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ON THE REFLECTING FUNCTION AND THE QUALITATIVE BEHAVIOR OF SOLUTIONS OF SOME NON-AUTONOMOUS DIFFERENTIAL EQUATIONS*

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Abstract In this article, we use the Mironenko's method to discuss the qualitative behavior of some non-autonomous differential equations. We study the structure of the reflecting functions of the simplest differential equations, and obtain some sufficient conditions under which these equations have the rational reflecting functions. We apply the obtained results to discuss the numbers of periodic solutions of the non-autonomous differential systems and derive some sufficient conditions for a critical point of theirs to be a center.

Keywords Reflecting function, periodic solution, center conditions, non-autonomous equations.

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1. Introduction

By [1] we know, for the polynomial differential system,

$$\begin{cases} \dot{x} = \displaystyle\sum_{i+j=1}^{n+1} a_{ij} x^i y^j, \\ \dot{y} = \displaystyle\sum_{i+j=1}^{n+1} b_{ij} x^i y^j, \end{cases} \tag{1.1}$$

where a_{ij} and b_{ij} are real constants, there has been a longstanding problem, called the Poincaré center-focus problem, for the system (1.1) find explicit conditions of a_{ij} and b_{ij} under which (1.1) has a center at the origin $(0,0)$, i.e., all the orbits nearby are closed. The problem is equivalent to an analogue for a corresponding periodic equation

$$\frac{dr}{d\theta} = \frac{\sum_{i=0}^{n} q_i(\theta) r^i}{\sum_{i=0}^{n} p_i(\theta) r^i} r = \frac{Q(\theta, r)}{P(\theta, r)} r = R(\theta, r). \tag{1.2}$$

To see this let us note that the phase curves of (1.1) near the origin $(0,0)$ in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ are determined by (1.2), where $p_i(\theta)$ and $q_i(\theta)$ $(i = 0, 1, 2, \dots, n)$ are polynomials in $\cos\theta$ and $\sin\theta$.

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In some sense, the equation (1.2) can be transformed to a scalar ordinary differential equation of the form [1,3,10]

$$\rho' = r_1(\theta)\rho + r_2(\theta)\rho^2 + \ldots + r_N(\theta)\rho^N, \tag{1.3}$$

where the $r_i(\theta)$ are polynomials in $\cos\theta$ and $\sin\theta$. When $N = 3$ it has been exploited in a number of previous papers [1,3,10]. Since the limit cycles of (1.1) correspond to 2π-periodic solutions of (1.2) (or (1.3)). The planar vector field (1.1) has a center at (0,0) if and only if equation (1.2) (or (1.3)) has a center at $r = 0$ $(or\rho = 0)$, i.e., all the solutions nearby $r = 0$ $(\rho = 0)$ are closed: $r(0) = r(2\pi)(\rho(0) = \rho(2\pi))$.

In this paper, we apply the theory of reflecting function to study directly the qualitative behavior of the solutions of equation (1.2), and obtain the sufficient conditions for $r = 0$ to be a center.

First of all, we study under which conditions the scalar differential equation (1.2) is the simplest equation with reflecting function $F(\theta, r)$ and discuss the structure of $F(\theta, r)$. Secondly, we find out the sufficient conditions under which these equations have the rational reflecting functions. Finally, we apply the obtained results to research the numbers of the periodic solutions of (1.2) and obtain the center conditions.

In the present section, we introduce the concept of the reflecting function, which will be used throughout the rest of this article.

Consider differential system

$$x' = X(t, x),\ (t \in I \subset R,\ x \in D \subset R^n,\ 0 \in I), \tag{1.4}$$

which has a continuously differentiable right-hand side and with a general solution $\varphi(t; t_0, x_0)$.

Definition 1.1 ([6]). We call the function $F(t, x) := \varphi(-t, t, x)$ **Reflecting Function** of (1.4).

By this, for any solution $x(t)$ of (1.4), we have $F(t, x(t)) = x(-t)$, $F(0, x) = x$. If system (1.4) is 2ω-periodic with respect to t, and $F(t, x)$ is its reflecting function, then $T(x) := F(-\omega, x) = \varphi(\omega; -\omega, x)$ is the Poincaré mapping of (1.4) over the period $[-\omega, \omega]$. Thus, the solution $x = \varphi(t; -\omega, x_0)$ of (1.4) defined on $[-\omega, \omega]$ is 2ω-periodic if and only if x_0 is a fixed point of $T(x) = F(-\omega, x)$.

Lemma 1.1 ([6]). *A differentiable function $F(t, x)$ is a reflecting function of system (1.4) if and only if it is a solution of the Cauchy problem*

$$F_t + F_x X(t, x) + X(-t, F) = 0,\ \ F(0, x) = x. \tag{1.5}$$

Each continuously differentiable function $F(t, x)$ that satisfies $F(-t, F(t, x)) = x, F(0, x) = x$, is a reflecting function of the whole class of system of the form

$$x' = -(F_x(t, x) + E)^{-1} F_t(t, x) + {F_x}^{-1}\Phi(t, x) - \Phi(-t, F(t, x)), \tag{1.6}$$

where $\Phi(t, x)$ is an arbitrary continuously differentiable vector function. Therefore, all systems of the form (1.4) are split into equivalence classes of the form (1.6) where each class is specified by a certain reflecting function. For all 2ω-periodic systems of one class, the shift operator [2,6] on interval $[-\omega, \omega]$ coincides, and the qualitative behavior of the periodic solutions of these systems are the same.

Definition 1.2 ([6]). The system

$$x' = -(F_x(t,x) + E)^{-1} F_t(t,x) \tag{1.7}$$

is called the **Simplest System** with reflecting function $F(t,x)$.

Thus, to study the behavior of the solutions of (1.6), only need to discuss the property of the solutions of the simplest system (1.7).

Lemma 1.2 ([6]). *If the system (1.4) is the simplest system with reflecting function $F(t,x)$, then $X(t,x) = X(-t, F(t,x))$.*

There are many papers which are also devoted to investigations of qualitative behavior of solutions of differential systems by help of reflecting functions [4–9, 11–13].

In the following, we will denote $p_i = p_i(\theta)$, $\bar{p}_i = p_i(-\theta)$, $P = P(\theta, r)$, $\bar{P} = P(-\theta, F)$, etc. The notation "$\delta \neq 0$" means that in some deleted neighborhood of $(0,0)$ and $\theta^2 + r^2$ being small enough δ is different from zero. We always assume that all equations in this paper have a continuously differentiable right-hand side and have a unique solution for their initial value problem.

2. Main Results

Let us consider differential equation (1.2), in which P and Q are coprime polynomials of degree n (n is a positive integer number) with respect to r.

Firstly, we will discuss the structure of the reflecting function F when the equation (1.2) is the simplest equation.

If system (1.2) is the simplest with reflecting function F, by Lemma 1.2 we have $R(\theta, r) = R(-\theta, F)$, i.e.,

$$A_0 + A_1 F + A_2 F^2 + \ldots + A_{n+1} F^{n+1} = 0, \tag{2.1}$$

where

$$A_0 = -r\bar{p}_0 Q, \quad A_{n+1} = \bar{q}_n P,$$
$$A_k = P\bar{q}_{k-1} - r\bar{p}_k Q, \quad k = 1, 2, \ldots, n.$$

Lemma 2.1. *If $F = \frac{R_l}{S_m}$ is a solution of (2.1), then $l = m$ or $l = m+1$ $(m \leq n)$. Where R_l, S_m are coprime polynomials with respect to r of degree l, m, respectively, l and m are nonnegative integers.*

Proof. As $F = \frac{R_l}{S_m}$ is the solution of (2.1),

$$S_m(A_0 S_m^n + A_1 S_m^{n-1} R_l + \ldots + A_n R_l^n) = -A_{n+1} R_l^{n+1}. \tag{2.2}$$

Since R_l, S_m are coprime polynomials, so from (2.2) implies that A_{n+1} is divisible by S_m and $m \leq n$. According to $(P, Q) = 1$ and equating the same powers of r of equation (2.2) follows $l = m$ or $l = m+1$. □

Theorem 2.1. *If equation (1.2) is the simplest equation with reflecting function F, then $F = \frac{R_m}{S_m}$, or $F = \frac{R_{m+1}}{S_m}$ $(m \leq n)$, where S_k, R_k are polynomials of degree k with respect to r.*

Proof. Without loss of generality, we may assume that $q_n \neq 0$. Otherwise, similarly, we can get the same conclusion.

The relation (2.1) can be rewritten as

$$\lambda_0 + \lambda_1 F + \ldots + \lambda_n F^n + F^{n+1} = 0, \tag{2.3}$$

where $\lambda_k = \frac{A_k}{A_{n+1}}, \lambda_{n+1} = 1,\ k = 0, 1, 2, \ldots, n.$

Differentiating relation (2.3) with respect to θ and taking into account that $F(\theta, r(\theta)) = r(-\theta)$, we get

$$B_0 + B_1 F + \ldots + B_n F^n = 0, \tag{2.4}$$

where $B_k = D\lambda_k - (k+1)\lambda_{k+1} R,\ k = 0, 1, 2, \ldots, n,\ D\lambda_k = \frac{\partial \lambda_k}{\partial \theta} + \frac{\partial \lambda_k}{\partial r} R.$

Case 1. If $\sum_{k=0}^{n} B_k^2 \equiv 0$, i.e.,

$$D\lambda_k = (k+1)\lambda_{k+1} R, \quad k = 0, 1, 2, \ldots, n. \tag{2.5}$$

Denoting

$$f(\theta, F) = \lambda_0 + \lambda_1 F + \ldots + F^{n+1} = u_0 + u_1 G + \ldots + u_{n+1} G^{n+1},$$

where $u_k = \frac{1}{k!} \frac{\partial^k f(\theta, F)}{\partial F^k} |_{F = r_0} (k = 0, 1, 2, \ldots, n+1)$, $r_0 = -\frac{\lambda_n}{n+1}$, $G = F - r_0$.

Using relation (2.5), we can check that $Du_k = \frac{\partial u_k}{\partial \theta} + \frac{\partial u_k}{\partial r} R = 0$, thus, for any solution $r(\theta)$ of (1.2), we have $u_k(\theta, r(\theta)) = u_k(-\theta, r(-\theta)) = u_k(-\theta, F)$.

Consequently, the relation (2.3) is equivalent to

$$u_0 + u_1 G + \ldots + u_n G^n + G^{n+1} = 0.$$

Replacing θ by $-\theta$ and using $u_k = \bar{u}_k$, it yields

$$u_0 + u_1 \bar{G} + \ldots + u_n \bar{G}^n + \bar{G}^{n+1} = 0.$$

These equations indicate that $G = \bar{G}$, i.e., $F - r_0 = r - \bar{r}_0$, and that $F = r + r_0 - \bar{r}_0 = r - \frac{\lambda_n}{n+1} + \frac{\bar{\lambda}_n}{n+1} = r + \frac{1}{n+1}(\frac{q_{n-1}}{q_n} - \frac{\bar{q}_{n-1}}{\bar{q}_n}) + \frac{1}{n+1}(\frac{\bar{p}_n}{\bar{q}_n} - \frac{p_n}{q_n})R = \frac{R_{n+1}}{S_n}$. Thus, in this case the conclusion of the present theorem is correct.

Case 2. If $\sum_{k=0}^{n} B_k^2 \neq 0$. Let's assume that $B_n \neq 0$. The equation (2.4) can be rewritten as

$$\mu_0 + \mu_1 F + \ldots + \mu_{n-1} F^{n-1} + F^n = 0, \tag{2.6}$$

where $\nu_k = \frac{B_k}{B_n},\ k = 0, 1, 2, \ldots, n-1.$

Substituting (2.6) into (2.3), we get

$$C_0 + C_1 F + \ldots + C_{n-1} F^{n-1} = 0, \tag{2.7}$$

where

$$C_0 = \lambda_0 - \lambda_n \nu_0 + \nu_{n-1} \nu_0,$$
$$C_k = \lambda_k - \lambda_n \nu_k - \nu_{k-1} + \nu_{n-1} \nu_k,\ \ k = 1, 2, \ldots, n-1.$$

1^0. If $\sum_{k=0}^{n-1} C_k^2 = 0$, i.e.,

$$\lambda_0 = \lambda_n \nu_0 + \nu_{n-1} \nu_0, \quad \lambda_k = \lambda_n \nu_k + \nu_{k-1} - \nu_k \nu_{n-1}, \quad k = 1, 2, \ldots, n-1.$$

Applying these relations and simply computing we get

$$\lambda_0+\lambda_1(\nu_{n-1}-\lambda_n)+\lambda_2(\nu_{n-1}-\lambda_n)^2+\ldots+\lambda_n(\nu_{n-1}-\lambda_n)^n+(\nu_{n-1}-\lambda_n)^{n+1}=0. \quad (2.8)$$

Using relations (2.3) and (2.8), we have $F=\nu_{n-1}-\lambda_n=\frac{B_{n-1}}{B_n}-\frac{A_n}{A_{n+1}}$. Substituting it into (2.1), we confirm that the conclusion of the present theorem is true.
2^0. If $\sum_{i=0}^{n-1} C_i^2 \neq 0$. Let's assume $C_{n-1}\neq 0$, then (2.7) becomes

$$\eta_0+\eta_1 F+\ldots+\eta_{n-2}F^{n-2}+F^{n-1}=0, \quad (2.9)$$

where $\eta_k=\frac{C_k}{C_{k-1}},\ k=0,1,2,\ldots,n-2$.

Substituting (2.9) into (2.6), we obtain

$$D_0+D_1F+\ldots+D_{n-2}F^{n-2}=0, \quad (2.10)$$

where

$$D_k=\nu_k-\eta_k\nu_{n-1}-\eta_{n-1}+\eta_{n-2}\eta_k,\ \ k=0,1,2,\ldots,n-2.$$

Similarly, we see that if $\sum_{k=0}^{n-2} D_k^2=0$, then $F=\eta_{n-2}-\nu_{n-1}$, by Lemma 2.1, the present theorem is true. If $\sum_{k=0}^{n-2} D_k^2\neq 0$, then $E_0+E_1F=0$ and $F=\frac{R_l}{S_m}$, according to Lemma 2.1, the present theorem is correct.

Therefore, the proof is finished. □

Corollary 2.1. *If equation (1.2) is the simplest equation with reflecting function F and $p_i(\theta+2\pi)=p_i(\theta), q_i(\theta+2\pi)=q_i(\theta)$, $i=0,1,2,\ldots,n$, then one of the following conclusions is correct.*

1) *The equation (1.2) has at most $n+1$ periodic solutions.*
2) *All the solutions of (1.2) defined on $[-\omega,\omega]$ are 2ω periodic, i.e., $r=0$ is a center of (1.2).*
3) *The equation (1.2) does not have any periodic solution.*

Proof. By Theorem 2.1, we know the reflecting function of (1.2) is in the form of $F=\frac{R_m}{S_m}$ or $F=\frac{R_{m+1}}{S_m}$ $(m\le n)$. Then the Poincaré mapping of periodic equation (1.2) is $T(r)=F(-\pi,r)$. The number of 2π-periodic solutions of (1.2) is equal to the number of roots of the fixed point equation $F(-\pi,r)=r$. From this follows the present conclusions. □

Remark 2.1. By [13] we know that if $F=\frac{\beta_0+\beta_1x+\beta_2x^2}{\alpha_0+\alpha_1x+\alpha_2x^2}$ is the reflecting function of one differential equation, then $F=\frac{\beta_0+\beta_1x}{\alpha_0+\alpha_1x}$ or $F=\beta_0+\beta_1x$. Thus, we can guess that if F is a polynomial or rational fraction function, then $F=f_0+f_1x$ or $F=\frac{\beta_0+\beta_1x}{\alpha_0+\alpha_1x}$. This conjecture need us to prove.

On the other hand, if $F=f_0+f_1x$ or $F=\frac{\beta_0+\beta_1x}{\alpha_0+\alpha_1x}$ is the reflecting function of (1.2), by Lemma 1.1 and the uniqueness of the solutions of the initial value problem of differential equation (1.5) implies $f_0\equiv 0, \beta_0\equiv 0$. So, in the following, we only discuss when the equation (1.2) has the reflecting function in the form of $F=f_1r$ and $F=\frac{\beta r}{1+\alpha r}$.

Theorem 2.2. *Suppose that $p_0\neq 0$,*

$$(q_0\bar{p}_0+p_0\bar{q}_0)\sum_{i+j=k} p_i\bar{p}_j f_1^j = p_0\bar{p}_0\sum_{i+j=k} q_i\bar{q}_j f_1^j,$$

$$k=0,1,2,\dots,2n,\ \textit{when}\ i>n,\ p_i=q_i=0,$$

where

$$f_1=e^{-\int_0^\theta(\frac{q_0}{p_0}+\frac{\bar q_0}{\bar p_0})d\theta}.$$

Then $F=f_1r$ is the reflecting function of (1.2).

Moreover, if $p_i(\theta+2\pi)=p_i(\theta)$, $q_i(\theta+2\pi)=q_i(\theta)$ $(i=0,1,2,\dots,n)$, then one of the following conclusions is correct.

1) *If $\int_{-\pi}^{\pi}\frac{q_0}{p_0}d\theta\neq 0$, the equation (1.2) has only one 2π-periodic solution which is asymptotically stable when $\int_{-\pi}^{\pi}\frac{q_0}{p_0}d\theta<0$, unstable when $\int_{-\pi}^{\pi}\frac{q_0}{p_0}d\theta>0$.*
2) *If $\int_{-\pi}^{\pi}\frac{q_0}{p_0}d\theta=0$, then all the solutions of (1.2) defined on $[-\pi,\pi]$ are 2π-periodic, i.e., $r=0$ is a center of (1.2).*

Proof. By the present conditions, it is not difficult to check that $F=f_1r$ is a solution of the Cauchy problem:

$$f_1'+f_1\frac{Q}{P}+\frac{Q(-\theta,f_1r)}{P(-\theta,f_1r)}f_1=0,\ f_1(0)=1.$$

Thus, $F=f_1r$ is the reflecting function of (1.2).

If the equation (1.2) is 2π-periodic, then the Poincaré mapping of (1.2) is $T(r)=F(-\pi,r)=f_1(-\pi)r=re^{\int_{-\pi}^{\pi}\frac{q_0(\theta)}{p_0(\theta)}d\theta}$. By this and [6] yields the present conclusions. □

Now we discuss when the equation (1.2) has the reflecting function $F=\frac{\beta r}{1+\alpha r}$.

Theorem 2.3. *Suppose that $p_0\neq 0$ and*

$$\begin{aligned}&\sum_{i+j=k}\big((\frac{q_0}{p_0}+\frac{\bar q_0}{\bar p_0})p_i\check p_j-(q_i\check p_j+p_i\check q_j)\big)\\ =&\sum_{i+j=k-1}(\alpha p_i\check q_i-(\alpha\frac{\bar q_0}{\bar p_0}+\beta\bar\delta+\delta)p_i\check p_j),\qquad (2.11)\\ &k=2,3,\dots,2n+1,\ \textit{when}\ i>n,\ p_i=0,\ q_i=0,\ \check p_i=0,\ \check q_i=0,\end{aligned}$$

where

$$\check p_k=\sum_{i=0}^{k}\bar p_iC_{n-i}^{k-i}\alpha^{k-i}\beta^i,\ k=0,1,2,\dots,n,$$

$$\check q_k=\sum_{i=0}^{k}\bar q_iC_{n-i}^{k-i}\alpha^{k-i}\beta^i,\ k=0,1,2,\dots,n,$$

$$\beta=e^{\sigma-\bar\sigma},\quad \sigma=-\int_0^\theta\frac{q_0}{p_0}d\theta,\qquad (2.12)$$

$$\alpha=e^{\sigma}\int_0^\theta(\delta e^{-\sigma}+\bar\delta e^{-\bar\sigma})d\theta,\quad \delta=\frac{p_0q_1-p_1q_0}{p_0^2}.\qquad (2.13)$$

Then $F=\frac{\beta r}{1+\alpha r}$ is the reflecting function of (1.2).

Moreover, if $p_i(\theta+2\pi)=p_i(\theta)$, $q_i(\theta+2\pi)=q_i(\theta)$ $(i=0,1,2,\dots,n)$, then one of the following conclusions is correct.

1) If $\alpha(-\pi) \neq 0$ and $\beta(-\pi) \neq 1$, then equation (1.2) has two 2π-periodic solutions.

2) If $\alpha(-\pi) = 0$, $\beta(-\pi) = 1$, then all the solutions of (1.2) defined on $[-\pi, \pi]$ are 2π-periodic, i.e., $r = 0$ is a center of (1.2).

3) If $\alpha(-\pi) = 0$, $\beta(-\pi) \neq 1$ or $\alpha(-\pi) \neq 0$, $\beta(-\pi) = 1$, then equation (1.2) has only one 2π-periodic solution, i.e., $r = 0$.

Proof. By Lemma 1.1, we see $F = \frac{\beta r}{1+\alpha r}$ is the reflecting function of (1.2) if and only if

$$(\beta' + r(\beta'\alpha - \alpha'\beta)) \sum_{i=0}^{n} p_i r^i \sum_{i=0}^{n} \bar{p}_i \beta^i r^i (1+\alpha r)^{n-i} + \beta \sum_{i=0}^{n} q_i r^i \sum_{i=0} \bar{p}_i \beta^i r^i (1+\alpha r)^{n-i}$$
$$+ \beta(1+\alpha r) \sum_{i=0}^{n} p_i r^i \sum_{i=0} \bar{q}_i \beta^i r^i (1+\alpha r)^{n-i} = 0,$$
$$\alpha(0) = 0, \ \beta(0) = 1.$$

Equating the coefficients of the same power of r implies

$$\begin{aligned}
&\beta' p_0 \check{p}_0 + \beta q_0 \check{q}_0 + \beta p_0 \check{q}_0 = 0, \quad \beta(0) = 1,\\
&\beta'(p_0\check{p}_1 + p_1\check{p}_0) + \alpha\beta' p_0\check{p}_0 - \alpha'\beta p_0\check{p}_0 + \beta(q_0\check{p}_1 + q_1\check{p}_0 + p_0\check{q}_1 + p_1\check{q}_0)\\
&+ \alpha\beta p_0\check{q}_0 = 0, \quad \alpha(0) = 0,\\
&\beta' \sum_{i+j=k} p_i\check{p}_j + (\alpha\beta' - \alpha'\beta) \sum_{i+j=k-1} p_i\check{p}_j + \beta \sum_{i+j=k} q_i\check{p}_j + \beta \sum_{i+j=k} p_i\check{q}_j\\
&+ \alpha\beta \sum_{i+j=k-1} p_i\check{q}_j = 0,\\
&k = 2, 3, \ldots, 2n+1, \text{ when } i > n, \ p_i = 0, \ q_i = 0, \ \check{p}_i = 0, \ \check{q}_i = 0.
\end{aligned}$$

From the first equation of the above we can get

$$\beta' = -(\frac{q_0}{p_0} + \frac{\bar{q}_0}{\bar{p}_0})\beta, \quad \beta(0) = 1,$$

which yields the relation (2.12).

Solving the second equation of the above we have

$$\alpha' = -\frac{q_0}{p_0}\alpha + \frac{q_1p_0 - q_0p_1}{p_0^2} + \frac{\bar{q}_1\bar{p}_0 - \bar{q}_0\bar{p}_1}{\bar{p}_0^2}\beta = -\frac{q_0}{p_0}\alpha + \delta + \bar{\delta}\beta, \quad \alpha(0) = 0,$$

which implies the relation (2.13).

Substituting these relations into the third relation of the above we can obtain (2.11). Therefore, under the assumption of the present theorem, $F = \frac{\beta r}{1+\alpha r}$ is the reflecting function of (1.2).

If the equation (1.2) is 2π-periodic, then the Poincaré mapping of (1.2) is $T(r) = F(-\pi, r) = \frac{\beta(-\pi)r}{1+\alpha(-\pi)r}$. So, the fixed point equation is $(\beta(-\pi) - 1 + \alpha(-\pi)r)r = 0$. Using this equation and by [6] yields the present conclusions. □

Remark 2.2. From the above theorems, we know that under certain conditions the equation (1.2) has a center at $r = 0$, that is said, under the same conditions, the planar vector field (1.1) has a center at (0,0).

Example 2.1. Taking $x = r\cos\theta, y = r\sin\theta$, the cubic system

$$\begin{cases} \dot{x} = -2y + x^2 - 2y^2 - y^3, \\ \dot{y} = 2x + 3xy + xy^2, \end{cases} \tag{2.14}$$

can be transformed to equation

$$r' = \frac{\cos\theta}{1 + (1 + r\sin\theta)^2} r^2. \tag{2.15}$$

As this equation is the simplest equation with reflecting function $F = \frac{r}{1+r\sin\theta}$ and $F(-\pi, r) \equiv r$, thus the critical point $(0,0)$ of (2.14) is a center point.

From the previous introduction and (1.6), we know the equation (2.15) is equivalent to equation

$$r' = \frac{c}{1 + (1 + rs)^2} r^2 + (1 + rs)^2 G(\theta, r) - G(-\theta, \frac{r}{1 + rs}),$$

where $G(\theta, r)$ is an arbitrary continuously differentiable function, $s := \sin\theta$, $c := \cos\theta$.

In particular, taking $G = \frac{r}{(1+rs)(1+(1+rs)^2)}$ and $G = \frac{r^3}{(1+rs)(1+(1+sr)^2)^2}$ we get the following equations, respectively

$$r' = \frac{c - s(1 + rs)}{1 + (1 + rs)^2} r^2, \tag{2.16}$$

$$r' = \frac{2c + 2rsc + (cs^2 - s)r^2 - s^2 r^3}{(2 + 2rs + r^2 s^2)^2} r^2. \tag{2.17}$$

Taking $r = \sqrt{x^2 + y^2}$, $\theta = \arctan\frac{y}{x}$, (2.16) and (2.17), respectively, reduces to systems

$$\begin{cases} \dot{x} = -2y + x^2 - xy - 2y^2 - xy^2 - y^3, \\ \dot{y} = 2x + 3xy - y^2 + xy^2 - y^3, \end{cases}$$

and

$$\begin{cases} \dot{x} = -4y + 2x^2 - 8y^2 - 8y^3 + 2x^2 y + x^2 y^2 - 4y^4 - x^3 y - xy^3 - x^3 y^2 - xy^4 - y^5, \\ \dot{y} = 4x + 10xy + 10xy^2 + 5xy^3 - x^2 y^2 - x^2 y^3 + xy^4 - y^4 - y^5. \end{cases}$$

Thus, the origin $(0,0)$ of the above systems is a center.

As $G(\theta, r)$ is an arbitrary continuously differentiable function, similarly, we can write infinitely many polynomial differential systems, their origin point (0,0) is a center.

From this example, we see that using the method of Mironenko (reflecting function) we not only solve a center-focus problem, but also at the same time, we open a class of differential equations with the same character of point $r = 0$. So, we can say, sometimes, the method of Mironeneko is more effective than Lyapunov's method. Therefore, if we can find out the reflecting function of a differential system, then the qualitative behavior of periodic solutions and the center-focus problem are solved. Unfortunately, looking for reflecting function is also very difficult task, so we need further study it.

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