_n^0 \xi}{\alpha_n^0} \right) d\xi,$$

$$I_{N_4}(x, t) = \int_0^{\mu^+(t)} W(t, \xi) \sum_{n=0}^N \frac{w_1(x, \lambda_n) \cos \lambda_n \xi}{\alpha_n} d\xi.$$

Let us show that

$$\cos \lambda \xi = \begin{cases} c_0(\xi, \lambda), & \xi < a, \\ \frac{1}{\alpha^+} c_0(a + \frac{\xi - a}{\alpha}, \lambda) - \frac{\alpha^-}{\alpha^+} c_0(2a - \xi, \lambda), & \xi > a. \end{cases} \quad (2.4)$$

Obviously, since

$$c_0(\xi, \lambda) = \begin{cases} \cos \lambda \xi, & \xi \leq a, \\ \frac{1}{2} \left(1 + \frac{1}{\alpha} \right) \cos \lambda \mu^+(\xi) + \frac{1}{2} \left(1 - \frac{1}{\alpha} \right) \cos \lambda \mu^-(\xi), & \xi > a, \end{cases}$$

because of $2a - \mu^+(\xi) < a$ we have

$$\cos \lambda \mu^+(\xi) = \frac{1}{\alpha^+} c_0(\xi, \lambda) - \frac{\alpha^-}{\alpha^+} c_0(2a - \mu^+(\xi), \lambda).$$

Now substitution $\mu^+(\xi) \rightarrow \xi$ gives Eq. (2.4). We put

$$F_0(x, t) = \sum_{n=0}^{\infty} \left(\frac{c_0(t, \lambda_n) \cos \lambda_n x}{\alpha_n} - \frac{c_0(t, \lambda_n) \cos \lambda_n x}{\alpha_n} \right) \quad (2.5)$$

and

$$F(x, t) = \frac{1}{2} \left(1 + \frac{1}{\sqrt{\rho(x)}} \right) F_0(\mu^+(x), t) + \frac{1}{2} \left(1 - \frac{1}{\sqrt{\rho(x)}} \right) F_0(\mu^-(x), t). \quad (2.6)$$

Clearly

$$F(x, t) = \sum_{n=0}^{\infty} \left(\frac{c_0(t, \lambda_n) c_0(x, \lambda_n)}{\alpha_n} - \frac{c_0(t, \lambda_n^0) c_0(x, \lambda_n^0)}{\alpha_n^0} \right). \quad (2.7)$$

Let $f(x) \in AC[0, \pi]$. By the Theorem 6 in [35] we have

$$f(x) = \sum_{n=0}^{\infty} \int_0^{\pi} \frac{\rho(t) f(t) w_1(x, \lambda_n) w_1(t, \lambda_n)}{\alpha_n} dt \quad (2.8)$$

and

$$f(x) = \sum_{n=0}^{\infty} \int_0^{\pi} \rho(t) f(t) \frac{c_0(x, \lambda_n^0) c_0(t, \lambda_n^0)}{\alpha_n^0} dt. \quad (2.9)$$

Using Eq. (2.8) and Eq. (2.9) we estimate

$$\begin{aligned} & \lim_{N \rightarrow \infty} \max_{0 \leq x \leq \pi} \left| \int_0^{\pi} \rho(t) f(t) \Phi_N(x, t) dt \right| \\ & \leq \lim_{N \rightarrow \infty} \max_{0 \leq x \leq \pi} \left| \int_0^{\pi} \rho(t) f(t) \sum_{n=0}^N \frac{w_1(x, \lambda_n) w_1(t, \lambda_n)}{\alpha_n} dt - f(x) \right|, \\ & \lim_{N \rightarrow \infty} \max_{0 \leq x \leq \pi} \left| \int_0^{\pi} \rho(t) f(t) \sum_{n=0}^N \frac{c_0(x, \lambda_n^0) c_0(t, \lambda_n^0)}{\alpha_n^0} dt - f(x) \right| = 0. \end{aligned} \quad (2.10)$$

Moreover, uniformly with respect to $x \in [0, \pi]$,

$$\lim_{N \rightarrow \infty} \int_0^\pi \rho(t) f(t) I_{N_1}(x, t) dt = \int_0^\pi \rho(t) f(t) F(x, t) dt. \quad (2.11)$$

Similarly, using Eq. (2.4) and Eq. (2.9) we have

$$\begin{aligned} & \lim_{N \rightarrow \infty} \int_0^\pi \rho(t) f(t) I_{N_2}(x, t) dt \\ &= \int_0^a W(x, t) f(t) dt + \frac{1}{\alpha^+} \int_a^x W(x, \mu^+(t)) f(t) dt - \frac{\alpha^-}{\alpha^+} \int_{\mu^-(x)}^a W(x, 2a-t) f(t) dt. \end{aligned}$$

Because $t > \mu^-(x)$ in the last integral of the right hand side of Eq. (2.10) and $W(x, 2a-t) \equiv 0$, for $2a-t > \mu^+(x)$ we have

$$\begin{aligned} & \lim_{N \rightarrow \infty} \int_0^\pi \rho(t) f(t) I_{N_2}(x, t) dt \\ &= \int_0^x W(x, \mu^+(t)) \frac{2\sqrt{\rho(t)}}{1 + \sqrt{\rho(t)}} f(t) dt + \int_0^x W(x, 2a-t) \frac{1 - \sqrt{\rho(2a-t)}}{1 + \sqrt{\rho(2a-t)}} f(t) dt, \end{aligned} \quad (2.12)$$

uniformly in $x \in [0, \pi]$. Further, using Eq. (2.5) we have

$$\lim_{N \rightarrow \infty} \int_0^\pi \rho(t) f(t) I_{N_3}(x, t) dt = \int_0^\pi \rho(t) f(t) \int_0^{\mu^+(x)} W(x, \xi) F_0(\xi, t) d\xi dt. \quad (2.13)$$

By Lemma 1 of [35] we also have

$$\frac{w_1(x, \lambda_n)}{\alpha_n} = \frac{2\lambda_n w_2(x, \lambda_n)}{\Delta(\lambda_n)}, \quad (2.14)$$

where $w_2(x, \lambda)$ is the solution of the equation of the problem L with the conditions

$$\omega_2(\pi, \lambda) = -1, \quad \omega_2'(\pi, \lambda) = h_1.$$

Now using the residue theorem we calculate

$$\begin{aligned} & \lim_{N \rightarrow \infty} \int_0^\pi \rho(t) f(t) I_{N_4}(x, t) dt \\ &= 2 \lim_{N \rightarrow \infty} \int_0^\pi \rho(t) f(t) \frac{1}{2\pi i} \oint_{\Gamma_N} \left[\frac{\lambda w_2(x, \lambda)}{\Delta(\lambda)} \int_0^{\mu^+(t)} W(t, \xi) \cos \lambda \xi \right] d\lambda dt, \end{aligned}$$

where $\Gamma_N = \{\lambda : |\lambda| = N\}$. Since

$$w_2(x, \lambda) = O\left(e^{|\operatorname{Im} \lambda(\mu^+(\pi) - \mu^+(x))|}\right)$$

and

$$|\Delta(\lambda)| \geq C_\delta |\lambda| e^{|\operatorname{Im} \lambda \mu^+(\pi)|}, \quad \lambda \in G_\delta \quad (2.15)$$

(see [35]), where $C_\delta > 0$, $G_\delta = \{\lambda : |\lambda - \lambda_n^0| \geq \delta\}$, for all $\lambda \in G_\delta$ we have

$$\left| \frac{\lambda w_2(x, \lambda)}{\Delta(\lambda)} \right| \leq \widetilde{C}_\delta e^{-|\operatorname{Im} \lambda(\mu^+(x) - \mu^+(t))|}, \quad \lambda \in G_\delta, \quad (2.16)$$

where $\widetilde{C}_\delta > 0$ is a constant. Because of $\mu^+(t) < \mu^+(x)$ we obtain

$$\lim_{|\lambda| \rightarrow \infty} \max_{0 \leq x \leq \pi} \frac{\lambda w_2(x, \lambda)}{\Delta(\lambda)} = 0. \quad (2.17)$$

In other hand

$$\lim_{|\lambda| \rightarrow \infty} \max_{0 \leq x \leq \pi} e^{-|\operatorname{Im} \lambda(\mu^+(t))|} \int_0^{\mu^+(t)} W(t, \xi) \cos \lambda \xi d\xi = 0 \quad (2.18)$$

by the Riemann-Lebesgue lemma (see [31]). Therefore Eqs. (2.16) and (2.17) imply that

$$\lim_{N \rightarrow \infty} \int_0^\pi \rho(t) f(t) I_{N_4}(x, t) dt = 0. \quad (2.19)$$

Now from Eqs. (2.10)-(2.13) and (2.19) we have

$$\begin{aligned} & \int_0^\pi \rho(t) f(t) F(x, t) dt + \int_0^x W(x, \mu^+(t)) \frac{2\sqrt{\rho(t)}}{1 + \sqrt{\rho(t)}} f(t) dt \\ & + \int_0^x W(x, 2a - t) \frac{1 - \sqrt{\rho(2a - t)}}{1 + \sqrt{\rho(2a - t)}} f(t) dt \\ & + \int_0^\pi \rho(t) f(t) \int_0^{\mu^+(x)} W(x, \xi) F_0(\xi, t) d\xi dt = 0. \end{aligned}$$

Since $f(x) \in AC[0, \pi]$ is arbitrary we obtain the following theorem:

Theorem 2.1. *For every fixed $x \in (0, \pi)$ the kernel function $W(x, t)$ of the integral representation of the solution $w_1(x, \lambda)$ satisfies the following linear-functional integral equation*

$$\begin{aligned} & \frac{2}{1 + \sqrt{\rho(t)}} W(x, \mu^+(t)) + \frac{1 - \sqrt{\rho(2a - t)}}{1 + \sqrt{\rho(2a - t)}} W(x, 2a - t) \\ & + F(x, t) + \int_0^{\mu^+(x)} W(x, \xi) F_0(\xi, t) d\xi = 0, \end{aligned} \quad (2.20)$$

where the functions $F_0(x, t)$ and $F(x, t)$ are defined by the formulas (2.5) and (2.6) respectively.

Note that here we have taken into our account the obvious equality

$$\frac{1 - \sqrt{\rho(2a - t)}}{1 + \sqrt{\rho(2a - t)}} = \rho(t) \frac{(1 - \sqrt{\rho(2a - t)})}{1 + \sqrt{\rho(2a - t)}}, \quad 0 < t < x. \quad (2.21)$$

Theorem 2.2. *Let numbers $\{\lambda_n\}_{n \geq 0}$ satisfying the asymptotic formula*

$$\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + \frac{k_n}{n}, \quad n \rightarrow \infty \quad (2.22)$$

are given. Here $\lambda_n \neq \lambda_k$ if $n \neq k$, $(\lambda_n^0)_{n \geq 0}$ are zeros of the function $c_0(x, \lambda)$, $(d_n)_{n \geq 0}$ is a bounded sequence and $(k_n)_{n \geq 0} \in \ell_2$. Then the system of functions $\{c_0(x, \lambda_n)\}_{n \geq 0}$ is a Riesz basis of the space $L_2(0, \pi, \rho)$.

Proof. Let $f(x) \in L_2(0, \pi, \rho)$ and

$$\int_0^\pi \rho(x)f(x)c_0(x, \lambda_n)dx = 0, \quad n \geq 0. \quad (2.23)$$

Consider the functions

$$\Delta(\lambda) = \mu^+(\pi)(\lambda_0^2 - \lambda^2) \prod_{n=1}^{\infty} \frac{\lambda_n^2 - \lambda^2}{(\lambda_n^0)^2},$$

$$F(\lambda) = \frac{1}{\Delta(\lambda)} \int_0^\pi \rho(x)f(x)c_0(x, \lambda)dx, \quad \lambda \neq \lambda_n.$$

From the equation (2.4) we have that the function $F(\lambda)$ becomes an entire function after removing the similarities. By virtue of (2.15) and the asymptotic behavior $c_0(x, \lambda) = O\left(e^{|\operatorname{Im} \lambda \mu^+(x)|}\right)$, $|\lambda| \rightarrow \infty$ we have

$$|F(\lambda)| \leq \frac{C_\delta}{|\lambda|}, \quad \lambda \in G_\delta, \quad |\lambda| \geq \lambda^*$$

for sufficiently large $\lambda^* > 0$. Using the maximum principle ([12], page 128) and Liouville's theorem ([12], page 177) we conclude that $F(\lambda) \equiv 0$, i.e.

$$\int_0^\pi \rho(x)f(x)c_0(x, \lambda)dx \equiv 0.$$

This yields that $f(x) = 0$. Therefore we have proved that the system of functions $\{c_0(x, \lambda_n)\}_{n \geq 0}$ is complete in the space $L_2(0, \pi, \rho)$. Further from the asymptotic formula (2.22) we obtain that the system $\{c_0(x, \lambda_n)\}_{n \geq 0}$ is quadratically close to the orthogonal basis $\{c_0(x, \lambda_n)\}_{n \geq 0}$, i.e.

$$\sum_{n=0}^{\infty} \|c_0(x, \lambda_n) - c_0(x, \lambda_n^0)\|_{L_2(0, \pi; \rho)}^2 < +\infty.$$

Then from the Proposition 1.8.5 in [42] we conclude that $\{c_0(x, \lambda_n)\}_{n \geq 0}$ is a Riesz basis of $L_2(0, \pi, \rho)$. Theorem 2.2 is proved. \square

Theorem 2.3. For each fixed $x \in (0, \pi]$, equation (2.20) has a unique solution $W(x, t)$ which belongs to $L_2(0, \mu(x))$.

Proof. If $x \leq a$ then Eq. (2.20) has a form

$$W(x, t) + F(x, t) + \int_0^x W(x, \xi)F_0(\xi, t)d\xi = 0, \quad (2.24)$$

which is a Fredholm integral equation and equivalent to the equation of type

$$(I + B)f = g, \quad (2.25)$$

where I is the unit operator, B is a compact operator in the space $L_2(0, \pi)$, $f, g \in L_2(0, \pi)$. Let us prove that in the case $x > a$ the equation (2.20) is also equivalent to an equation of type (2.25).

If $x > a$ the equation (2.20) can be written as

$$L_x W(x, \cdot) + K_x W(x, \cdot) = -F(x, \cdot),$$

where

$$(L_x f)(t) = \frac{2}{1 + \sqrt{\rho(t)}} f(\mu^+(t)) + \frac{1 - \sqrt{\rho(2a-t)}}{1 + \sqrt{\rho(2a-t)}} f(2a-t), \quad 0 < t < x, \quad (2.26)$$

$$(K_x f)(t) = \int_0^{\mu^+(x)} f(\xi) F_0(\xi, t) d\xi, \quad 0 < t < x. \quad (2.27)$$

It was shown in [2] that the operator L_x has a bounded inverse in the space $L_2(0, \pi)$ and

$$L_x^{-1} f(t) = \begin{cases} f(t) - \frac{1-\alpha}{2} f\left(\frac{-t+\alpha a+a}{\alpha}\right), & t < a, \\ \frac{1+\alpha}{2} f\left(\frac{t+\alpha a-a}{\alpha}\right), & t > a. \end{cases} \quad (2.28)$$

Consequently the equation (2.20) is equivalent to the equation

$$W(x, \cdot) + L_x^{-1} K_x W(x, \cdot) = -L_x^{-1} F(x, \cdot). \quad (2.29)$$

Since L_x^{-1} is a bounded and K_x is a compact operator in $L_2(0, \pi)$ then the operator $B = L_x^{-1} K_x$ is compact in $L_2(0, \pi)$. The right hand side of (2.29) also belongs to $L_2(0, \pi)$ since L_x is invertible in $L_2(0, \pi)$. Therefore the equation (2.29) is a Fredholm integral equation of type (2.25) and it is sufficient to prove that the homogeneous equation

$$L_x W(x, t) + \int_0^{\mu^+(x)} W(x, \xi) F_0(\xi, t) d\xi = 0 \quad (2.30)$$

has only trivial solution $W(x, t) = 0$. Let $W(t) := W(x, t)$ be a solution of Eq. (2.30).

Then

$$\int_0^x \rho(t) (L_x W(t))^2 dt + \int_0^x \rho(t) L_x W(t) \int_0^{\mu^+(x)} W(\xi) F_0(\xi, t) d\xi dt = 0. \quad (2.31)$$

Now consider the formulas (2.4) and (2.5). Taking into the account $W(2a - \xi) = 0$ for $\xi < \mu^-(x)$ we have

$$K_x W(t) = \int_0^{\mu^+(x)} W(\xi) F_0(\xi, t) d\xi = \int_0^x \rho(\xi) L_x W(\xi) F(\xi, t) d\xi.$$

Therefore (2.31) takes the form

$$\begin{aligned} & \int_0^x \rho(t) (L_x W(t))^2 dt + \sum_{n=0}^{\infty} \frac{1}{\alpha_n} \left(\int_0^x \rho(t) c_0(t, \lambda_n) L_x W(t) dt \right)^2 \\ & - \sum_{n=0}^{\infty} \frac{1}{\alpha_n^0} \left(\int_0^x \rho(t) c_0(t, \lambda_n^0) L_x W(t) dt \right)^2 = 0. \end{aligned} \quad (2.32)$$

Now using the Parseval's equality ([35])

$$\int_0^x \rho(t) f^2(t) dt = \sum_{n=0}^{\infty} \frac{1}{\alpha_n^0} \left(\int_0^x \rho(t) f(t) c_0(t, \lambda_n^0) dt \right)^2,$$

for the function

$$f(t) = \begin{cases} L_x W(t), & 0 < t < x, \\ 0, & t > x, \end{cases}$$

which belongs to the space $L_2(0, x)$, we have

$$\int_0^x \rho(t) (L_x W(t)) c_0(t, \lambda_n) dt = 0, \quad n \geq 0.$$

Since the system of function $\{c_0(t, \lambda_n)\}_{n \geq 0}$ is complete in $L_2(0, \pi)$ by the Theorem 2.2 we have $L_x W(t) = 0$. Because the operator L_x has inverse in the space $L_2(0, \pi)$ we obtain $W(t) \equiv W(x, \cdot) = 0$. Theorem 2.3 is proved. \square

From Theorem 2.2 and Theorem 2.3 we obtain the following theorem:

Theorem 2.4. *The spectral data $\{\lambda_n^2, \alpha_n\}_{n \geq 0}$ uniquely determines the boundary value problem L .*

The integral equation (2.20) is called main integral equation of GLM (Gelfand-Levitan-Marchenko) type for the problem L .

3. On Properties of the functions $F_0(x, t)$, $F(x, t)$ and the solution $W(x, t)$ of the main integral equation

Lemma 3.1. *Let numbers $\{\lambda_n, \alpha_n\}_{n \geq 0}$ are given, where $\lambda_n \neq \lambda_m$ if $n \neq m$, $\alpha_n > 0$ for all $n \geq 0$, and the following asymptotic formulas are satisfied:*

$$\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + \frac{k_n}{n}, \quad \alpha_n = \alpha_n^0 + \frac{t_n}{n}, \quad n \rightarrow \infty. \quad (3.1)$$

Here $\{\lambda_n^0\}_{n \geq 0}$ are zeros of the function $c_0(x, \lambda)$, (d_n) is a bounded sequence,

$$\alpha_n^0 = \int_0^\pi \rho(x) c_0^2(x, \lambda_n^0) dx \quad (n \geq 0),$$

$(k_n), (t_n) \in \ell_2$. If

$$b(x) = \sum_{n=0}^{\infty} \left(\frac{\cos \lambda_n x}{\alpha_n} - \frac{\cos \lambda_n^0 x}{\alpha_n^0} \right), \quad (3.2)$$

then $b(x) \in W_2^1(0, 2\pi)$, $F_0(x, x) \in W_2^1(0, 2\pi)$, $F(x, x) \in W_2^1(0, 2\pi)$.

Proof. Denote $\varepsilon_n = \lambda_n - \lambda_n^0$. Then

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(\frac{\cos \lambda_n x}{\alpha_n} - \frac{\cos \lambda_n^0 x}{\alpha_n^0} \right) \\ &= \sum_{n=0}^{\infty} \left[\frac{\cos \lambda_n x - \cos \lambda_n^0 x}{\alpha_n^0} + \left(\frac{1}{\alpha_n} - \frac{1}{\alpha_n^0} \right) \cos \lambda_n x \right]. \end{aligned} \quad (3.3)$$

We have

$$\cos \lambda_n x - \cos \lambda_n^0 x = -\varepsilon_n x \sin \lambda_n^0 x - (\sin \varepsilon_n x - \varepsilon_n x) \sin \lambda_n^0 x - 2 \sin^2 \frac{\varepsilon_n x}{2} \cos \lambda_n^0 x. \quad (3.4)$$

Consequently, using Eqs. (3.2)-(3.4) and $\varepsilon_n = \frac{d_n}{\lambda_n^0} + \frac{k_n}{n}$ ([35]) we obtain

$$b(x) = B_1(x) + B_2(x),$$

where

$$B_1(x) = \sum_{n=1}^{\infty} \frac{d_n x \sin \lambda_n^0 x}{\alpha_n^0 \lambda_n^0}, \quad (3.5)$$

$$\begin{aligned} B_2(x) = & \sum_{n=1}^{\infty} \left(\frac{1}{\alpha_n} - \frac{1}{\alpha_n^0} \right) \cos \lambda_n x + \frac{1}{\alpha_n^0} (\cos \lambda_0 x - \cos \lambda_0^0 x) \\ & - \sum_{n=1}^{\infty} \frac{K_n x \sin \lambda_n^0 x}{\alpha_n^0 n} - \sum_{n=1}^{\infty} \frac{1}{\alpha_n^0} (\sin \varepsilon_n x - \varepsilon_n x) \sin \lambda_n^0 x \\ & - 2 \sum_{n=1}^{\infty} \frac{1}{\alpha_n^0} \sin^2 \frac{\varepsilon_n x}{2} \cos \lambda_n^0 x. \end{aligned} \quad (3.6)$$

Using

$$\alpha_n^0 = \int_0^{\pi} \rho(x) c_0^2(x, \lambda_n^0) dx, \quad (3.7)$$

where

$$c_0(x, \lambda_n) = \frac{1}{2} \left(1 + \frac{1}{\sqrt{\rho(x)}} \right) \cos \lambda \mu^+(x) + \frac{1}{2} \left(1 - \frac{1}{\sqrt{\rho(x)}} \right) \cos \lambda \mu^-(x),$$

we can show that

$$\alpha_n^0 = \frac{a}{2} + \frac{(\pi - a)(1 + \alpha^2)}{4} + O\left(\frac{1}{n}\right), \quad n \rightarrow \infty. \quad (3.8)$$

Indeed,

$$\alpha_n^0 = \frac{a}{2} + \frac{(1 + \alpha^2)(\pi - a)}{4} + \frac{P_n}{\lambda_n^0}, \quad (3.9)$$

where

$$P_n = \frac{1}{4} \alpha (\alpha^+ \sin 2\lambda_n^0 \mu^+(\pi) - \alpha^- \sin 2\lambda_n^0 \mu^-(\pi)).$$

By virtue of

$$\lambda_n^0 = \frac{n}{\mu^+(\pi)} + \theta_n, \quad \sup |\theta_n| < +\infty,$$

we have that the sequence (P_n) is bounded. Then from Eq. (3.9) we have Eq. (3.8). Further, since

$$\varepsilon_n = O\left(\frac{1}{n}\right), \quad \alpha_n^0 = O(1), \quad n \rightarrow \infty,$$

we obtain

$$\frac{1}{\alpha_n} - \frac{1}{\alpha_n^0} = \frac{\gamma_n}{n}, \quad (\gamma_n) \in \ell_2,$$

and consequently $B_1(x), B_2(x) \in W_2^1(0, 2\pi)$ i.e. $B(x) \in W_2^1(0, 2\pi)$. It easy to verify that

$$F_0(x, t) = \frac{1}{4} \left(1 + \frac{1}{\sqrt{\rho(t)}} \right) [b(x - \mu^+(t)) + b(x + \mu^+(t))] + \frac{1}{4} \left(1 - \frac{1}{\sqrt{\rho(t)}} \right) [b(x - \mu^-(t)) + b(x + \mu^-(t))]. \quad (3.10)$$

Therefore, $F_0(x, x) \in W_2^1(0, 2\pi)$ and by the formula (2.6) we have $F(x, x) \in W_2^1(0, 2\pi)$. The lemma is proved. \square

Now using the main integral equation (2.20), the formulas (2.24), (2.25), (2.27), (3.10) and (2.6) we obtain the following theorem:

Theorem 3.1. *The kernel function $W(x, t)$ of the main integral equation and the functions $F(x, t), F_0(x, t)$ satisfy the following relations:*

$$\frac{\partial^2 F_0(x, t)}{\partial t^2} = \rho(t) \frac{\partial^2 F_0(x, t)}{\partial x^2}, \quad \rho(t) \frac{\partial^2 F(x, t)}{\partial x^2} = \rho(x) \frac{\partial^2 F(x, t)}{\partial t^2}, \quad (3.11)$$

$$F_0(x, t)|_{t=0} = b(x),$$

$$F(x, t)|_{t=0} = \frac{1}{2} \left(1 + \frac{1}{\sqrt{\rho(t)}} \right) b(\mu^+(x)) + \frac{1}{2} \left(1 - \frac{1}{\sqrt{\rho(t)}} \right) b(\mu^-(x)), \quad (3.12)$$

$$\frac{\partial F_0(x, t)}{\partial t}|_{t=0} = 0, \quad \frac{\partial F(x, t)}{\partial t}|_{t=0}, \quad (3.13)$$

$$\frac{\partial}{\partial x} F_0(\mu^\pm(x), t) = \pm \sqrt{\rho(x)} \frac{\partial}{\partial \xi} F_0(\xi, t)|_{t=\mu^\pm(x)}, \quad (3.14)$$

$$\frac{\partial}{\partial t} W(x, 0) = 0, \quad (3.15)$$

$$\frac{\sqrt{\rho(x)} - 1}{\sqrt{\rho(x)} + 1} \frac{d}{dx} W(x, \mu^+(x)) = \frac{d}{dx} \{W(x, \mu^-(x) + 0) - W(x, \mu^-(x) - 0)\}. \quad (3.16)$$

4. Solution of the Inverse Problem

In this section we investigate the necessary and sufficient condition for solvability of the inverse problem with respect to the spectral data. We have proved the following theorem.

Theorem 4.1. *For real numbers $\{\lambda_n^2, \alpha_n\}_{n \geq 0}$ to be the spectral data for a certain boundary value problem $L = L(q(x), h, h_1)$ with $q(x) \in L_2(0, \pi)$ it is necessary and sufficient the relations*

$$\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + \frac{k_n}{n}, \quad (k_n) \in \ell_2, \quad (4.1)$$

$$\alpha_n = \alpha_n^0 + \frac{t_n}{n}, \quad (t_n) \in \ell_2, \quad (4.2)$$

where λ_n^0 are zeros of the characteristic function $\Delta_0(\lambda) = c'_0(\pi, \lambda)$, (d_n) is the

bounded sequence

$$d_n = \frac{2h^+ \cos \lambda_n^0 \mu^+(\pi) + 2h^- \cos \lambda_n^0 \mu^-(\pi)}{(\alpha + 1)\mu^+(\pi) \cos \lambda_n^0 \mu^+(\pi) - (\alpha - 1)\mu^-(\pi) \cos \lambda_n^0 \mu^-(\pi)},$$

$$\alpha_n^0 = \int_0^\pi \rho(x) c_0^2(x, \lambda_n) dx,$$

$$h^\pm = \alpha^\pm \left(h_1 + \frac{1}{2} \int_0^\pi q(t) dt \right) \pm \alpha^\pm \alpha \left(h + \frac{\alpha}{2} \int_0^a q(t) dt \right).$$

Proof. The necessity part of Theorem 3.1 is proved in [35, Theorem 1], here we prove the sufficiency. Let real numbers $\{\lambda_n, \alpha_n\}_{n \geq 0}$ of the form (2.1)-(2.4) be given. We construct function $F_0(x, t)$ and $F(x, t)$ by the formulas (2.5) and (2.6) of the Section 3 and consider the main integral equation (2.20). Let the function $W(x, t)$ is the solution of (2.20). We construct the function $w_1(x, \lambda)$ by the formula

$$w_1(x, \lambda) = c_0(x, \lambda) + \int_0^{\mu^+(x)} W(x, t) \cos \lambda t dt, \quad (4.3)$$

and the function $q(x)$ and the numbers h, h_1 by formulas

$$q(x) = \frac{h\rho(x)}{\sqrt{\rho(x)} + 1} \frac{d}{dx} W(x, \mu^+(x)), \quad (4.4)$$

$$h = W(0, 0), \quad (4.5)$$

$$h_1 = h^+ + h^- - \frac{\alpha}{2} \int_0^\pi \frac{q(t)}{\sqrt{\rho(x)} + 1} dt, \quad (4.6)$$

where h^+ and h^- are defined using (d_n) . To prove the theorem we need some axillary assertions. \square

Lemma 4.1. *The following relations hold:*

$$-w_1''(x, \lambda) + q(x)w_1(x, \lambda) = \lambda^2 \rho(x)w_1(x, \lambda), \quad (4.7)$$

$$w_1(0, \lambda) = 1, \quad w_1'(0, \lambda) = h. \quad (4.8)$$

Proof. 1) First we assume that $b(x) \in W_2^2(0, \pi)$, where $b(x)$ is defined by Eq. (3.1) of the previous section. Differentiating the identity

$$J(x, t) := \frac{2}{1 + \sqrt{\rho(t)}} W(x, \mu^+(t)) + \frac{1 - \sqrt{\rho(2a-t)}}{1 + \sqrt{\rho(2a-t)}} W(x, 2a-t)$$

$$+ F(x, t) + \int_0^{\mu^+(x)} W(x, \xi) F_0(\xi, t) d\xi = 0, \quad 0 < t < x, \quad (4.9)$$

we calculate

$$J_{tt}(x, t) = \frac{2\rho(t)}{1 + \sqrt{\rho(t)}} W_{tt}(x, \mu^+(t)) + \frac{1 - \sqrt{\rho(2a-t)}}{1 + \sqrt{\rho(2a-t)}} W_{tt}(x, 2a-t)$$

$$+ F_{tt}(x, t) + \int_0^{\mu^+(x)} W(x, \xi) \frac{\partial^2 F_0(\xi, t)}{\partial t^2} d\xi = 0, \quad (4.10)$$

$$\begin{aligned}
J_{xx}(x, t) &= \frac{2}{1 + \sqrt{\rho(t)}} W_{xx}(x, \mu^+(t)) + \frac{1 - \sqrt{\rho(2a-t)}}{1 + \sqrt{\rho(2a-t)}} W_{xx}(x, 2a-t) \\
&+ F_{xx}(x, t) + \int_0^{\mu^+(x)} W_{xx}(x, \xi) F_0(\xi, t) d\xi \\
&+ \sqrt{\rho(x)} F_0(\mu^+(x), t) \frac{\partial W(x, \xi)}{\partial x} \Big|_{\xi=\mu^+(x)} \\
&+ \sqrt{\rho(x)} F_0(\mu^+(x), t) \left[\frac{\partial W(x, \xi)}{\partial x} \Big|_{\xi=\mu^+(x)+0} - \frac{\partial W(x, \xi)}{\partial x} \Big|_{\xi=\mu^+(x)-0} \right] \\
&+ \sqrt{\rho(x)} F_0(\mu^+(x), t) \frac{d}{dx} W(x, \mu^+(x)) + \sqrt{\rho(x)} W(x, \mu^+(x)) \frac{\partial F_0(\mu^+(x), t)}{\partial x} \\
&+ \sqrt{\rho(x)} (W(x, \mu^-(x)+0) - W(x, \mu^-(x)-0)) \frac{\partial F_0(\mu^-(x), t)}{\partial x} \\
&+ \sqrt{\rho(x)} F_0(\mu^-(x), t) \frac{d}{dx} \{W(x, \mu^-(x)+0) - W(x, \mu^-(x)-0)\} = 0.
\end{aligned} \tag{4.11}$$

Using (3.11) we rewrite Eq. (4.10) as

$$\begin{aligned}
J_{tt}(x, t) &= \frac{2\rho(t)}{1 + \sqrt{\rho(t)}} W_{tt}(x, \mu^+(t)) + \frac{1 - \sqrt{\rho(2a-t)}}{1 + \sqrt{\rho(2a-t)}} W_{tt}(x, 2a-t) \\
&+ F_{tt}(x, t) + \rho(t) \int_0^{\mu^+(x)} W(x, \xi) \frac{\partial^2 F_0(\xi, t)}{\partial \xi^2} d\xi = 0.
\end{aligned} \tag{4.12}$$

Using the formula (2.21) finally we have

$$\begin{aligned}
\frac{1}{\rho(t)} J_{tt}(x, t) &= \frac{2}{1 + \sqrt{\rho(t)}} W_{tt}(x, \mu^+(t)) \\
&+ \frac{1 - \sqrt{\rho(2a-t)}}{1 + \sqrt{\rho(2a-t)}} W_{tt}(x, 2a-t) + \frac{1}{\rho(t)} F_{tt}(x, t) \\
&+ (W(x, \mu^-(x)-0) - W(x, \mu^-(x)+0)) \frac{\partial}{\partial \xi} F_0(\xi, t) \Big|_{\xi=\mu^-(x)} \\
&+ A(x, \mu^+(x)) \frac{\partial}{\partial \xi} F_0(\xi, t) \Big|_{\xi=\mu^+(x)} - F_0(x, \mu^-(x)) \frac{\partial W(x, \xi)}{\partial \xi} \Big|_{\xi=\mu^-(x)-0} \\
&+ F_0(x, 0) \frac{\partial W(x, \xi)}{\partial \xi} \Big|_{\xi=0} - F_0(x, \mu^+(x)) \frac{\partial W(x, \xi)}{\partial \xi} \Big|_{\xi=\mu^+(x)} \\
&+ F_0(x, \mu^-(x)) \frac{\partial W(x, \xi)}{\partial \xi} \Big|_{\xi=\mu^-(x)+0} + \int_0^{\mu^+(x)} \frac{\partial^2 W(x, \xi)}{\partial \xi^2} F_0(\xi, t) d\xi.
\end{aligned} \tag{4.13}$$

Now it follows from (4.9), (4.10), (4.13) and the identity

$$J_{xx}(x, t) - \rho(x) J_{tt}(x, t) - q(x) J(x, t) \equiv 0.$$

Using this identity according to the formulas (2.6), (3.11)-(4.1) from the previous

section and the formula (4.5), we have

$$\begin{aligned} & \frac{2}{1 + \sqrt{\rho(x)}} W_{xx}(x, \mu^+(t)) + \frac{1 - \sqrt{\rho(2a-t)}}{1 + \sqrt{\rho(2a-t)}} W_{xx}(x, 2a-t) \\ & - \rho(x) \left[\frac{2}{1 + \sqrt{\rho(x)}} W_{tt}(x, \mu^+(t)) + \frac{1 - \sqrt{\rho(2a-t)}}{1 + \sqrt{\rho(2a-t)}} W_{tt}(x, 2a-t) \right] \\ & - q(x) \left[\frac{2}{1 + \sqrt{\rho(x)}} W(x, \mu^+(t)) + \frac{1 - \sqrt{\rho(2a-t)}}{1 + \sqrt{\rho(2a-t)}} W(x, 2a-t) \right] \\ & + \int_0^{\mu^+(x)} [W_{xx}(x, \xi) - \rho(x)W_{\xi\xi}(x, \xi) - q(x)W(x, \xi)] F_0(\xi, t) d\xi = 0. \end{aligned} \quad (4.14)$$

By the Theorem 2.3 of the previous section the homogenous equation (4.14) has only trivial solution i.e.

$$W_{xx}(x, t) - \rho(x)W_{tt}(x, t) - q(x)W(x, t) = 0, \quad 0 < t < x. \quad (4.15)$$

Now differentiating Eq. (4.3) twice, we get

$$\begin{aligned} w_1'(x, \lambda) = & c_0'(x, \lambda) + \int_0^{\mu^+(x)} W_x(x, t) \cos \lambda t dt + \sqrt{\rho(x)} W(x, \mu^+(x)) \cos \lambda \mu^+(x) \\ & + \sqrt{\rho(x)} \cos \lambda \mu^-(x) [W(x, \mu^-(x) + 0) - W(x, \mu^-(x) - 0)], \end{aligned} \quad (4.16)$$

$$\begin{aligned} w_1''(x, \lambda) = & c_0''(x, \lambda) + \int_0^{\mu^+(x)} W_{xx}(x, t) \cos \lambda t dt \\ & + \sqrt{\rho(x)} \frac{\partial}{\partial x} W(x, t) |_{t=\mu^+(x)} \cos \lambda \mu^+(x) \\ & + \sqrt{\rho(x)} \left[\frac{\partial}{\partial x} W(x, t) |_{t=\mu^+(x)+0} - \frac{\partial}{\partial x} W(x, t) |_{t=\mu^-(x)-0} \right] \cos \lambda \mu^-(x) \\ & + \sqrt{\rho(x)} \cos \lambda \mu^+(x) \frac{d}{dx} (W(x, \mu^+(x))) \lambda \rho(x) W(x, \mu^+(x)) \sin \lambda \mu^+(x) \\ & + \sqrt{\rho(x)} \cos \lambda \mu^+(x) \frac{d}{dx} \{W(x, \mu^-(x) + 0) - W(x, \mu^-(x) - 0)\} \\ & + \lambda \rho(x) \{W(x, \mu^-(x) + 0) - W(x, \mu^-(x) - 0)\} \sin \lambda \mu^-(x). \end{aligned} \quad (4.17)$$

In other hand, integrating by parts twice, we obtain

$$\begin{aligned} & \lambda^2 \rho(x) w_1(x, \lambda) \\ = & \lambda^2 \rho(x) c_0(x, \lambda) + \lambda^2 \rho(x) \int_0^{\mu^+(x)} W(x, t) \cos \lambda t dt \\ = & c_0''(x, \lambda) + \lambda \rho(x) \sin \lambda \mu^-(x) [W(x, \mu^-(x) - 0) - W(x, \mu^-(x) + 0)] \\ & + \lambda \rho(x) \sin \lambda \mu^+(x) W(x, \mu^+(x)) + \rho(x) \frac{\partial W(x, 0)}{\partial t} \\ & + \rho(x) \cos \lambda \mu^-(x) \left[\frac{\partial W(x, t)}{\partial t} |_{t=\mu^-(x)+0} - \frac{\partial W(x, t)}{\partial t} |_{t=\mu^-(x)-0} \right] \\ & - \rho(x) \cos \lambda \mu^+(x) \frac{\partial W(x, t)}{\partial t} |_{t=\mu^+(x)} + \rho(x) \int_0^{\mu^+(x)} \frac{\partial^2 W(x, t)}{\partial t^2} \cos \lambda t dt. \end{aligned} \quad (4.18)$$

Together with the formulas (2.6), (3.11)-(4.1) and (4.15) this gives

$$w_1''(x, \lambda) + \rho(x)\lambda^2 w_1(x, \lambda) = q(x)w_1(x, \lambda).$$

Hence, the formula (4.7) is proved. The relations (4.8) are obtained from eq. (4.4) and (4.16). Lemma 3.1 is proved for the case $b(x) \in W_2^2(0, \pi)$. Now for the case $b(x) \in W_2^1(0, \pi)$ the lemma is proved by the standard method (see [42]). \square

Lemma 4.2. For each function $g(x) \in L_2(0, \pi)$ is following relation holds:

$$\int_0^\pi \rho(x)g^2(x)dx = \sum_{n=0}^{\infty} \frac{1}{\alpha_n} \left(\int_0^\pi g(t)w_1(t, \lambda_n)dt \right)^2. \quad (4.19)$$

Proof. Using the formulas (2.4)-(2.6) of the previous section it is easy to transform solution

$$w_1(x, \lambda) = c_0(x, \lambda) + \int_0^x \rho(x)\phi(x, t)c_0(t, \lambda)dt \quad (4.20)$$

and the main integral equation (2.20) from the previous section to the form

$$\phi(x, t) + F(x, t) + \int_0^x \rho(\xi)\phi(x, \xi)F(\xi, t)d\xi = 0, \quad (4.21)$$

where

$$\phi(x, t) = \frac{2}{1 + \sqrt{\rho(t)}}W(x, \mu^+(t)) + \frac{1 - \sqrt{\rho(2a-t)}}{1 + \sqrt{\rho(2a-t)}}W(x, 2a-t).$$

Solving the equation (4.20) with respect to $c_0(x, \lambda)$ we obtain

$$c_0(x, \lambda) = w_1(x, \lambda) + \int_0^x \rho(t)H(x, t)w_1(x, \lambda)dt. \quad (4.22)$$

By the standard method (see [42]) is can be proved that

$$H(x, t) = F(x, t) + \int_0^t \rho(\xi)\phi(t, \xi)F(x, \xi)d\xi, \quad 0 \leq t \leq x. \quad (4.23)$$

Denote $Q(\lambda) = \int_0^\pi \rho(t)g(t)w_1(t, \lambda)$. Then using (4.20) we have

$$Q(\lambda) = \int_0^\pi \rho(t)h(t)c_0(t, \lambda)dt,$$

where

$$h(t) = g(t) + \int_t^\pi \rho(\xi)g(\xi)\phi(t, \xi)d\xi. \quad (4.24)$$

By the similar way, using the formula (4.22) we obtain

$$g(t) = h(t) + \int_t^\pi \rho(\xi)h(\xi)H(\xi, t)d\xi. \quad (4.25)$$

Now according to Eq. (4.24) we have

$$\begin{aligned} \int_0^\pi \rho(t)h(t)F(x, t)dt &= \int_0^x \rho(t)g(t) \left[F(x, t) + \int_0^t \rho(\xi)\Phi(t, \xi)F(x, \xi)d\xi \right] dt \\ &+ \int_x^\pi \rho(t)g(t) \left[F(x, t) + \int_0^t \rho(\xi)\Phi(t, \xi)F(x, \xi)d\xi \right] dt. \end{aligned}$$

Consequently, by the formulas (4.21) and (4.23) we obtain

$$\int_0^\pi \rho(t)h(t)F(x,t)dt = \int_0^x \rho(t)g(t)H(x,t)dt - \int_x^\pi \rho(t)g(t)\Phi(t,x)dt. \quad (4.26)$$

From Eq. (4.5) and the Parsevall equality we have

$$\begin{aligned} & \int_0^\pi \rho(t)h^2(t)dt + \int_0^\pi \rho(t)\rho(t)h(t)h(x)F(x,t)dxdt \\ &= \sum_{n=0}^{\infty} \frac{1}{\alpha_n} \left(\int_0^\pi h(t)g(t)c_0(t, \lambda_n)dt \right) \\ &= \sum_{n=0}^{\infty} \frac{Q^2(\lambda_n)}{\alpha_n}. \end{aligned}$$

Using (4.26) we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{Q^2(\lambda_n)}{\alpha_n} &= \int_0^\pi \rho(t)h^2(t)dt + \int_0^\pi \rho(t)g(t) \left[\int_t^\pi \rho(x)h(x)H(x,t)dx \right] dt \\ &\quad - \int_0^\pi \rho(x)h(x) \left[\int_x^\pi \rho(t)g(t)\Phi(t,x)dt \right] dx. \end{aligned}$$

Finally, in view of (4.24) and (4.25) we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{Q^2(\lambda_n)}{\alpha_n} &= \int_0^\pi \rho(t)h^2(t)dt + \int_0^\pi \rho(t)g(t) (g(t) - h(t)) dt \\ &\quad - \int_0^\pi \rho(x)h(x) (h(x) - g(x)) dx = \int_0^\pi \rho(t)g^2(t)dt. \end{aligned}$$

The lemma is proved. \square

Corollary 4.1. For arbitrary functions $f(x), g(x) \in L_2(0, \pi; \rho)$ the following relation holds:

$$\int_0^\pi \rho(x)f(x)g(x)dx = \sum_{n=0}^{\infty} \frac{1}{\alpha_n} \int_0^\pi f(t)w_1(t, \lambda_n)dt \int_0^\pi g(t)w_1(t, \lambda_n)dt. \quad (4.27)$$

Lemma 4.3. The following relation holds:

$$\int_0^\pi w_1(t, \lambda_k)w_1(t, \lambda_n)dt = \begin{cases} 0, & n \neq k, \\ \alpha_n, & n = k. \end{cases} \quad (4.28)$$

Proof. 1) Let $f(x) \in W_2^2[0, \pi]$; consider the series

$$f^*(x) = \sum_{n=0}^{\infty} c_n w_1(x, \lambda_n), \quad (4.29)$$

where

$$c_n = \frac{1}{\alpha_n} \int_0^\pi f(x)w_1(x, \lambda_n)dx. \quad (4.30)$$

Using Lemma 3.1 and integration by parts we calculate

$$c_n = \frac{1}{\alpha_n \lambda_n^2} \left(hf(0) - f'(0) + w_1(\pi, \lambda_n) f'(\pi) - w_1(\pi, \lambda_n) f(\pi) \right. \\ \left. + \int_0^\pi w_1(x, \lambda_n) [-f''(x) + g(x)f(x)] dx \right).$$

From the asymptotic formulas for the $w_1(x, \lambda)$ and λ_n (see [35]) we have

$$c_n = O\left(\frac{1}{n^2}\right), \quad w_1(x, \lambda_n) = O(1)$$

uniformly for $x \in [0, \pi]$. Consequently the series (4.29) converges absolutely and uniformly on $[0, \pi]$. According to (4.27) and (4.30) we obtain

$$\int_0^\pi \rho(x) f(x) g(x) dx = \int_0^\pi g(t) \sum_{n=0}^{\infty} c_n w_1(t, \lambda_n) dt = \int_0^\pi g(t) f^*(t) dt.$$

Since $g(x)$ is arbitrary, we obtain $f^*(x) = f(x)$, i.e.

$$f(x) = \sum_{n=0}^{\infty} c_n w_1(x, \lambda_n). \quad (4.31)$$

2) Fix $k \geq 0$ and take $f(x) = w_1(x, \lambda_k)$. Then, by virtue of (4.31)

$$w_1(x, \lambda_k) = \sum_{n=0}^{\infty} c_{nk} w_1(x, \lambda_n), \quad c_{nk} = \frac{1}{\alpha_n} \int_0^\pi w_1(x, \lambda_k) w_1(x, \lambda_n) dx.$$

Further the system $\{c_0(x, \lambda_n)\}_{n \geq 0}$ is minimal in $L_2(0, \pi; \rho)$ (see Theorem 2.2 of the previous section), and consequently, in view of (4.3) the system $\{w_1(x, \lambda_n)\}_{n \geq 0}$ is also minimal in $L_2(0, \pi; \rho)$. Hence, $c_{nk} = \delta_{nk}$, where δ_{nk} is a Kronecker symbol. The Lemma 4.3 is proved. \square

Lemma 4.4. For all $n, m \geq 0$

$$\frac{w_1'(\pi, \lambda_n)}{w_1(\pi, \lambda_n)} = \frac{w_1'(\pi, \lambda_m)}{w_1(\pi, \lambda_m)}. \quad (4.32)$$

Proof. By the standard method (see [42]) we derive that

$$(\lambda_n^2 - \lambda_m^2) \int_0^\pi \rho(x) w_1(x, \lambda_n) w_1(x, \lambda_m) dx \\ = [w_1(x, \lambda_n) w_1'(x, \lambda_m) - w_1'(x, \lambda_n) w_1(x, \lambda_m)] \Big|_0^\pi.$$

Taking into the account the Eq. (4.28), we set

$$w_1(\pi, \lambda_n) w_1'(\pi, \lambda_m) - w_1'(\pi, \lambda_n) w_1(\pi, \lambda_m) = 0. \quad (4.33)$$

Clearly, if $w_1(\pi, \lambda_m) = 0$ for a certain m , then $w_1'(\pi, \lambda_m) \neq 0$, and in view of (79) $w_1(\pi, \lambda_n) = 0$ for all n , which is impossible since $w_1(\pi, \lambda_n) = c_0(\pi, \lambda_n) +$

$\int_0^\pi W(\pi, t) \cos \lambda_n t dt \sim c_0(\pi, \lambda_n) \neq 0$ for $n \rightarrow \infty$. Hence, $w_1(\pi, \lambda_n) \neq 0$ for all $n \geq 0$. Then Eq. (4.33) implies (4.32). Denote

$$\tilde{H} = -\frac{w_1(\pi, \lambda_n)}{w_1(\pi, \lambda_n)}.$$

From Eq. (3.8) we have that \tilde{H} is independent of n . Hence

$$w_1'(\pi, \lambda_n) + \tilde{H}w_1(\pi, \lambda_n) = 0, \quad n \geq 0.$$

Together with Lemma 3.1 and Lemma 4.2 this gives that numbers $\{\lambda_n, \alpha_n\}_{n \geq 0}$ are the spectral data for the constructed boundary value problem $L(q(x), h, h_1)$. Clearly $\tilde{H} = h_1$, where h_1 is defined by Eq. (4.6). Thus Lemma 4.4 is proved. \square

From the proof of the Lemma 4.4 we have the following algorithm for the construction the problem $L(q(x), h, h_1)$ by the spectral data $\{\lambda_n, \alpha_n\}_{n \geq 0}$:

1) Construct the function $F_0(x, t)$ and $F(x, t)$ by the formulas (2.5), (2.6).

2) Construct the function $W(x, t)$ as the unique solution of the main integral equation.

3) Calculate the function $q(x)$ and numbers h, h_1 by the formulas (4.4)-(4.6).

Note that the similar results can be obtained for the boundary value problems $L^0 = L^0(q(x), h_1)$ and $L_1 = L_1(q(x), h)$ for equation (2.1) with the boundary conditions

$$y(0) = 0, \quad y'(\pi) + h_1 y(\pi) = 0$$

and

$$y'(0) - h y(0) = 0, \quad y(\pi) = 0$$

respectively.

5. Recovery of differential operators from two spectra

Boundary value problems L, L^0 , and L_1^0 defined in the previous section are also can be recovered by its spectral data, in particular the following theorem hold:

Theorem 5.1. *For real numbers $\{\mu_n^2, \alpha_{n_1}\}_{n \geq 0}$ to be the spectral data for a certain boundary value problem $L_1(q(x), h)$ with $q(x) \in L_2(0, \pi)$, it is necessary and sufficient that $\mu_n \neq \mu_m$ ($n \neq m$), $\alpha_{n_1} > 0$ and*

$$\mu_n = \mu_n^0 + \frac{d_{n_1}}{\mu_n^0} + \frac{k_{n_1}}{n}, \quad k_{n_1} \in \ell_2, \quad d_{n_1} \in \ell_\infty, \quad (5.1)$$

$$\alpha_{n_1} = \alpha_{n_1}^0 + \frac{t_{n_1}}{n}, \quad t_{n_1} \in \ell_2, \quad (5.2)$$

are satisfied.

Let $\{\lambda_n^2\}$ and $\{\mu_n^2\}$ be the eigenvalues of L and L_1 respectively. Then the asymptotic formulas (3.4), (3.5),

$$\lambda_n = \lambda_n^0 + \frac{d_n}{\lambda_n^0} + \frac{k_n}{n}, \quad k_n \in \ell_2, \quad d_n \in \ell_\infty, \quad (5.3)$$

$$\alpha_n = \alpha_n^0 + \frac{t_n}{n}, \quad t_n \in \ell_2 \quad (5.4)$$

hold, and we have the representations

$$\Delta(\lambda) = \mu^+(\pi)(\lambda_0^2 - \lambda_n^2) \prod_{n=0}^{\infty} \frac{\lambda_n^2 - \lambda^2}{(\lambda_0)^2}, \quad (5.5)$$

$$d(\lambda) = \prod_{n=0}^{\infty} \frac{\mu_n^2 - \lambda^2}{(\mu_0)^2}, \quad (5.6)$$

for the characteristic functions $\Delta(\lambda)$ and $d(\lambda)$ of the problems L and L_1 respectively.

Theorem 5.2. *For real numbers $\{\lambda_n, \mu_n\}_{n \geq 0}$ to be the spectra for certain boundary value problems L and L_1 with $q(x) \in L_2(0, \pi)$ it is necessary and sufficient that Eq. (5.1), (5.2), (5.3), (5.4) and*

$$\lambda_n < \mu_n < \lambda_{n+1}, \quad n \geq 0 \quad (5.7)$$

hold. The function $q(x)$ and the numbers h and h_1 can be constructed by the following algorithm:

i) From the given numbers $\{\lambda_n, \mu_n\}_{n \geq 0}$ calculate the numbers α_n by the formula

$$\alpha_n = -\frac{\dot{\Delta}(\lambda_n)d(\lambda_n)}{2\lambda_n}. \quad (5.8)$$

ii) From the spectral data $\{\lambda_n, \alpha_n\}_{n \geq 0}$ construct $q(x)$, h , and h_1 by the formulas (4.4)-(4.6).

Proof. The necessity part of Theorem 5.1 has been proved in [35] and in [6]. Let numbers $\{\lambda_n^2, \mu_n^2\}_{n \geq 0}$ satisfying the conditions (5.1), (5.2), (5.3), (5.4) and (5.7) are given. We construct the functions $\Delta(\lambda)$ and $d(\lambda)$ by the formulas (5.5) and (5.6) respectively, and calculate the numbers α_n by the formula (5.8). By using the asymptotic formulas for the characteristic functions $\Delta(\lambda)$ and $d(\lambda)$ we can prove that

$$\alpha_n > 0, \quad \alpha_n = \alpha_n^0 + \frac{t_n}{n}, \quad t_n \in \ell_2.$$

Then, by Theorem 3.1 there exists a boundary value problem $L = L(q(x), h, H)$ with $q(x) \in L_2(0, \pi)$ such that $\{\lambda_n^2, \alpha_n\}_{n \geq 0}$ are spectral data of L . Denote by $\{\mu_n^2\}_{n \geq 0}$ the eigenvalues of the boundary value problem $\tilde{L}(q(x), h)$. Let us prove that $\mu_n = \tilde{\mu}_n$. Let $\tilde{d}(\lambda)$ be the characteristic function of \tilde{L}_1 . Then together with (5.8) we also have

$$\alpha_n = -\frac{\dot{\Delta}(\lambda_n)\tilde{d}(\lambda_n)}{2\lambda_n} \quad (n \geq 0)$$

and we conclude $d(\lambda_n) = \tilde{d}(\lambda_n)$, $n \geq 0$.

Consequently, the function

$$Z(\lambda) := \frac{d(\lambda) - \tilde{d}(\lambda)}{\Delta(\lambda)}$$

is entire in λ . On the other hand using the estimations

$$|d(\lambda)| \leq C \exp(|\operatorname{Im} \lambda| \mu^+(\pi)), \quad |\tilde{d}(\lambda)| \leq C \exp(|\operatorname{Im} \lambda| \mu^+(\pi))$$

for certain $C > 0$, and the estimation

$$|d(\lambda)| \geq \widetilde{C}_\delta |\lambda| e^{|\operatorname{Im} \lambda \mu^+(\pi)|}, \quad \lambda \in G_\delta, \quad |\lambda| > \lambda^*$$

for some fixed $\delta > 0$, we get

$$|Z(\lambda)| \leq \frac{C}{|\lambda|}, \quad \lambda \in G_\delta, \quad |\lambda| > \lambda^*.$$

Then using the maximum principle and Liouville's theorem we obtain $Z(\lambda) \equiv 0$, i.e. $d(\lambda) \equiv \widetilde{d}(\lambda)$, and consequently $\mu_n = \widetilde{\mu}_n$, $n \geq 0$. Theorem is proved. \square

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