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MULTIPLICITY RESULTS OF FOURTH-ORDER SINGULAR NONLINEAR DIFFERENTIAL EQUATION WITH A PARAMETER*

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Abstract In this paper, we investigate a class of fourth-order singular nonlinear differential equation with superlinearity or sublinearity assumptions at infinity for an appropriately chosen parameter. By applications of Green's function and the Krasnoselskii fixed point theorem, sufficient conditions for the existence of positive periodic solutions are established.

Keywords Fourth-order, superlinear and sublinear, singular, Krasnoselskii fixed point theorem, periodic solution.

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1. Introduction

The family of the so-called Lazer-Solimini equations

$$x'' + \frac{1}{x^\gamma} = p(t), \tag{1.1}$$

and

$$x'' - \frac{1}{x^\gamma} = p(t), \tag{1.2}$$

where $\gamma > 0$ and $p(t)$ is a periodic function with period ω. They are perhaps the simplest examples combining singular nonlinearity and a periodic dependence of the coefficients. In a renowned paper from 1987, Lazer and Solimini [14] investigated the problem of existence of positive ω-periodic solutions for these model equations.

Lazer and Solimini's work has attracted the attention of many scholars in differential equations. More recently, the method of lower and upper solutions [1, 12, 16], the Poincaré-Birkhoff twist theorem [4, 11, 23], topological degree theory [2, 25] the Schauder's fixed point theorem [17, 20, 24], the Leray-Schauder alternative principle [5, 6], the Krasnoselskii fixed point theorem in a cone [8, 21], the fixed point index theory [19] have been employed to investigate the existence of positive periodic solutions of singular second order and third-order differential equations.

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At the beginning, most of work concentrated on second-order and third-order singular differential equations, as in the references we mentioned above. Recently there have been published some results on fourth-order differential equation (see [3,7,9,10,15,26]). In 2003, Conti, Terracini and Verzini [9] study the fourth-order equation

$$u^{(4)}(t) - cu''(t) = f(t, u(t)), \quad t \in [0, T],$$

with periodic boundary conditions, where $c \geq -(\pi/T)^2$, $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous, T-periodic in t and has a superlinear behavior at 0 and at infinity. Under these assumptions, they shown that for each positive integer $n \geq 1$ the problem admits a T-periodic solution having precisely $2n$ simple zeroes in $[0, T]$. The proof was inspired by Nehari's argument of combining variational methods and nodal properties of solutions. However, here a new and subtle min-max procedure is built, allowing one to interpret nodal properties of solutions of the problem as a topological property and to get these solutions by means of a variational principle with two constraints. Afterwards, by constructing a special cone and using cone compression and expansion fixed point theorem, Cui and Zou [10] considered the existence and uniqueness of solutions are established for the following singular fourth-order boundary value problems:

$$\begin{cases} x^{(4)}(t) = f(t, x(t), -x''(t)), & 0 < t < 1, \\ x(0) = x(1) = x''(0) = x''(1) = 0, \end{cases}$$

where $f(t, x, y)$ may be singular at $t = 0,\ 1;\ x = 0$ and $y = 0$.

In the above papers, the authors investigated fourth-order equations. However, the study on the fourth-order singular equation is relatively infrequent. Motivated by [9,10,26], in this paper, we further consider a fourth-order singular differential equation with a parameter as follows,

$$x^{(4)}(t) + ax'''(t) + bx''(t) + cx'(t) + dx(t) = \mu g(t) f(x(t)) + \mu e(t), \tag{1.3}$$

with $\mu > 0$ is a positive parameter, and $e(t)$ may takes positive value or negative value. $a,\ b,\ c,\ d \in \mathbb{R}$, $g(t)$ and $e(t)$ are ω-periodic continuous scalar functions in $t \in \mathbb{R}$. The nonlinear term f of (1.3) can be with a singularity at origin, i.e.,

$$\lim_{x \to 0^+} f(x) = +\infty, \qquad (\text{or } \lim_{x \to 0^+} f(x) = -\infty), \qquad \text{uniformly in } t.$$

It is said that (1.3) is of repulsive type (resp. attractive type) if $f(x) \to +\infty$ (resp. $f(x) \to -\infty$) as $x \to 0^+$.

As far as we know, studies on fourth-order nonlinear differential equations are rather infrequent, especially those focused on the research of singular fourth-order nonlinear differential equations with a parameter. The main difficulty lies in the calculation of the Green's function of the fourth-order differential equation, being more complicated than in the second-order and third-order cases. Therefore, in Section 2, the Green's function for the fourth-order linear differential equation

$$x^{(4)}(t) + ax'''(t) + bx''(t) + cx'(t) + dx(t) = h(t) \tag{1.4}$$

will be given. Here $h \in C(R, (0, +\infty))$ is an ω-periodic function. Some useful properties for the Green's function are obtained. In Section 3, we define a cone and discuss several properties of the equivalent operator on the cone. In order to simplify the proof in section 3, we establish a series of lemmas and corollaries to estimate the operator. All the corollaries are the corresponding results for $e(t)$ taking negative values. In Section 4, by employing Green's function and the Krasnoselskii fixed point theorem, we state and prove the existence of positive periodic solutions for singular fourth-order differential equation with superlinearity or sublinearity assumptions at infinity for an appropriately chosen parameter.

2. Green's function of fourth-order differential equation

Firstly, we consider

$$\begin{cases} x^{(4)}(t) + ax'''(t) + bx''(t) + cx'(t) + dx(t) = h(t), \\ x^{(i)}(0) = x^{(i)}(\omega), \qquad i = 0, 1, 2, 3, \end{cases} \tag{2.1}$$

where $h \in C(\mathbb{R}, \mathbb{R}^+)$ is an ω-periodic function. Obviously, the calculation of the Green's function of (2.1) is very complicated, so, by analysis of the fourth-order differential equation (2.1), we consider the following six cases.

Case (I): There exist real constants α, β, γ and $\rho > 0$ such that $a = \alpha + \rho$, $b = \beta + \alpha\rho$, $c = \gamma + \beta\rho$, $d = \gamma\rho$. Then, (2.1) is transformed into

$$\begin{cases} y'(t) + \rho y(t) = h(t), \\ y(0) = y(\omega), \end{cases} \tag{2.2}$$

and

$$\begin{cases} x'''(t) + \alpha x''(t) + \beta x'(t) + \gamma x(t) = h(t), \\ x^{(i)}(0) = x^{(i)}(\omega), \quad i = 0, 1, 2. \end{cases} \tag{2.3}$$

Solution of (2.2) is written as

$$y(t) = \int_0^\omega G_1(t,s)h(s)ds, \tag{2.4}$$

where

$$G_1(t,s) = \begin{cases} \frac{e^{-\rho(t-s)}}{1-e^{-\omega\rho}}, & 0 \le s \le t \le \omega, \\ \frac{e^{-\rho(\omega+t-s)}}{1-e^{-\omega\rho}}, & 0 \le t < s \le \omega. \end{cases}$$

Solution of (2.3) is written as

$$x(t) = \int_0^\omega G_{2i}(t,s)y(s)ds, \qquad i = 1, 2, 3, 4. \tag{2.5}$$

Next, we will consider $G_{2i}(t,s)$, which can be found in [17]. The associated homogeneous equation of (2.3) is

$$x''' + \alpha x'' + \beta x' + \gamma x = 0. \tag{2.6}$$

Its characteristic equation is

$$\lambda^3 + \alpha\lambda^2 + \beta\lambda + \gamma = 0. \tag{2.7}$$

Obviously the roots λ_1, λ_2, λ_3 of (2.7) satisfy one of the four cases:

1. $\lambda_1 \neq \lambda_2 \neq \lambda_3,\ \lambda_1,\ \lambda_2,\ \lambda_3 \in R$.
2. $\lambda_1 = \lambda_2 \neq \lambda_3,\ \lambda_1,\ \lambda_2,\ \lambda_3 \in R$.
3. $\lambda_1 = \lambda_2 = \lambda_3 = \lambda \in R$.
4. $\lambda_1 = \alpha + i\beta,\ \lambda_2 = \alpha - i\beta,\ \lambda_3 = \lambda,\ \alpha,\ \beta,\ \lambda \in R$.

If $\gamma = 0$, then at least one of the roots of (2.6) is 0. In this case we call equation (2.3) degenerate. This case will be discussed elsewhere. In this paper, we always assume $\gamma \neq 0$.

Lemma 2.1. *If $\lambda_1 \neq \lambda_2 \neq \lambda_3,\ \lambda_1,\ \lambda_2,\ \lambda_3 \in R$, then the equation* (2.3) *has a unique ω-periodic solution*

$$x(t) = \int_t^{t+\omega} G_{21}(t,s)y(s)\mathrm{d}s, \tag{2.8}$$

where

$$\begin{aligned} G_{21}(t,s) = {} & \frac{\exp(\lambda_1(t+\omega-s))}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(1-\exp(\lambda_1\omega))} + \frac{\exp(\lambda_2(t+\omega-s))}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)(1-\exp(\lambda_2\omega))} \\ & + \frac{\exp(\lambda_3(t+\omega-s))}{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)(1-\exp(\lambda_3\omega))} \qquad \text{for } s \in [t, t+\omega]. \end{aligned} \tag{2.9}$$

Lemma 2.2. *If $\lambda_1 = \lambda_2 \neq \lambda_3,\ \lambda_1,\ \lambda_2,\ \lambda_3 \in R$, then the equation of* (2.3) *has a unique ω-periodic solution*

$$x(t) = \int_t^{t+\omega} G_{22}(t,s)y(s)\mathrm{d}s, \tag{2.10}$$

where

$$\begin{aligned} G_{22}(t,s) = {} & \frac{\exp(\lambda_1(t+\omega-s))[(1-\exp(\lambda_1\omega))((s-t)(\lambda_3-\lambda_1)-1)-(\lambda_3-\lambda_1)\omega]}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_1\omega))^2} \\ & + \frac{\exp(\lambda_3(t+\omega-s))}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_3\omega))} \qquad \text{for } s \in [t, t+\omega]. \end{aligned} \tag{2.11}$$

Lemma 2.3. *If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda \in R$, then the equation of* (2.3) *has a unique ω-periodic solution*

$$x(t) = \int_t^{t+\omega} G_{23}(t,s)y(s)\,\mathrm{d}s, \tag{2.12}$$

where

$$G_{23}(t,s) = \frac{[(s-t)\exp(\lambda\omega)+\omega-s+t]^2+\omega^2\exp(\lambda\omega)}{2(1-\exp(\lambda\omega))^3}\exp(\lambda(t+\omega-s)) \tag{2.13}$$

for $s \in [t, t+\omega]$.

Now take the abbreviations

$$l_1(t,s) = \cos\beta(t+\omega-s) - \exp(\alpha\omega)\cos\beta(t-s),$$

$$l_2(t,s) = \sin\beta(t+\omega-s) - \exp(\alpha\omega)\sin\beta(t-s),$$

we have

Lemma 2.4. *If $\lambda_1 = \alpha + i\beta,\ \lambda_2 = \alpha - i\beta,\ \lambda_3 = \lambda,\ \alpha,\ \beta,\ \lambda \in R$, then the equation of* (2.3) *has a unique ω-periodic solution*

$$x(t) = \int_t^{t+\omega} G_{24}(t,s)y(s)\mathrm{d}s, \tag{2.14}$$

where

$$\begin{aligned} G_{24}(t,s) =& \frac{\exp(\alpha(t+\omega-s))[(\alpha-\lambda)l_2(t,s) - \beta l_1(t,s)]}{\beta[(\alpha-\lambda)^2+\beta^2](1+\exp(2\alpha\omega) - 2\exp(\alpha\omega)\cos\beta\omega)} \\ &+ \frac{\exp(\lambda(t+\omega-s))}{(1-\exp(\lambda\omega))[(\alpha-\lambda)^2+\beta^2]} \qquad \textit{for } s \in [t, t+\omega]. \end{aligned} \tag{2.15}$$

We give properties of Green's function in the following:
Case (I*): $\lambda_1 \neq \lambda_2 \neq \lambda_3,\ \lambda_1,\ \lambda_2,\ \lambda_3 \in R$.
For sake of convenience, we use the following abbreviations

$$\begin{aligned} A_1 =& \frac{\exp(\lambda_1\omega)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(1-\exp(\lambda_1\omega))} + \frac{1}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)(1-\exp(\lambda_2\omega))} \\ &+ \frac{\exp(\lambda_3\omega)}{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)(1-\exp(\lambda_3\omega))}, \\ B_1 =& \frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(1-\exp(\lambda_1\omega))} + \frac{\exp(\lambda_2\omega)}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)(1-\exp(\lambda_2\omega))} \\ &+ \frac{1}{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)(1-\exp(\lambda_3\omega))}, \\ p_2 =& (\lambda_1+\lambda_2-2\lambda_3)\exp(\lambda_1\omega) + (2\lambda_1-\lambda_2-\lambda_3)\exp(\lambda_3\omega) \\ &+ (\lambda_1-\lambda_3)\exp((\lambda_1+\lambda_2+\lambda_3)\omega), \\ q_2 =& (\lambda_1-\lambda_3) + (\lambda_1-\lambda_2)\exp((\lambda_2+\lambda_3)\omega) + (\lambda_2-\lambda_3)\exp((\lambda_1+\lambda_2)\omega) \\ &+ 2(\lambda_1-\lambda_3)\exp((\lambda_1+\lambda_3)\omega). \end{aligned}$$

Lemma 2.5. *If $p_2 > q_2$ and one of the following conditions*
(i) $\lambda_3 < \lambda_2 < \lambda_1 < 0$; *(ii) $\lambda_1 > \lambda_2 > 0$ and $\lambda_3 < 0$,*
is satisfied, then $0 < A_1 \le G_{21}(t,s) \le B_1$.

Case (I):** $\lambda_1 = \lambda_2 \neq \lambda_3,\ \lambda_1,\ \lambda_2,\ \lambda_3 \in R$.
For convenience, define the abbreviations

$$\begin{aligned} A_2 =& \frac{\exp(\lambda_1\omega) - 1 + (\lambda_1-\lambda_3)\omega}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_1\omega))^2} + \frac{\exp(\lambda_3\omega)}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_3\omega))}, \\ B_2 =& \frac{(\exp(2\lambda_1\omega) - \exp(\lambda_1\omega)) + (\lambda_1-\lambda_3)\omega\exp(2\lambda_1\omega)}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_1\omega))^2} \\ &+ \frac{1}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_3\omega))}, \\ p_3 =& \exp(\lambda_1\omega) + (\lambda_1-\lambda_3)\omega + (\exp(\lambda_1\omega)-3)\exp((\lambda_1+\lambda_3)\omega) \\ &+ (2+(\lambda_3-\lambda_1)\omega)\exp(\lambda_3\omega). \end{aligned}$$

Lemma 2.6. *If $\lambda_1 > 0, \lambda_3 < 0$, then $0 < A_2 \le G_{22}(t,s) \le B_2$.*

Lemma 2.7. *If $\lambda_1 < \lambda_3 < 0$ and $p_3 > 1$, then $0 < A_2 \le G_{22}(t,s) \le B_2$.*

Case (I*):** $\lambda_1 = \lambda_2 = \lambda_3 = \lambda \in R$.
For convenience, define

$$A_5 = \frac{\omega^2 \exp(2\lambda\omega)(1+\exp(\lambda\omega))}{2(1-\exp(\lambda\omega))^3} \quad \text{and} \quad B_5 = \frac{\omega^2(1+\exp(\lambda\omega))}{2(1-\exp(\lambda\omega))^3}.$$

Lemma 2.8. *If $\lambda < 0$, then $0 < A_5 \le G_{23}(t,s) \le B_5$.*

Case (I**):** $\lambda_1 = \alpha + i\beta,\ \lambda_2 = \alpha - i\beta,\ \lambda_3 = \lambda,\ \alpha,\ \beta,\ \lambda \in R$.
For the sake of convenience, define

$$\begin{aligned}
A_6 &= \frac{-\exp(\alpha\omega)}{\beta\sqrt{[(\alpha-\lambda)^2+\beta^2](1+\exp(2\alpha\omega)-2\cos(\beta\omega)\exp(\alpha\omega))}} \\
&\quad + \frac{\exp(\lambda\omega)}{[(\alpha-\lambda)^2+\beta^2](1-\exp(\lambda\omega))}, \\
B_6 &= \frac{\exp(\alpha\omega)}{\beta\sqrt{[(\alpha-\lambda)^2+\beta^2](1+\exp(2\alpha\omega)-2\cos(\beta\omega)\exp(\alpha\omega))}} \\
&\quad + \frac{1}{[(\alpha-\lambda)^2+\beta^2](1-\exp(\lambda\omega))}, \\
A_7 &= \frac{-1}{\beta\sqrt{[(\alpha-\lambda)^2+\beta^2](1+\exp(2\alpha\omega)-2\cos(\beta\omega)\exp(\alpha\omega))}} \\
&\quad + \frac{\exp(\lambda\omega)}{[(\alpha-\lambda)^2+\beta^2](1-\exp(\lambda\omega))}, \\
B_7 &= \frac{1}{\beta\sqrt{[(\alpha-\lambda)^2+\beta^2](1+\exp(2\alpha\omega)-2\cos(\beta\omega)\exp(\alpha\omega))}} \\
&\quad + \frac{1}{[(\alpha-\lambda)^2+\beta^2](1-\exp(\lambda\omega))}.
\end{aligned}$$

Lemma 2.9. *If $\alpha > 0,\ \beta > 0,\ \lambda < 0$, and*

$$\frac{1+\exp(2\alpha\omega)-2\exp(\alpha\omega)\cos(\beta\omega)}{\exp(2\alpha\omega)} > \frac{[(\alpha-\lambda)^2+\beta^2](1-\exp(\lambda\omega))^2}{\beta^2\exp(2\lambda\omega)}, \tag{2.16}$$

then $0 < A_6 \le G_{24}(t,s) \le B_6$.

Lemma 2.10. *If $\alpha < 0,\ \lambda < 0,\ \beta > 0$ and*

$$(1+\exp(2\alpha\omega)-2\cos(\beta\omega)\exp(\alpha\omega)) > \frac{[(\alpha-\lambda)^2+\beta^2](1-\exp(\lambda\omega))^2}{\beta^2\exp(2\lambda\omega)}, \tag{2.17}$$

then $0 < A_7 \le G_{24}(t,s) \le B_7$.

Therefore, we know that the solution of (2.1) is written as

$$\begin{aligned}
x(t) &= \int_0^\omega G_{2i}(t,\tau)\int_0^\omega G_1(\tau,s)h(s)dsd\tau \\
&= \int_0^\omega \int_0^\omega G_{2i}(t,\tau)G_1(\tau,s)h(s)dsd\tau \\
&= \int_0^\omega \left[\int_0^\omega G_{2i}(t,s)G_1(s,\tau)ds\right] h(\tau)d\tau \\
&= \int_0^\omega \left[\int_0^\omega G_{2i}(t,\tau)G_1(\tau,s)d\tau\right] h(s)ds, \qquad i = 1,2,3,4.
\end{aligned}$$

Therefore, by writing

$$G^{1i}(t,s) = \int_0^\omega G_{2i}(t,\tau)G_1(\tau,s)d\tau, \quad i = 1,2,3,4, \tag{2.18}$$

we can get

$$x(t) = \int_0^\omega G^{1i}(t,s)h(s)ds. \tag{2.19}$$

Theorem 2.1. *Assume that* $\lambda_3 < \lambda_2 < \lambda_1 < 0$ *(or* $\lambda_1 > \lambda_2 > 0, \quad \lambda_3 < 0$*), and* $p_2 > q_2$ *hold. Then* $G^{11}(t,s) > 0$ *for all* $(t,s) \in [0,\omega] \times [0,\omega]$.

Proof. From Lemma 2.5, we know $G_{21}(t,s) > 0$. Since $G_1(t,s) > 0$, from (2.18) we see that $G^{11}(t,s) > 0$ for all $(t,s) \in [0,\omega] \times [0,\omega]$. □

Theorem 2.2. *Assume that* $\lambda_1 > 0, \lambda_3 < 0$ *(or* $\lambda_1 < \lambda_3 < 0, \quad p_3 > 1$*) hold. Then* $G^{12}(t,s) > 0$ *for all* $(t,s) \in [0,\omega] \times [0,\omega]$.

Theorem 2.3. *Assume that* $\lambda < 0$ *holds. Then* $G^{13}(t,s) > 0$ *for all* $(t,s) \in [0,\omega] \times [0,\omega]$.

Theorem 2.4. *Assume that* $\alpha > 0, \quad \beta > 0, \quad \lambda < 0,$ (2.16) *(or* $\alpha < 0, \quad \beta > 0, \quad \lambda < 0,$ (2.17)*) hold. Then* $G^{14}(t,s) > 0$ *for all* $(t,s) \in [0,\omega] \times [0,\omega]$.

Case (II): There exist positive real constants m and ρ such that $a = \rho$, $b = 0$, $c = -m^3$, $d = -\rho m^3$. Then, (2.1) is transformed into

$$\begin{cases} y'(t) + \rho y(t) = h(t), \\ y(0) = y(\omega), \end{cases} \tag{2.20}$$

and

$$\begin{cases} x'''(t) - m^3 x(t) = y(t), \\ x^{(i)}(0) = x^{(i)}(\omega), \quad i = 0,1,2. \end{cases} \tag{2.21}$$

Then, solution of (2.20) is written as

$$y(t) = \int_0^\omega G_1(t,s)h(s)ds. \tag{2.22}$$

Solution of (2.21) is written as

$$x(t) = \int_0^\omega G_3(t,s)y(s)ds, \tag{2.23}$$

where

$$G_3(t,s) = \begin{cases} \dfrac{2\exp\left(\frac{1}{2}m(s-t)\right)\left[\sin\left(\frac{\sqrt{3}}{2}m(t-s)+\frac{\pi}{6}\right)-\exp\left(-\frac{1}{2}m\omega\right)\sin\left(\frac{\sqrt{3}}{2}m(t-s-\omega)+\frac{\pi}{6}\right)\right]}{3m^2\left(1+\exp(-m\omega)-2\exp\left(-\frac{m\omega}{2}\right)\cos\left(\frac{\sqrt{3}}{2}m\omega\right)\right)} \\ +\dfrac{\exp(m(t-s))}{3m^2(\exp(m\omega)-1)}, \quad 0 \le s \le t \le \omega, \\[2ex] \dfrac{2\exp\left(\frac{1}{2}m(s-t-\omega)\right)\left[\sin\left(\frac{\sqrt{3}}{2}m(t-s+\omega)+\frac{\pi}{6}\right)-\exp\left(-\frac{1}{2}m\omega\right)\sin\left(\frac{\sqrt{3}}{2}m(t-s)+\frac{\pi}{6}\right)\right]}{3m^2\left(1+\exp(-m\omega)-2\exp\left(-\frac{m\omega}{2}\right)\cos\left(\frac{\sqrt{3}}{2}m\omega\right)\right)} \\ +\dfrac{\exp(m(t+\omega-s))}{3m^2(\exp(m\omega)-1)}, \quad 0 \le t < s \le \omega. \end{cases}$$

By the following lemma, which can be found in [18], we will consider the sign of $G_3(t,s)$. Let

$$l=\frac{1}{3m^2(\exp(m\omega)-1)},\qquad L=\frac{3+2\exp\left(-\frac{m\omega}{2}\right)}{3m^2\left(1-\exp\left(-\frac{m\omega}{2}\right)\right)^2}.$$

Lemma 2.11. *Assume that $\sqrt{3}m\omega<\frac{4}{3}\pi$ holds. Then $0<l<G_3(t,s)\le L$ for all $t\in[0,\omega]$ and $s\in[0,\omega]$.*

Similarly to (2.19), we know that the solution of (2.1) can be written as

$$x(t)=\int_0^\omega G^2(t,s)h(s)ds, \tag{2.24}$$

where $G^2(t,s)=\int_0^\omega G_3(t,\tau)G_1(\tau,s)d\tau$. And we get the following Theorem.

Theorem 2.5. *Assume that $\sqrt{3}m\omega<\frac{4}{3}\pi$ holds. Then $G^2(t,s)\ge 0$ for all $(t,s)\in[0,\omega]\times[0,\omega]$.*

Case (III): There exist positive real constants m and ρ such that $a=\rho$, $b=0$, $c=m^3$, $d=\rho m^3$. Then, (2.1) is transformed into

$$\begin{cases} y'(t)+\rho y(t)=h(t),\\ y(0)=y(\omega),\end{cases} \tag{2.25}$$

and

$$\begin{cases} x'''(t)+m^3x(t)=y(t),\\ x^{(i)}(0)=x^{(i)}(\omega),\quad i=0,1,2.\end{cases} \tag{2.26}$$

Then, solution of (2.25) is written as

$$y(t)=\int_0^\omega G_1(t,s)h(s)ds. \tag{2.27}$$

Solution of (2.26) is written as

$$x(t)=\int_0^\omega G_4(t,s)y(s)ds, \tag{2.28}$$

where

$$G_4(t,s)=\begin{cases}\dfrac{2\exp\left(\frac12 m(t-s)\right)\left[\sin\left(\frac{\sqrt3}{2}m(t-s)-\frac{\pi}{6}\right)-\exp\left(\frac12 m\omega\right)\sin\left(\frac{\sqrt3}{2}m(t-s-\omega)-\frac{\pi}{6}\right)\right]}{3m^2\left(1+\exp(m\omega)-2\exp\left(\frac12 m\omega\right)\cos\left(\frac{\sqrt3}{2}m\omega\right)\right)}\\ +\dfrac{\exp(m(s-t))}{3\rho^2(1-\exp(-\rho\omega))},\quad 0\le s\le t\le\omega,\\ \\ \dfrac{2\exp\left(\frac12 m(t+\omega-s)\right)\left[\sin\left(\frac{\sqrt3}{2}m(t+\omega-s)-\frac{\pi}{6}\right)-\exp\left(\frac12 m\omega\right)\sin\left(\frac{\sqrt3}{2}m(t-s)-\frac{\pi}{6}\right)\right]}{3m^2\left(1+\exp(m\omega)-2\exp\left(\frac12 m\omega\right)\cos\left(\frac{\sqrt3}{2}m\omega\right)\right)}\\ +\dfrac{\exp(m(s-t-\omega))}{3m^2(1-\exp(-m\omega))},\quad 0\le t<s\le\omega.\end{cases}$$

By the following lemma, which can be found in [18], we will consider the sign of $G_4(t,s)$.

Lemma 2.12. *Assume that $\sqrt{3}m\omega < \frac{4}{3}\pi$ holds. Then $0 < l < G_4(t,s) \le L$ for all $t \in [0,\omega]$ and $s \in [0,\omega]$.*

Similarly to (2.19), we know that the solution of (2.1) can be written as

$$x(t) = \int_0^\omega G^3(t,s)h(s)ds, \tag{2.29}$$

where $G^3(t,s) = \int_0^\omega G_4(t,\tau)G_1(\tau,s)d\tau$. And we get the following Theorem.

Theorem 2.6. *Assume that $\sqrt{3}m\omega < \frac{4}{3}\pi$ holds. Then $G^3(t,s) \ge 0$ for all $(t,s) \in [0,\omega] \times [0,\omega]$.*

Case (IV): There exists a positive real constant ρ such that $a = -2\rho$, $b = 3\rho^3$, $c = -2\rho^3$, $d = \rho^4$. Then, (2.1) is transformed into

$$\begin{cases} y''(t) - \rho y'(t) + \rho^2 y(t) = h(t), \\ y^{(i)}(0) = y^{(i)}(\omega), \quad i = 1,2, \end{cases} \tag{2.30}$$

and

$$\begin{cases} x''(t) - \rho x'(t) + \rho^2 x(t) = y(t), \\ x^{(i)}(0) = x^{(i)}(\omega), \quad i = 0,1. \end{cases} \tag{2.31}$$

Then, solution of (2.30) is written as

$$y(t) = \int_0^\omega G_5(t,s)h(s)ds.$$

Solution of (2.31) is written as

$$x(t) = \int_0^\omega G_5(t,s)y(s)ds.$$

Lemma 2.13 (see [19]). *The boundary problem (2.30) is equivalent to integral equation*

$$y(t) = \int_0^\omega G_5(t,s)h(s)ds,$$

where

$$G_5(t,s) = \begin{cases} \dfrac{2e^{\frac{\rho}{2}(t-s)}\left[\sin\frac{\sqrt{3}}{2}\rho(\omega-t+s)+e^{-\frac{\rho\omega}{2}}\sin\frac{\sqrt{3}}{2}\rho(t-s)\right]}{\sqrt{3}\rho\left(e^{\frac{\rho\omega}{2}}+e^{-\frac{\rho\omega}{3}}-2\cos\frac{\sqrt{3}}{2}\rho\omega\right)}, & 0 \le s \le t \le \omega, \\ \dfrac{2e^{\frac{\rho}{2}(\omega+t-s)}\left[\sin\frac{\sqrt{3}}{2}\rho(s-t)+e^{-\frac{\rho\omega}{2}}\sin\frac{\sqrt{3}}{2}\rho(\omega-s+t)\right]}{\sqrt{3}\rho\left(e^{\frac{\rho\omega}{2}}+e^{-\frac{\rho\omega}{3}}-2\cos\frac{\sqrt{3}}{2}\rho\omega\right)}, & 0 \le t < s \le \omega. \end{cases}$$

Moreover, for $G_5(t,s)$, if $\rho < \frac{2\pi}{\sqrt{3}\omega}$, we have the estimates

$$0 \le l_2' := \frac{2\sin(\frac{\sqrt{3}}{2}\rho\omega)}{\sqrt{3}\rho(e^{\frac{\rho\omega}{2}}+1)^2} \le G_5(t,s) \le \frac{2}{\sqrt{3}\sin(\frac{\sqrt{3}}{2}\rho\omega)} := L_2', \qquad \forall\ s,t \in [0,\omega].$$

Similarly to (2.19), we know that the solution of (2.1) can be written as

$$x(t) = \int_0^\omega G^4(t,s)h(s)ds, \tag{2.32}$$

where $G^4(t,s) = \int_0^\omega G_5(t,\tau)G_5(\tau,s)d\tau$. And we get the following Theorem.

Theorem 2.7. *Assume that* $\rho < \frac{2\pi}{\sqrt{3}\omega}$ *holds. Then* $G^4(t,s) \geq 0$ *for all* $(t,s) \in [0,\omega] \times [0,\omega]$.

Case (V): There exists a positive real constant ρ such that $a = \rho$, $b = 0$, $c = \rho^3$, $d = \rho^4$. Then, (2.1) is transformed into

$$\begin{cases} u'(t) + \rho u(t) = h(t), \\ u(0) = u(\omega), \end{cases} \tag{2.33}$$

and

$$\begin{cases} y'(t) + \rho y(t) = u(t), \\ y(0) = y(\omega), \end{cases} \tag{2.34}$$

and

$$\begin{cases} x''(t) - \rho x'(t) + \rho^2 x(t) = y(t), \\ x^{(i)}(0) = x^{(i)}(\omega), \quad i = 0, 1. \end{cases} \tag{2.35}$$

Similarly to (2.19), we know that the solution of (2.1) can be written as

$$x(t) = \int_0^\omega G^5(t,s)h(s)ds, \tag{2.36}$$

where $G^5(t,s) = \int_0^\omega \int_0^\omega G_5(t,\tau_2)G_1(\tau_2,\tau_1)G_1(\tau_1,s)d\tau_1 d\tau_2$. And we get the following Theorem.

Theorem 2.8. *Assume that* $\rho < \frac{2\pi}{\sqrt{3}\omega}$ *holds. Then* $G^5(t,s) \geq 0$ *for all* $(t,s) \in [0,\omega] \times [0,\omega]$.

Case (VI): There exists a positive real constant ρ such that $a = 4\rho$, $b = 6\rho^2$, $c = 4\rho^3$, $d = \rho^4$. Then, (2.1) is transformed into

$$\begin{cases} u'(t) + \rho u(t) = h(t), \\ u(0) = u(\omega), \end{cases} \tag{2.37}$$

and

$$\begin{cases} v'(t) + \rho v(t) = u(t), \\ v(0) = v(\omega), \end{cases} \tag{2.38}$$

and

$$\begin{cases} y'(t) + \rho y(t) = v(t), \\ y(0) = y(\omega), \end{cases} \tag{2.39}$$

and

$$\begin{cases} x'(t) + \rho x(t) = y(t), \\ y(0) = y(\omega). \end{cases} \tag{2.40}$$

Similarly to (2.19), we know that the solution of (2.1) can be written as

$$x(t) = \int_0^\omega G^6(t,s)h(s)ds, \tag{2.41}$$

where $G^6(t,s) = \int_0^\omega \int_0^\omega \int_0^\omega G_1(t,\tau_3)G_1(\tau_3,\tau_2)G_1(\tau_2,\tau_1)G_1(\tau_1,s)d\tau_1 d\tau_2 d\tau_3$. And we get the following Theorem.

Theorem 2.9. $G^6(t,s) \geq 0$ *for all* $(t,s) \in [0,\omega] \times [0,\omega]$.

3. Preliminary Lemmas

Firstly, we establish the existence of positive periodic solutions for fourth-order differential equation (1.4) by using fixed point theorem, which can be found in [13].

Lemma 3.1 ([13]). *Let X be a Banach space and K a cone in X. Assume that $\Omega_1,\ \Omega_2$ are bounded open subsets of X with $0 \in \Omega_1,\ \bar{\Omega}_1 \subset \Omega_2$, and let*

$$T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

be completely continuous operator such that either

(i) $\|Tu\| \geq \|u\|,\ u \in K \cap \partial\Omega_1$ *and* $||Tu|| \leq ||u||, u \in K \cap \partial\Omega_2$; *or*

(ii) $\|Tu\| \leq \|u\|,\ u \in K \cap \partial\Omega_1$ *and* $||Tu|| \geq ||u||, u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

For the sake of convenience, we list the following assumptions which will be used repeatedly in the sequel:

(H_1) $f(x)$ is a scalar continuous function defined for $|x| > 0$, and $f(x) > 0$ for $|x| > 0$.

(H_2) $g(t) \geq 0,\ t \in [0, \omega],\ \int_0^\omega g(t)dt > 0$.

(H_3) $g(t) > 0$ for $t \in [0, \omega]$.

Case (I): There exist real constants $\alpha,\ \beta,\ \gamma$ and $\rho > 0$ such that $a = \alpha + \rho$, $b = \beta + \alpha\rho$, $c = \gamma + \beta\rho$, $d = \gamma\rho$. The following are the main existence results in this section.

Under Theorems 2.1-2.4, we always denote

$$m_{1i} = \min_{0 \leq s,t \leq \omega} G^{1i}(t, s),\ M_{1i} = \max_{0 \leq s,t \leq \omega} G^{1i}(t, s).\ \sigma_{1i} = m_{1i}/M_{1i},\ 1 = 1, 2, 3, 4.$$

Obviously, $M_{1i} > m_{1i} > 0$ and $0 < \sigma_{1i} < 1$.

Case (i): $\lambda_1 \neq \lambda_2 \neq \lambda_3$, and $\lambda_1,\ \lambda_2,\ \lambda_3 \in R$.

Define the cone K in X by

$$K = \{x \in X : x(t) \geq 0 \ \text{ for all } \ t \ \in [0, \omega] \ \text{ and } \ \min_{t \in R} x(t) \geq \sigma_{11}||x||\}.$$

We take $X = C_\omega$ with $||x|| = \max_t |x(t)|$. Also, for $r > 0$, let

$$\Omega_r = \{x \in K : ||x|| < r\}.$$

Define the operator $T : K \setminus \{0\} \to X$

$$(T_\mu x)(t) = \mu \int_0^\omega G^{11}(t, s)(g(s)f(x(s)) + e(s))ds. \tag{3.1}$$

When e is nonnegative, $g(s)f(x(s))+e(s)$ is nonnegative. If e takes negative values, we will choose $x(s)$ so that $g(s)f(x(s))+e(s)$ is nonnegative. This is possible because $\lim_{x\to 0} f(x) = \infty$ or $\lim_{|x|\to\infty} f(x) = \infty$.

Now if x is a fixed point of T_μ in $K \setminus \{0\}$, then x is a positive solution of (1.4). Also note that each component $x(t)$ of any nonnegative periodic solution x is strictly positive for all t because of the positiveness of the Green functions and assumptions (H_1) and (H_2). We now look at several properties of the operator.

Lemma 3.2. *Assume that $\lambda_3 < \lambda_2 < \lambda_1 < 0$ (or $\lambda_1 > \lambda_2 > 0, \quad \lambda_3 < 0$), $p_2 > q_2$, (H_1), (H_2) hold and $e(t) \geq 0,\ t \in [0, \omega]$. Then $T_\mu(K \setminus \{0\}) \subset K$ and $T_\mu : K \setminus \{0\} \to K$ is completely continuous.*

Proof. If $x \in K \setminus \{0\}$, then $\min\limits_{t\in[0,\omega]} |x(t)| \geq \sigma_{11}||x|| > 0$, and then T_μ is defined. Now we have that,

$$\begin{aligned}
\min_{t\in[0,\omega]} T_\mu x(t) &\geq m_{11}\mu \int_0^\omega (g(s)f(x(s)) + e(s))ds \\
&= \mu\sigma_{11}M_{11} \int_0^\omega (g(s)f(x(s)) + e(s))ds \\
&\geq \sigma_{11} \sup_{t\in[0,\omega]} T_\mu x(t) \\
&= \sigma_{11}||T_\mu x||.
\end{aligned}$$

Thus, $T_\mu(K \setminus \{0\}) \subset K$. It is easy to verify that T_μ is completely continuous. □

If $e(t)$ takes negative values, we need to choose appropriate domains so that $g(s)f(x(s)) + e(s)$ become nonnegative. The proof of $T_\mu(K \setminus \{0\}) \subset K$ and $T_\mu(K \setminus \Omega_R) \subset K$ in Lemma 3.3 is the same as in Lemma 3.2.

Lemma 3.3. *Assume that $\lambda_3 < \lambda_2 < \lambda_1 < 0$ (or $\lambda_1 > \lambda_2 > 0, \quad \lambda_3 < 0$), $p_2 > q_2$, (H_1), (H_3) hold.*

(a) If $\lim\limits_{x\to 0} f(x) = \infty$, there is a $\delta > 0$ such that if $0 < r < \delta$, then T_μ is defined on $\bar{\Omega}_r \setminus \{0\}$, $T_\mu(\bar{\Omega}_r \setminus \{0\}) \subset K$, and $T_\mu : \bar{\Omega}_r \setminus \{0\} \to K$ is completely continuous.

(b) If $\lim\limits_{x\to\infty} f(x) = \infty$, there is a $\triangle > 0$ such that if $R > \triangle$, then T_μ is defined on $K \setminus \Omega_R$, $T_\mu(K \setminus \Omega_R) \subset K$ and $T_\mu : K \setminus \Omega_R \to K$ is completely continuous.

Proof. We split $g(t)f(x(t))+e(t)$ into the two terms $\frac{1}{2}g(t)f(x(t)$ and $\frac{1}{2}g(t)f(x(t))+e(t)$. The first term is always nonnegative and used to carry out the estimates of the operator in the lemma and corollaries in this section. We will make the second term $\frac{1}{2}g(t)f(x(t)) + e(t)$ nonnegative by choosing appropriate domains of f. The choice of the even split of $g(t)f(x(t))$ here is not necessarily optimal in terms of obtaining maximal μ-intervals for the existence of periodic solutions of the equation.

Noting that $g(t)$ is positive on $[0, \omega]$, $\lim\limits_{x\to 0} f(x) = \infty$, implies that there exists a constant $\delta > 0$ such that

$$f(x) \geq 2\frac{\max\limits_{t\in[0,\omega]}\{|e(t)| + 1\}}{\min\limits_{t\in[0,\omega]} g(t)},$$

for $0 < |x| < \delta$. Now for $x \in \bar{\Omega}_r \setminus \{0\}$ and $0 < r < \delta$, noting that

$$\delta > r \geq |x(t)| \geq \min_{t\in[0,\omega]} |x(t)| \geq \sigma_1||x|| > 0, \qquad t \in [0, \omega],$$

and therefore, we have, for $t \in [0, \omega]$,

$$\begin{aligned} g(t)f(x(t)) + e(t) &\geq \frac{1}{2}g(t)f(x(t)) + e(t) \\ &\geq g(t)\frac{\max\limits_{t\in[0,\omega]}\{|e(t)|+1\}}{\min\limits_{t\in[0,\omega]} g(t)} + e(t) \\ &> 0. \end{aligned}$$

Thus, it is clear that $T_\mu x(t)$ in (3.1) is well defined and positive, and now it is easy to see that $T_\mu(\bar{\Omega}_r \setminus \{0\} \subset K$ and $T_\mu : \bar{\Omega}_r \setminus \{0\} \to K$ is completely continuous.

On that other hand, if $\lim\limits_{x\to\infty} f(x) = \infty$, there is an $R'' > 0$ such that

$$f(x) \geq 2\frac{\max\limits_{t\in[0,\omega]}\{|e(t)|+1\}}{\min\limits_{t\in[0,\omega]} g(t)},$$

for $|x| \geq R''$. Now let $\triangle = \frac{R''}{\sigma_{11}}$. Then for $x \in K \setminus \Omega_R$, $R > \triangle$, we have that $\min\limits_{t\in[0,\omega]} x(t) \geq \sigma_{11}||x|| > R''$, and therefore,

$$g(t)f(x(t)) + e(t) \geq \frac{1}{2}g(t)f(x(t)) + e(t) > 0, \qquad t \in [0, \omega].$$

Now $T_\mu x(t)$ in (3.1) is well defined and positive. It is clear that $T_\mu(K \setminus \Omega_R) \subset K$ and $T_\mu : K \setminus \Omega_R \to K$ is completely continuous. □

Now let

$$\Gamma = \min\{\frac{1}{2}m_{11}\sigma_{11}\int_0^\omega g(s)ds\} > 0.$$

Lemma 3.4. *Assume that $\lambda_3 < \lambda_2 < \lambda_1 < 0$ (or $\lambda_1 > \lambda_2 > 0$, $\lambda_3 < 0$), $p_2 > q_2$, (H_1), (H_2) hold and $e(t) \geq 0$, $t \in [0, \omega]$. Let $r > 0$ and if there exists $\eta > 0$ such that*

$$f(x(t)) \geq \eta x(t) \qquad \textit{for} \;\; t \in [0, \omega],$$

for $x(t) \in \partial\Omega_r$, then the following inequality holds,

$$||T_\mu x|| \geq \mu\Gamma\eta||x||.$$

Proof. From the definition of $T_\mu x$ it follows that

$$\begin{aligned} ||T_\mu x|| &\geq \max_{t\in[0,\omega]} T_\mu x(t) \\ &\geq \frac{1}{2}\mu m_{11}\int_0^\omega g(s)f(x(s))ds \\ &\geq \frac{1}{2}\mu m_{11}\int_0^\omega g(s)\eta x(s)ds \\ &\geq \frac{1}{2}\mu m_{11}\sigma_{11}\int_0^\omega g(s)ds\eta||x|| \\ &= \mu\Gamma\eta||x||. \end{aligned}$$

□

If $e(t)$ takes negative values, we need to adjust δ and $\triangle$ in Lemma 3.3 to guarantee that $g(t)f(x(t)) + e(t)$ is nonnegative.

Corollary 3.1. *Assume that* $\lambda_3 < \lambda_2 < \lambda_1 < 0$ *(or* $\lambda_1 > \lambda_2 > 0, \quad \lambda_3 < 0$*),* $p_2 > q_2$*,* (H_1)*,* (H_3) *hold.*

(a) If $\lim_{x\to 0} f(x) = \infty$*, then Lemma 3.4 is true if, in addition,* $0 < r < \delta$*, where* δ *is defined in Lemma 3.3.*

(b) If $\lim_{x\to\infty} f(x) = \infty$*, then Lemma 3.4 is true if, in addition,* $\triangle > 0$*, where* $\triangle$ *is defined in Lemma 3.3.*

Proof. We split $g(t)f(x(t))+e(t)$ into the two terms $\frac{1}{2}g(t)f(x(t))$ and $\frac{1}{2}g(t)f(x(t))+e(t)$. By choosing δ and $\triangle$ in Lemma 3.3, $g(t)f(x(t))+e(t)$ become nonnegative. The estimate in Corollary 3.1 can be carried out by the first terms as in Lemma 3.4. □

Let $\hat{f}(\theta) : [1, \infty) \to \mathbb{R}_+$ be the function given by

$$\hat{f}(\theta) = \max\{f(u) : u \in \mathbb{R}_+ \ \text{ and } \ 1 \le |u| \le \theta\}.$$

It is easy to see that $\hat{f}(\theta)$ is a nondecreasing function on $[1, \infty)$. The following lemma is essentially the same as Lemma 2.8 in [22].

Lemma 3.5 (see [22])**.** *Assume* (H_1) *holds. If* $\lim_{|x|\to\infty} \frac{f(x)}{|x|}$ *exists (which can be infinity), then* $\lim_{\theta\to\infty} \frac{\hat{f}(\theta)}{\theta}$ *exists and* $\lim_{\theta\to\infty} \frac{\hat{f}(\theta)}{\theta} = \lim_{|x|\to\infty} \frac{f(x)}{|x|}$.

Lemma 3.6. *Assume that* $\lambda_3 < \lambda_2 < \lambda_1 < 0$ *(or* $\lambda_1 > \lambda_2 > 0, \quad \lambda_3 < 0$*),* $p_2 > q_2$*,* (H_1)*,* (H_2) *hold and* $e(t) \ge 0, \ t \in [0, \omega]$*. Let* $r > \max\{\frac{1}{\sigma_{11}}, 2\mu M_{11} \int_0^\omega |e(s)|ds\}$ *and if there exists an* $\varepsilon > 0$ *such that*

$$\hat{f}(r) \le \varepsilon r,$$

then

$$||T_\mu x|| \le \mu \widehat{C} \varepsilon ||x|| + \frac{1}{2}||x|| \quad \text{for } \ x \in \partial\Omega_r,$$

where the constant $\widehat{C} = M_{11} \int_0^\omega g(s)ds$.

Proof. From the definition of T_μ, we have for $x \in \partial\Omega_r$,

$$\begin{aligned} ||T_\mu x|| &= \max_{t\in[0,\omega]} T_\mu x(t) \\ &\le \mu M_{11} \int_0^\omega g(s)f(x(s))ds + \mu M_{11} \int_0^\omega |e(s)|ds \\ &\le \mu M_{11} \int_0^\omega g(s)\hat{f}(r)ds + \frac{r}{2} \\ &\le \mu M_{11} \int_0^\omega g(s)dsr\varepsilon + \frac{r}{2} \\ &= \mu \widehat{C} \varepsilon ||x|| + \frac{1}{2}||x||. \end{aligned}$$

□

If $e(t)$ takes negative values, we need to restrict the domain of T_μ to guarantee that $g(t)f(x(t)) + e(t)$ is nonnegative.

Corollary 3.2. *Assume that* $\lambda_3 < \lambda_2 < \lambda_1 < 0$ *(or* $\lambda_1 > \lambda_2 > 0, \ \ \lambda_3 < 0$*),* $p_2 > q_2$*,* (H_1)*,* (H_3) *hold. If* $\lim_{x\to\infty} f(x) = \infty$*, Lemma 3.6 is true if, in addition,* $r > \triangle$*, where* $\triangle$ *is defined in Lemma 3.3.*

Proof. If we choose $\triangle$ defined in Lemma 3.3, then T_μ is well defined and $g(t)f(x(t)) + e(t)$ is nonnegative, and Corollary 3.2 can be shown in the same way as Lemma 3.6. □

The conclusions of Lemma 3.4 and 3.6 are based on the inequality assumptions between $f(x)$ and x. If these assumption are not necessarily true, we will have the following results.

Lemma 3.7. *Assume that* $\lambda_3 < \lambda_2 < \lambda_1 < 0$ *(or* $\lambda_1 > \lambda_2 > 0, \ \ \lambda_3 < 0$*),* $p_2 > q_2$*,* (H_1)*,* (H_2) *hold and* $e(t) \geq 0, \ t \in [0, \omega]$*. Let* $r > 0$*. Then*

$$||T_\mu x|| \geq \mu \frac{m_{11}\hat{m}_{r11}}{2} \int_0^\omega g(s)ds,$$

for all $x \in \partial\Omega_r$*, where* $\hat{m}_{r11} = \min\{f(x) : x \in \mathbb{R}_+$ *and* $\sigma_{11} r \leq |x| \leq r\} > 0$*.*

Proof. If $x(t) \in \partial\Omega_r$, then $\sigma_{11} r \leq |x(t)| \leq r$, for $t \in [0, \omega]$. Therefore $f(x(t)) \geq \hat{m}_{r1}$ for $t \in [0, \omega]$. By the definition of T_μ, we have

$$\begin{aligned} ||T_\mu x|| &= \max_{t\in[0,\omega]} T_\mu x(t) \\ &\geq \frac{1}{2}\mu m_{11} \int_0^\omega g(s)f(x(s))ds \\ &\geq \mu \frac{m_{11}\hat{m}_{r11}}{2} \int_0^\omega g(s)ds. \end{aligned}$$

□

Now we consider the cases that $e(t)$ may take negative values. We need to restrict the domain of T_μ to guarantee that $g(t)f(x(t)) + e(t)$ is nonnegative. $\frac{1}{2}g(t)f(x(t))$ is used to carry out the estimates is Lemma 3.7.

Corollary 3.3. *Assume that* $\lambda_3 < \lambda_2 < \lambda_1 < 0$ *(or* $\lambda_1 > \lambda_2 > 0, \ \ \lambda_3 < 0$*),* $p_2 > q_2$*,* (H_1)*,* (H_3) *hold.*

(a) If $\lim_{x\to 0} f(x) = \infty$*, then Lemma 3.7 is true if, in addition,* $0 < r < \delta$*, where* $\delta > 0$ *is defined in Lemma 3.3.*

(b) If $\lim_{x\to\infty} f(x) = \infty$*, then Lemma 3.7 is true if, in addition,* $r > \triangle$*, where* $\triangle$ *is defined in Lemma 3.3.*

Proof. By selecting δ and $\triangle$ defined in Lemma 3.3, T_μ is well defined and $g(t)f(x(t)) + e(t)$ is nonnegative, and then Corollary 3.3 can be shown as Lemma 3.7. □

Lemma 3.8. *Assume that* $\lambda_3 < \lambda_2 < \lambda_1 < 0$ *(or* $\lambda_1 > \lambda_2 > 0, \ \ \lambda_3 < 0$*),* $p_2 > q_2$*,* (H_1)*,* (H_2) *hold and* $e(t) \geq 0, \ t \in [0, \omega]$*. Let* $r > 0$*. Then*

$$||T_\mu x|| \leq \mu \left(M_{11} \int_0^\omega g(s)\hat{M}_{r11} ds + M_{11} \int_0^\omega |e(s)|ds \right),$$

for all $x \in \partial\Omega_r$*, where* $\hat{M}_{r11} = \max\{f(x) : x \in \mathbb{R}_+$ *and* $\sigma_{11} r \leq |x| \leq r\} > 0$*.*

Proof. If $x \in \partial\Omega_r$, then $\sigma_{11} r \leq |x(t)| \leq r,\ t \in [0,\omega]$. Therefore $f(x(t)) \leq \hat{M}_{r11}$ for $t \in [0,\omega]$. Thus we have that

$$\begin{aligned}
||T_\mu x|| &= \max_{t\in[0,\omega]} T_\mu x(t) \\
&\leq \mu M_{11} \int_0^\omega g(s) f(x(s)) ds + \mu M_{11} \int_0^\omega e(s) ds \\
&\leq \mu M_{11} \int_0^\omega g(s) f(x(s)) ds + \mu M_{11} \int_0^\omega |e(s)| ds \\
&\leq \mu M_{11} \int_0^\omega g(s) \hat{M}_{r11} ds + \mu M_{11} \int_0^\omega |e(s)| ds \\
&= \mu \left(M_{11} \int_0^\omega g(s) \hat{M}_{r11} ds + M_{11} \int_0^\omega |e(s)| ds \right).
\end{aligned}$$

□

Again, if $e(t)$ takes negative values, we need to restrict r and R to guarantee $g(t)f(x(t)) + e(t)$ is nonnegative.

Corollary 3.4. *Assume that* $\lambda_3 < \lambda_2 < \lambda_1 < 0$ *(or* $\lambda_1 > \lambda_2 > 0,\ \ \lambda_3 < 0$*),* $p_2 > q_2$*,* (H_1)*,* (H_3) *hold.*

- *(a) If* $\lim_{x\to 0} f(x) = \infty$*, then Lemma 3.8 is true if, in addition,* $0 < r < \delta$*, where* $\delta > 0$ *is defined in Lemma 3.3.*
- *(b) If* $\lim_{x\to\infty} f(x) = \infty$*, then Lemma 3.8 is true if, in addition,* $r > \triangle$*, where* $\triangle$ *is defined in Lemma 3.3.*

Proof. By selecting δ and $\triangle$ defined in Lemma 3.3, T_μ is well defined and $g(t)f(x(t)) + e(t)$ is nonnegative, and then Corollary 3.4 can be shown as Lemma 3.8. □

4. Main Results

In this section, we present out main results for the existence and multiplicity of positive periodic solutions of singular fourth-order equation of repulsive type (1.4). We state our theorems as follows.

Theorem 4.1. *Let* $\lambda_3 < \lambda_2 < \lambda_1 < 0$ *(or* $\lambda_1 > \lambda_2 > 0,\ \ \lambda_3 < 0$*),* $p_2 > q_2$*,* (H_1)*,* (H_2) *hold and* $e(t) \geq 0,\ t \in [0,\omega]$*. Assume that* $\lim_{x\to 0} f(x) = \infty$*.*

- *(a) If* $\lim_{|x|\to\infty} \frac{f(x)}{|x|} = 0$*, then, for all* $\mu > 0$*, (1.4) has a positive periodic solution.*
- *(b) If* $\lim_{|x|\to\infty} \frac{f(x)}{|x|} = \infty$*, then, for all sufficiently small* $\mu > 0$*, (1.4) has two positive periodic solutions.*
- *(c) There exists a* μ_1 *such that (1.4) has a positive periodic solution for* $0 < \mu < \mu_1$*.*

Proof. (a) Since $e(t) \geq 0$, T_μ is defined on $K \setminus \{0\}$ and $g(t)f(x(t)) + e(t)$ is nonnegative. Noting $\lim_{|x|\to\infty} \frac{f(x)}{|x|} = 0$, it follows from Lemma 3.5 that $\lim_{\theta\to\infty} \frac{\hat{f}(\theta)}{\theta} = 0$.

Therefore, we can choose $r_1 > \max\{\frac{1}{\sigma_{11}}, 2\mu M_{11}\int_0^\omega |e(s)|ds\}$ so that $\hat{f}(r_1) \le \varepsilon r_1$, where the constant $\varepsilon > 0$ satisfies

$$\mu\hat{C}\varepsilon < \frac{1}{2},$$

and $\widehat{C}$ is the positive constant defined in Lemma 3.6. We have by Lemma 3.6 that

$$||T_\mu x|| \le \left(\mu\widehat{C}\varepsilon + \frac{1}{2}\right)||x|| < ||x|| \qquad \text{for } x \in \partial\Omega_{r_1}.$$

On the other hand, by the condition $\lim_{x\to 0} f(x) = \infty$, there is a positive number $r_2 < r_1$ such that

$$f(x) \ge \eta|x|,$$

for $x \in \mathbb{R}_+ \setminus \{0\}$ and $|x| \le r_2$, where $\eta > 0$ is chosen so that

$$\mu\Gamma\eta > 1.$$

It is easy to see that, for $x \in \partial\Omega_{r_2},\ t \in [0,\omega]$,

$$f(x) \ge \eta x(t).$$

Lemma 3.4 implies that

$$||T_\mu x|| \ge \mu\Gamma\eta||x|| > ||x|| \qquad \text{for } x \in \partial\Omega_{r_2}.$$

By Lemma 3.1, T_μ has a fixed point $x \in \bar{\Omega}_{r_1} \setminus \Omega_{r_2}$. The fixed point $x \in \bar{\Omega}_{r_1} \setminus \Omega_{r_2}$ is the desired positive periodic solution of (1.4).

(b) Again since $e(t) \ge 0$, T_μ is defined on $K \setminus \{0\}$ and $g(t)f(x(t)) + e(t)$ is nonnegative. Fix two numbers $0 < r_3 < r_4$, there exists a $\mu_0 > 0$ such that

$$\mu_0 < \frac{r_3}{M_{11}\int_0^\omega g(s)\hat{M}_{r_{31}}ds + M_{11}\int_0^\omega |e(s)|ds},$$

and

$$\mu_0 < \frac{r_4}{M_{11}\int_0^\omega g(s)\hat{M}_{r_{41}}ds + M_{11}\int_0^\omega |e(s)|ds},$$

where $\hat{M}_{r_{31}}$ and $\hat{M}_{r_{41}}$ are defined in Lemma 3.8 implies that, for $0 < \mu < \mu_0$,

$$|T_\mu x|| < ||x|| \qquad \text{for } x \in \partial\Omega_{r_j} \quad (j = 3, 4).$$

On the other hand, in view of the assumptions $\lim_{x\to\infty} \frac{f(x)}{|x|} = \infty$ and $\lim_{x\to 0} f(x) = \infty$, there are positive numbers $0 < r_2 < r_3 < r_4 < r_1'$ such that

$$f(x) \ge \eta|x|$$

for $x \in \mathbb{R}_+$ and $0 < |x| \le r_2$ or $|x| > r_1'$ where $\eta > 0$ is chosen so that

$$\mu\Gamma\eta > 1.$$

Thus if $x \in \partial\Omega_{r_2}$, then

$$f(x(t)) \ge \eta x(t), \qquad t \in [0,\omega].$$

Let $r_1 = \max\{2r_4, \frac{1}{\sigma_{11}} r'_1\}$. If $x \in \partial\Omega_{r_1}$, then

$$\min_{t\in[0,\omega]} x(t) \geq \sigma_{11}||x|| = \sigma_{11} r_1 \geq r'_1,$$

which implies that

$$f(x(t)) \geq \eta x(t) \quad \text{ for } t \in [0, \omega].$$

Thus Lemma 3.4 implies that

$$||T_\mu x|| \geq \mu\Gamma\eta||x|| > ||x|| \quad \text{ for } x \in \partial\Omega_{r_1},$$

and

$$||T_\mu x|| \geq \mu\Gamma\eta||x|| > ||x|| \quad \text{ for } x \in \partial\Omega_{r_2}.$$

It follows from Lemma 3.1, that T_μ has two fixed points $x_1(t)$ and $x_2(t)$ such that $x_1(t) \in \bar{\Omega}_{r_3} \setminus \Omega_{r_2}$ and $x_2(t) \in \bar{\Omega}_{r_1} \setminus \Omega_{r_4}$, which are the desired distinct positive periodic solutions of (1.4) for $\mu < \mu_0$ satisfying

$$r_2 < ||x_1|| < r_3 < r_4 < ||x_2|| < r_1.$$

(c) First we note that T_μ is defined on $K\setminus\{0\}$ and $g(t)f(x(t))+e(t)$ is nonnegative since $e(t) \geq 0$. Fix a number $r_3 > 0$. Lemma 3.8 implies that there exists a $\mu_1 > 0$ such that we have, for $0 < \mu < \mu_1$,

$$||T_\mu x|| < ||x|| \quad \text{ for } x \in \partial\Omega_{r_3}.$$

On the other hand, in view of the assumption $\lim_{x\to 0} f(x) = \infty$, there is a positive number $0 < r_2 < r_3$ such that

$$f(x) \geq \eta|x|,$$

for $x \in \mathbb{R}_+$ and $0 < |x| \leq r_2$ where $\eta > 0$ is chosen so that

$$\mu\Gamma\eta > 1.$$

Thus if $x \in \partial\Omega_{r_2}$, then

$$f(x(t)) \geq \eta x(t), \quad t \in [0, \omega].$$

Thus Lemma 3.4 implies that

$$||T_\mu x|| \geq \mu\Gamma\eta||x|| > ||x|| \quad \text{for } x \in \partial\Omega_{r_2}.$$

Lemma 3.1 implies that T_μ has a fixed point $x \in \bar{\Omega}_{r_3} \setminus \Omega_{r_2}$. The fixed point $x \in \bar{\Omega}_{r_3} \setminus \Omega_{r_2}$ is the desired positive periodic solution of (1.4). □

When $e(t)$ takes negative values, we give the following theorem.

Theorem 4.2. *Let* $\lambda_3 < \lambda_2 < \lambda_1 < 0$ *(or* $\lambda_1 > \lambda_2 > 0, \quad \lambda_3 < 0$*),* $p_2 > q_2$, (H_1), (H_3) *hold. Assume that* $\lim_{x\to 0} f(x) = \infty$.

(a) If $\lim_{|x|\to\infty} f(x) = \infty$ *and* $\lim_{|x|\to\infty} \frac{f(x)}{|x|} = 0$, *then there exists* $\mu_0 > 0$ *such that* (1.4) *has a positive periodic solution for* $\mu > \mu_0$.

(b) If $\lim_{|x|\to\infty} \frac{f(x)}{|x|} = \infty$, *then, for all sufficiently small* $\mu > 0$, (1.4) *has two positive periodic solutions.*

(c) There exists a $\mu_1 > 0$ *such that* (1.4) *has a positive periodic solution for* $0 < \mu < \mu_1$.

Proof. (a) Since $\lim_{|x|\to\infty} f(x) = \infty$, By Lemma 3.3, there is a $\triangle > 0$ such that if $R > \triangle$, then $g(t)f(x(t)) + e(t)$ is nonnegative and $T_\mu : K \setminus \Omega_R \to K$ is defined. Now for a fixed number $r_1 > \triangle$, Corollary 3.3 implies that there exists a $\mu_0 > 0$ such that, for $\mu > \mu_0$.

$$||T_\mu x|| > ||x|| \qquad \text{for } x \in \partial\Omega_{r_1}.$$

On the other hand, since $\lim_{|x|\to\infty} \frac{f(x)}{|x|} = 0$, it follows Lemma 3.5 that $\lim_{\theta\to\infty} \frac{\hat{f}(\theta)}{\theta} = 0$. Therefore, we can choose

$$r_2 > \max\left\{2r_1, \frac{1}{\sigma_{11}}, 2\mu M_{11}\int_0^\omega |e(s)|ds\right\} > \triangle,$$

so that $\hat{f}(r_2) \le \varepsilon r_2$, where the constant $\varepsilon > 0$ satisfies

$$\mu\hat{C}\varepsilon < \frac{1}{2}.$$

We have, by Corollary 3.2, that

$$||T_\mu x|| \le \left(\mu\hat{C}\varepsilon + \frac{1}{2}\right)||x|| < ||x|| \qquad \text{for } x \in \partial\Omega_{r_2}.$$

By Lemma 3.1, T_μ has a fixed point $x \in \bar{\Omega}_{r_2} \setminus \Omega_{r_1}$. The fixed point $x \in \bar{\Omega}_{r_2} \setminus \Omega_{r_1}$ is the desired positive periodic solution of (1.4).

(b) First, since $\lim_{x\to 0} f(x) = \infty$, by Lemma 3.3, there is $\delta > 0$ such that if $0 < r < \delta$, T_μ is defined on $\hat{\Omega} \setminus \{0\}$ and $g(t)f(x(t)) + e(t)$ is nonnegative. Furthermore, $T_\mu(\hat{\Omega}_r \setminus \{0\}) \subset K$. Now for a fixed number $r_1 < \delta$, Corollary 3.4 implies that there exists a $\mu_1 > 0$ such that we have, for $\mu < \mu_1$,

$$||T_\mu x|| < ||x|| \qquad \text{for } x \in \partial\Omega_{r_1}.$$

In view of the assumption $\lim_{x\to 0} f(x) = \infty$, there is a positive number $0 < r_3 < r_1$ such that

$$f(x) \ge \eta|x|$$

for $x \in \mathbb{R}_+$ and $0 < |x| \le r_3$ where $\eta > 0$ is chosen so that

$$\mu\Gamma\eta > 1.$$

Thus if $x \in \partial\Omega_{r_3}$, then

$$f(x(t)) \ge \eta x(t), \qquad t \in [0, \omega].$$

Thus Corollary 3.1 implies that

$$||T_\mu x|| \ge \mu\Gamma\eta||x|| > ||x|| \qquad \text{for } x \in \partial\Omega_{r_3}.$$

It follows from Lemma 3.1, T_μ has a fixed point $x_1(t) \in \bar{\Omega}_{r_1} \setminus \Omega_{r_3}$ which is a positive periodic solutions of (1.4) for $\mu < \mu_1$ satisfying

$$r_3 < ||x_1|| < r_1.$$

On the other hand, since $\lim\limits_{|x|\to\infty} \frac{f(x)}{|x|} = \infty$, by Lemma 3.3, there is $\triangle > 0$ such that if $R > \triangle$, T_μ is defined on $K \setminus \Omega_R$ and $g(t)f(x(t)) + e(t)$ is nonnegative. Furthermore, $T_\mu(K \setminus \Omega_R) \subset K$. For a fixed number $r_2 > \max\{\triangle, r_1\}$, and Corollary 3.4 implies that there exists a $0 < \mu_0 < \mu_1$ such that we have, for $\mu < \mu_0$,

$$||T_\mu x|| < ||x|| \quad \text{for } x \in \partial\Omega_{r_2}.$$

Since $\lim\limits_{|x|\to\infty} \frac{f(x)}{|x|} = \infty$, there is a positive number r' such that

$$f(x) \geq \eta|x|$$

for $x \in \mathbb{R}_+$ and $|x| \geq r'$ where $\eta > 0$ is chosen so that

$$\mu\Gamma\eta > 1.$$

Let $r_4 = \max\{2r_2, \frac{1}{\sigma_{11}} r'\} > \triangle$. If $x \in \partial\Omega_{r_4}$, then

$$\min_{t\in[0,\omega]} x(t) \geq \sigma_{11}||x|| = \sigma_{11} r_4 \geq r',$$

which implies that

$$f(x(t)) \geq \eta x(t) \quad \text{for } t \in [0, \omega].$$

Again Corollary 3.1 implies that

$$||T_\mu x|| \geq \mu\Gamma\eta||x|| > ||x|| \qquad \text{for } x \in \partial\Omega_{r_4}.$$

It follows from Lemma 3.1, T_μ has a fixed point $x_2(t) \in \bar{\Omega}_{r_4} \setminus \Omega_{r_2}$, which is a positive periodic solutions of (1.4) for $\mu < \mu_0$ satisfying

$$r_2 < ||x_2|| < r_4.$$

Noting that

$$r_3 < ||x_1|| < r_1 < r_2 < ||x_2|| < r_4,$$

we can conclude that x_1 and x_2 are the desired distinct positive solutions of (1.4) for $\mu < \mu_0$.

(c) Since $\lim\limits_{x\to 0} f(x) = \infty$, by Lemma 3.3, there is a $\delta > 0$ such that if $0 < r < \delta$, then T_μ is defined and $g(t)f(x(t)) + e(t)$ is nonnegative. Now for a fixed number $r_1 < \delta$, Corollary 3.4 implies that there exists a $\mu_1 > 0$ such that we have, for $\mu < \mu_1$,

$$||T_\mu x|| < ||x|| \qquad for\ x \in \partial\Omega_{r_1}.$$

On the other hand, in view of the assumption $\lim\limits_{x\to 0} f(x) = \infty$, there is a positive number $0 < r_2 < r_1 < \delta$ such that

$$f(x) \geq \eta|x|$$

for $x \in \mathbb{R}_+$ and $0 < |x| \le r_2$, where $\eta > 0$ is chosen so that

$$\mu\Gamma\eta > 1.$$

Thus if $x \in \partial\Omega_{r_2}$, then

$$f(x(t)) \ge \eta x(t), \qquad t \in [0, \omega].$$

Thus Corollary 3.1 implies that

$$||T_\mu x|| \ge \mu\Gamma\eta||x|| > ||x|| \qquad \text{for } x \in \partial\Omega_{r_2}.$$

Lemma 3.1 implies that T_μ has a fixed point $x_1 \in \bar{\Omega}_{r_1} \setminus \Omega_{r_2}$. The fixed point $x_1 \in \bar{\Omega}_{r_1} \setminus \Omega_{r_2}$ is the desired positive periodic solution of (1.4). □

Case (I):** $\lambda_1 = \lambda_2 \ne \lambda_3,\ \lambda_1,\ \lambda_2,\ \lambda_3 \in R$.

In this case, replacing above assumptions $\lambda_3 < \lambda_2 < \lambda_1 < 0$ (or $\lambda_1 > \lambda_2 > 0,\ \ \lambda_3 < 0$), and $p_2 > q_2$ by assumption $\lambda_1 > 0, \lambda_3 < 0$ (or $\lambda_1 < \lambda_3 < 0,\ \ p_3 > 1$), we can get similar existence results which we omit here.

Case (I*):** $\lambda_1 = \lambda_2 = \lambda_3 = \lambda \in R$.

In this case, replacing above assumptions $\lambda_3 < \lambda_2 < \lambda_1 < 0$ (or $\lambda_1 > \lambda_2 > 0,\ \ \lambda_3 < 0$), and $p_2 > q_2$ by assumption $\lambda < 0$, we can get similar existence results which we omit here.

Case (I**):** $\lambda_1 = \alpha + i\beta,\ \lambda_2 = \alpha - i\beta,\ \lambda_3 = \lambda,\ \alpha,\ \beta,\ \lambda \in R$.

In this case, replacing above assumptions $\lambda_3 < \lambda_2 < \lambda_1 < 0$ (or $\lambda_1 > \lambda_2 > 0,\ \ \lambda_3 < 0$), and $p_2 > q_2$ by assumption $\alpha > 0,\ \ \beta > 0,\ \ \lambda < 0$, (2.16) (or $\alpha < 0,\ \ \beta > 0,\ \ \lambda < 0$, (2.17)), we can get similar existence results which we omit here.

Case (II): There exist positive real constants m and ρ such that $a = \rho$, $b = 0$, $c = -m^3$, $d = -\rho m^3$.

In this case, replacing above assumptions $\lambda_3 < \lambda_2 < \lambda_1 < 0$ (or $\lambda_1 > \lambda_2 > 0,\ \ \lambda_3 < 0$), and $p_2 > q_2$ by assumption $\sqrt{3}m\omega < \frac{4}{3}\pi$, we can get similar existence results which we omit here.

Case (III): There exist positive real constants m and ρ such that $a = \rho$, $b = 0$, $c = m^3$, $d = \rho m^3$.

In this case, replacing above assumptions $\lambda_3 < \lambda_2 < \lambda_1 < 0$ (or $\lambda_1 > \lambda_2 > 0,\ \ \lambda_3 < 0$), and $p_2 > q_2$ by assumption $\sqrt{3}m\omega < \frac{4}{3}\pi$, we can get similar existence results which we omit here.

Case (IV): There exists a positive real constant ρ such that $a = -2\rho$, $b = 3\rho^3$, $c = -2\rho^3$, $d = \rho^4$.

In this case, replacing above assumptions $\lambda_3 < \lambda_2 < \lambda_1 < 0$ (or $\lambda_1 > \lambda_2 > 0,\ \ \lambda_3 < 0$), and $p_2 > q_2$ by assumption $\rho < \frac{2\pi}{\sqrt{3}\omega}$, we can get similar existence results which we omit here.

Case (V): There exists a positive real constant ρ such that $a = \rho$, $b = 0$, $c = \rho^3$, $d = \rho^4$.

In this case, replacing above assumptions $\lambda_3 < \lambda_2 < \lambda_1 < 0$ (or $\lambda_1 > \lambda_2 > 0,\ \ \lambda_3 < 0$), and $p_2 > q_2$ by assumption $\rho < \frac{2\pi}{\sqrt{3}\omega}$, we can get similar existence results which we omit here.

Case (VI): There exists a positive real constant ρ such that $a = 4\rho$, $b = 6\rho^2$, $c = 4\rho^3$, $d = \rho^4$.

In this case, delete to assumptions $\lambda_3 < \lambda_2 < \lambda_1 < 0$ (or $\lambda_1 > \lambda_2 > 0,\ \ \lambda_3 < 0$), and $p_2 > q_2$, we can get similar existence results which we omit here.

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