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ON THE GLOBAL WELL-POSEDNESS OF THE 3D VISCOUS PRIMITIVE EQUATIONS*

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Abstract Here we consider the global well-posedness of the 3D viscous primitive equations of the large-scale ocean. Inspired by the methods in Cao etc [2] and Guo etc [5], we prove the global well-posedness and the long-time dynamics for the primitive equations.

Keywords Primitive equations, Navier-stokes equations, weakly strong solutions, global well-posedness.

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1. Introduction

The primitive equations of the large-scale ocean are derived from the incompressible Navier-Stokes equations with Coriolis force by taking into account both the Boussinesq and hydrostatic approximations, see e.g. [7,9]. Starting with a series of works by Lions etc [7,8], the primitive equations of the ocean or the atmosphere have been extensively studied from the mathematical point of view, cf. e.g. [1,2,5].

In the present paper, we are interested in studying the existence and uniqueness of global weakly strong solutions to the initial boundary value problem of large-scale oceanic primitive equations considered by Cao etc [2]. Here we give the definition of the weakly strong solution and our main results.

Definition 1.1. Let $U_0 = (v_0, T_0) \in X$, and let $\mathcal{T}$ be a fixed positive time. $U = (v, T)$ is called a **weakly strong solution** of the system (2.11)–(2.17) on the time interval $[0, \mathcal{T}]$ if it satisfies (2.11)–(2.12) in weak sense such that

$$
\begin{aligned}
& v \in L^2(0, \mathcal{T}; V_1) \cap L^\infty(0, \mathcal{T}; H_1), \\
& \tilde{v} \in L^\infty(0, \mathcal{T}; (L^4(\Omega))^2), \\
& \partial_z v \in L^\infty(0, \mathcal{T}; (L^2(\Omega))^2) \cap L^2(0, \mathcal{T}; (H^1(\Omega))^2), \\
& T \in L^\infty(0, \mathcal{T}; L^4(\Omega)) \cap L^2(0, \mathcal{T}; V_2), \\
& \partial_z T \in L^\infty(0, \mathcal{T}; L^2(\Omega)) \cap L^2(0, \mathcal{T}; H^1(\Omega)), \\
& \frac{\partial v}{\partial t} \in L^2(0, V_1'), \ \frac{\partial T}{\partial t} \in L^2(0, \mathcal{T}; V_2'),
\end{aligned}
$$

where V_i' is the dual space of V_i for $i = 1,\ 2$, and the primitive equations and the working spaces are give in section 2.

Now we formulate our main results in the present paper.

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Theorem 1.1 (Existence of global weakly strong solutions for (IBVP))**.** *Let* $Q \in H^1(\Omega)$, $U_0 = (v_0, T_0) \in X$. *Then for any* $\mathcal{T} > 0$ *given, there exists a weakly strong solution* U *to the system (2.11)–(2.17) on the interval* $[0, \mathcal{T}]$.

Theorem 1.2 (Uniqueness of global weakly strong solutions for (IBVP))**.** *Let* $Q \in H^1(\Omega)$, $U_0 = (v_0, T_0) \in X$. *Then for any* $\mathcal{T} > 0$ *given, the weakly strong solution* U *of the system (2.11)–(2.17) on the interval* $[0, \mathcal{T}]$ *is unique. Moreover, the weakly strong solution* U *is dependent continuously on the initial data.*

Assuming that the initial data $U_0 = (v_0, T_0)$ satisfy: $v_0 \in (L^4(\Omega))^2$, $T_0 \in L^4(\Omega)$, $\partial_z v_0 \in (L^2(\Omega))^2$, $\partial_z T_0 \in L^2(\Omega)$, which is weaker than that in [2], we prove the global well-posedness for the primitive equations of the large-scale ocean. As a byproduct, we study the long-time behavior of weakly strong solutions (the result is posed as Proposition 3.3). The main steps in our paper are to obtain uniform estimates of $\|\tilde{v}(t)\|_{(L^4(\Omega))^2}$ and $\|\partial_z U(t)\|_{(L^2(\Omega))^3}$, where $\tilde{v}$ is the fluctuation of horizontal velocity v. First, inspired by the methods of [5], we prove that L^4-norm of $\tilde{v}$ is bounded uniformly in t after we obtain uniform estimates about L^3-norm of $\tilde{v}$. Second, on the basis of uniform estimates of $\|\tilde{v}(t)\|_{(L^4(\Omega))^2}$, we can prove that $\|\partial_z U(t)\|_{(L^2(\Omega))^3}$ is bounded uniformly in t.

The paper is organized as follows. In section 2, we give the primitive equations of the large-scale ocean and our working spaces. We prove main results of our paper in sections 3, 4.

2. The 3-D viscous primitive equations of the large-scale ocean

In this section, we recall the model considered in Cao etc [2]. The three-dimensional viscous primitive equations of the large-scale ocean in a Cartesian coordinate system(for details, we refer the reader to [9] and references therein) is written as

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v + \omega \frac{\partial v}{\partial z} + f k \times v + \nabla p - \frac{1}{Re_1} \triangle v - \frac{1}{Re_2} \frac{\partial^2 v}{\partial z^2} = f(t, x), \tag{2.1}$$

$$\frac{\partial p}{\partial z} + T = 0, \tag{2.2}$$

$$\text{div} v + \frac{\partial \omega}{\partial z} = 0, \tag{2.3}$$

$$\frac{\partial T}{\partial t} + v \cdot \nabla T + \omega \frac{\partial T}{\partial z} - \frac{1}{Rt_1} \triangle T - \frac{1}{Rt_2} \frac{\partial^2 T}{\partial z^2} = Q, \tag{2.4}$$

where the unknown functions are v, ω, p, T, $v = (v^{(1)}, v^{(2)})$ the horizontal velocity, ω vertical velocity, p the pressure, T temperature, $f = f_0(\beta + y)$ the Coriolis parameter, k vertical unit vector, Re_1, Re_2, Rt_1, Rt_2 Reynolds numbers, Q a given function on a cylindrical domain Ω defined later, $\nabla = (\partial_x, \partial_y)$, $\triangle = \partial_x^2 + \partial_y^2$, $\text{div} v = \partial_x v + \partial_y v$.

The space domain of the equations (2.1)–(2.4) is

$$\Omega = \{(x, y, z) : \ (x, y) \in M \text{ and } z \in (-h(x, y), 0)\},$$

where M is a smooth bounded domain in $\mathbb{R}^2$. Here we assume $h = 1$, that is, $\Omega = M \times (-1, 0)$. For general non-constant functions $h(x, y)$, in order to obtain our results, we need some regular conditions on $h(x,y)$, for example $h(x,y) \in C^3(\overline{M})$. For simplicity and without loss generality, the boundary value conditions are given by

$$\frac{\partial v}{\partial z} = 0,\ \omega = 0,\ \frac{\partial T}{\partial z} = -\alpha_s T \qquad \text{on } M \times \{0\} = \Gamma_u, \tag{2.5}$$

$$\frac{\partial v}{\partial z} = 0,\ \omega = 0,\ \frac{\partial T}{\partial z} = 0 \qquad \text{on } M \times \{-1\} = \Gamma_b, \tag{2.6}$$

$$v \cdot \vec{n} = 0,\ \frac{\partial v}{\partial \vec{n}} \times \vec{n} = 0,\ \frac{\partial T}{\partial \vec{n}} = 0 \qquad \text{on } \partial M \times [-1,0] = \Gamma_l, \tag{2.7}$$

where α_s is a positive constant and $\vec{n}$ is the norm vector to Γ_l.

Remark 2.1. Like [2], the salinity diffusion equation is omitted here. However, our results are valid when the salinity is taken into account and the boundary value conditions $\frac{\partial v}{\partial z}|_{z=0} = -\alpha_s T,\ \frac{\partial T}{\partial z}|_{z=0} = 0$ are replaced by $\frac{\partial v}{\partial z}|_{z=0} = \tau,\ \frac{\partial T}{\partial z}|_{z=0} = -\alpha_s(T - T^*)$ for smooth enough $\tau,\ T^*$.

Integrating (2.3) and using the boundary conditions (2.5), (2.6), we have

$$\omega(t;x,y,z) = W(v)(t;x,y,z) = -\int_{-1}^{z} \operatorname{div} v(t;x,y,z')\, dz', \tag{2.8}$$

$$\int_{-1}^{0} \operatorname{div} v\, dz = 0. \tag{2.9}$$

Suppose that p_s is a certain unknown function at the bottom $M \times \{-1\}$. By integrating (2.2),

$$p(t;x,y,z) = p_s(t;x,y) - \int_{-1}^{z} T\, dz'. \tag{2.10}$$

In this article, we assume that the constants $Re_1,\ Re_2,\ Rt_1,\ Rt_2$ are all equal to 1, which can not change our results. Then the equations (2.1)–(2.4) can be written as

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v + W(v)\frac{\partial v}{\partial z} + fk \times v + \nabla p_s - \int_{-1}^{z} \nabla T dz' - \triangle v - \frac{\partial^2 v}{\partial z^2} = 0, \tag{2.11}$$

$$\frac{\partial T}{\partial t} + (v \cdot \nabla)T + W(v)\frac{\partial T}{\partial z} - \triangle T - \frac{\partial^2 T}{\partial z^2} = Q, \tag{2.12}$$

$$\int_{-1}^{0} \operatorname{div} v\, dz = 0. \tag{2.13}$$

The boundary value conditions of the equations (2.11)–(2.13) are given by

$$\frac{\partial v}{\partial z} = 0, \quad \frac{\partial T}{\partial z} = -\alpha_s T \qquad \text{on } \Gamma_u, \tag{2.14}$$

$$\frac{\partial v}{\partial z} = 0, \quad \frac{\partial T}{\partial z} = 0 \qquad \text{on } \Gamma_b, \tag{2.15}$$

$$v \cdot \vec{n} = 0,\ \frac{\partial v}{\partial \vec{n}} \times \vec{n} = 0,\ \frac{\partial T}{\partial \vec{n}} = 0 \qquad \text{on } \Gamma_l, \tag{2.16}$$

and the initial value conditions can be given as

$$U|_{t=0} = (v|_{t=0}, T|_{t=0}) = U_0 = (v_0, T_0). \tag{2.17}$$

We call (2.11)–(2.17) as the initial boundary value problem of the new formulation of the 3-D viscous primitive equations of large-scale ocean, which is denoted by (IBVP).

Now we define the fluctuation $\tilde{v}$ of horizontal velocity and find the equations satisfied by $\tilde{v}$ and $\bar{v}$ as that in [2]. Let $\bar{v} = \int_{-1}^{0} v dz$, and denote the fluctuation of the horizontal velocity by $\tilde{v} = v - \bar{v}$. We notice that

$$\bar{\tilde{v}} = \int_{-1}^{0} \tilde{v} dz = 0, \quad \nabla \cdot \bar{v} = 0. \tag{2.18}$$

By integrating the momentum equation (2.11) with respect to z from -1 to 0, using the boundary value conditions (2.14)–(2.16) and (2.18), we get

$$\begin{aligned} &\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla)\bar{v} + \overline{\tilde{v}\mathrm{div}\tilde{v} + (\tilde{v} \cdot \nabla)\tilde{v}} + fk \times \bar{v} + \nabla p_s - \int_{-1}^{0}\int_{-1}^{z} \nabla T dz' dz \\ &- \triangle \bar{v} = 0 \ \text{ in } M. \end{aligned} \tag{2.19}$$

Subtracting (2.19) from (2.11), we know that the fluctuation $\tilde{v}$ satisfies the following equation and boundary value conditions

$$\begin{aligned} &\frac{\partial \tilde{v}}{\partial t} + (\tilde{v} \cdot \nabla)\tilde{v} + W(\tilde{v})\frac{\partial \tilde{v}}{\partial z} + (\tilde{v} \cdot \nabla)\bar{v} + (\bar{v} \cdot \nabla)\tilde{v} - \overline{(\tilde{v}\mathrm{div}\tilde{v} + (\tilde{v} \cdot \nabla)\tilde{v})} + fk \times \tilde{v} \\ &- \int_{-1}^{z} \nabla T dz^{'} + \int_{-1}^{0}\int_{-1}^{z} \nabla T dz^{'} dz - \triangle \tilde{v} - \frac{\partial^2 \tilde{v}}{\partial z^2} = 0 \ \text{ in } \Omega, \end{aligned} \tag{2.20}$$

$$\frac{\partial \tilde{v}}{\partial z} = 0 \text{ on } \Gamma_u, \ \frac{\partial \tilde{v}}{\partial z} = 0 \text{ on } \Gamma_b, \ \tilde{v} \cdot \vec{n} = 0, \ \frac{\partial \tilde{v}}{\partial \vec{n}} \times \vec{n} = 0 \text{ on } \Gamma_l. \tag{2.21}$$

Now we give our working spaces.

$L^p(\Omega) := \{u;\ u : \Omega \to \mathbb{R}, \int_\Omega |u|^p < +\infty\}$ with the norm $|u|_p = (\int_\Omega |u|^p)^{\frac{1}{p}}$, $1 \le p < \infty$. $\int_\Omega \cdot d\Omega$ and $\int_M \cdot dM$ are denoted by $\int_\Omega \cdot$ and $\int_M \cdot$ respectively. $H^m(\Omega)$ is the usual Sobolev space(m is a positive integer) with the norm

$$\|u\|_m = (\int_\Omega (\sum_{1 \le k \le m} \sum_{i_j = 1,2,3; j=1,\cdots,k} |\nabla_{i_1} \cdots \nabla_{i_k} u|^2 + |u|^2))^{\frac{1}{2}},$$

where $\nabla_1 = \frac{\partial}{\partial x}, \ \nabla_2 = \frac{\partial}{\partial y}, \ \nabla_3 = \frac{\partial}{\partial z}$.

We define our working spaces for the problem (IBVP). Let

$$\begin{aligned} \widetilde{\mathcal{V}_1} := \{&v \in (C^\infty(\Omega))^2; \frac{\partial v}{\partial z}|_{z=0} = 0, \frac{\partial v}{\partial z}|_{z=-1} = 0, v \cdot \vec{n}|_{\Gamma_s} = 0, \frac{\partial v}{\partial \vec{n}} \times \vec{n}|_{\Gamma_s} = 0, \\ &\nabla \cdot \bar{v} = 0\}, \end{aligned}$$

$$\widetilde{\mathcal{V}_2} := \{T;\ T \in C^\infty(\Omega),\ \frac{\partial T}{\partial z}|_{z=0} = -\alpha_s T,\ \frac{\partial T}{\partial z}|_{z=-1} = 0,\ \frac{\partial T}{\partial \vec{n}}|_{\Gamma_s} = 0\},$$

$V_1 =$ the closure of $\widetilde{\mathcal{V}_1}$with respect to the norm $\| \cdot \|_1$ ($\|v\|_m^m = \|v^{(1)}\|_m^m + \|v^{(2)}\|_m^m$),

$V_2 =$ the closure of $\widetilde{\mathcal{V}_2}$with respect to the norm $\| \cdot \|_1$,

$H_1 =$ the closure of $\widetilde{\mathcal{V}_1}$with respect to the norm $| \cdot |_2$,

$H_2 =$ the closure of $\widetilde{\mathcal{V}_2}$with respect to the norm $| \cdot |_2$,

$X_1 =$ the closure of $\widetilde{\mathcal{V}_1}$with respect to the norm $\|\cdot\|_{X_1} = |\cdot|_4 + |\partial_z \cdot|_2$,
$X_2 =$ the closure of $\widetilde{\mathcal{V}_2}$with respect to the norm $\|\cdot\|_{X_2} = |\cdot|_4 + |\partial_z \cdot|_2$,
$V = V_1 \times V_2,\ H = H_1 \times H_2,\ X = X_1 \times X_2.$

The inner products and norms on $V_1,\ V_2,\ V$ are given by

$$\begin{aligned}
&(v, v_1)_{V_1} = \int_\Omega (\partial_x v \cdot \partial_x v_1 + \partial_y v \cdot \partial_y v_1 + \partial_z v \cdot \partial_z v_1 + v \cdot v_1),\\
&\|v\| = (v,v)_{V_1}^{\frac{1}{2}}, \quad \forall v,\ v_1 \in V_1,\\
&(T, T_1)_{V_2} = \int_\Omega (\nabla T \cdot \nabla T_1 + \frac{\partial T}{\partial z}\frac{\partial T_1}{\partial z} + T T_1),\\
&\|T\| = (T,T)_{V_2}^{\frac{1}{2}}, \quad \forall T,\ T_1 \in V_2,\\
&(U, U_1) = (v^{(1)}, v_1^{(1)}) + (v^{(2)}, v_1^{(2)}) + (T, T_1),\\
&(U, U_1)_V = (v, v_1)_{V_1} + (T, T_1)_{V_2},\\
&\|U\| = (U,U)_V^{\frac{1}{2}},\ |U|_2 = (U,U)^{\frac{1}{2}},\ \forall U = (v,T),\ U_1 = (v_1, T_1) \in V,
\end{aligned}$$

where $(\cdot,\cdot)$ denotes the L^2 inner products in $L^2(\Omega)$.

3. Global existence of weakly strong solutions and long-time dynamics

3.1. Global existence

Proof of Theorem 1.1. We prove Theorem 1.1 by the well-known Faedo-Galerkin method. Since the procedure is similar to the proof of the existence of Leray-Hopf weak solutions to Navier-Stokes system in Lions [6, Theorem 6.1], we only give *a priori* estimates of approximation solutions. By the usual Faedo-Galerkin method, let $U_n = (v_n, T_n)$ be approximate weakly strong solutions to the system (2.11)–(2.16) with the initial value conditions $U_n|_{t=0} = (v_n|_{t=0}, T_n|_{t=0}) = U_{n0} = (v_{n0}, T_{n0})$ on the interval $[0, \mathcal{T}]$, where $U_{n0} \to U_0$ in X as $n \to +\infty$.
L^2 estimates about T_n, v_n Taking the inner product of equation (2.12) with T_n in $L^2(\Omega)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d|T_n|_2^2}{dt} + \int_\Omega |\nabla T_n|^2 + \int_\Omega |\frac{\partial T_n}{\partial z}|^2 + \frac{\alpha_s}{Rt_2}|T_n|_{z=0}|_2^2\\
&= -\int_\Omega [(v_n \cdot \nabla) T_n + W(v_n)\frac{\partial T_n}{\partial z}] T_n + \int_\Omega Q T_n.
\end{aligned} \tag{3.1}$$

By integration by parts, $T_n(x,y,z) = -\int_z^0 \frac{\partial T_n}{\partial z'} dz' + T_n|_{z=0}$, using Hölder inequality, Cauchy-Schwarz inequality and Young inequality, we derive from (3.1)

$$\frac{d|T_n|_2^2}{dt} + \int_\Omega |\nabla T_n|^2 + \int_\Omega |\frac{\partial T_n}{\partial z}|^2 + \alpha_s |T_n|_{z=0}|_2^2 \le c|Q|_2^2. \tag{3.2}$$

In this article, c denote positive constants and can be determined in concrete conditions. ε given later is a small enough positive constant. By (3.2) and the Gronwall inequality,

$$|T_n|_2^2 \le e^{-c_0 t}|T_{n0}|_2^2 + c|Q|_2^2 \le E_0, \tag{3.3}$$

where $c_0 = \min\{\frac{1}{2}, \frac{\alpha_s}{2}\} > 0,\ t \ge 0$ and E_0 is a positive constant independent of n. From (3.2) and (3.3), we get

$$\begin{aligned}
&c_1 \int_t^{t+r} [\int_\Omega (|\nabla T_n|^2 + |\frac{\partial T_n}{\partial z}|^2 + |T_n|^2) + |T_n|_{z=0}|_2^2] + |T_n(t)|_2^2 \\
\le& 2e^{-c_0 t}|T_{n0}|_2^2 + 3c|Q|_2^2 \\
\le& E_1,
\end{aligned} \tag{3.4}$$

where $c_1 = \min\{1, \frac{1}{3}, \frac{\alpha_s}{2}\},\ t \ge 0,\ 1 \ge r > 0$ given, E_1 is a positive constant independent of n, and $\int_t^{t+r} \cdot ds$ is denoted by $\int_t^{t+r} \cdot$.

Choosing v_n as a test function in equation (2.11), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d|v_n|_2^2}{dt} + \int_\Omega |\nabla v_n|^2 + \int_\Omega |\frac{\partial v_n}{\partial z}|^2 \\
=& -\int_\Omega [(v_n \cdot \nabla)v_n + W(v_n)\frac{\partial v_n}{\partial z} + fk \times v_n + \nabla p_{sn}] \cdot v_n + \int_\Omega (\int_{-1}^z \nabla T_n dz') \cdot v_n,
\end{aligned} \tag{3.5}$$

where $|\nabla v_n|^2 = |\partial_x v_n|^2 + |\partial_y v_n|^2$.

With integration by parts, (2.13)–(2.16), $(fk \times v_n) \cdot v_n = 0$ and Young inequality, from (3.5), we have

$$\frac{d|v_n|_2^2}{dt} + \int_\Omega |\nabla v_n|^2 + \int_\Omega |\frac{\partial v_n}{\partial z}|^2 \le c|T_n|_2^2. \tag{3.6}$$

By $\|v_n\|_{L^2(M)}^2 \le C_M \|\nabla v_n\|_{L^2(M)}^2$ (cf. Galdi [3, p55]) and Gronwall inequality, we derive from (3.6)

$$|v_n(t)|_2^2 \le e^{-\frac{t}{C_M}}|v_{n0}|_2^2 + cE_1, \tag{3.7}$$

where $t \ge 0$. From (3.6) and (3.7), we get

$$\begin{aligned}
c_2 \int_t^{t+r} [\int_\Omega (|\nabla v_n|^2 + |\frac{\partial v_n}{\partial z}|^2 + |v_n|^2)] + |v_n(t)|_2^2 &\le 2e^{-\frac{t}{C_M}}|v_{n0}|_2^2 + cE_1 \\
&\le E_2,
\end{aligned} \tag{3.8}$$

where $c_2 = \min\{\frac{1}{2C_M},\ \frac{1}{2}, 1\}$, $t \ge 0$, E_2 is a positive constant independent of n. By Minkowski inequality and Hölder inequality, for any $t \ge 0$ we have

$$\|\bar{v}_n(t)\|_{L^2(M)}^2 \le |v_n(t)|_2^2.$$

Similarly,

$$\int_M |\nabla \bar{v}_n|^2 \le \int_\Omega |\nabla v_n|^2.$$

So, from (3.8),

$$c_1\int_t^{t+r}\int_M(|\nabla\bar{v}_n|^2+|\bar{v}_n|^2)+\|\bar{v}_n(t)\|^2_{L^2(M)}\leq E_2,\forall t\geq 0. \tag{3.9}$$

By the interpolation inequalities, we derive from (3.9)

$$\begin{aligned}\int_t^{t+r}|\bar{v}_n|_4^4&=\int_t^{t+r}\|\bar{v}_n\|^4_{L^4(M)}\\&\leq\int_t^{t+r}\|\bar{v}_n\|^2_{L^2(M)}\|\bar{v}_n\|^2_{H^1(M)}\\&\leq cE_2^2.\end{aligned}\tag{3.10}$$

L^4 **estimates about** T_n We take the inner product of equation (2.12) with $|T_n|^2T_n$ in $L^2(\Omega)$ and obtain

$$\begin{aligned}&\frac{1}{4}\frac{d|T_n|_4^4}{dt}+3\int_\Omega|\nabla T_n|^2|T_n|^2+3\int_\Omega|\frac{\partial T_n}{\partial z}|^2|T_n|^2+\alpha_s\int_M|T_n|_{z=0}|^4\\=&\int_\Omega Q|T_n|^2T_n-\int_\Omega[(v_n\cdot\nabla)T_n-(\int_{-1}^z\operatorname{div}v_ndz^{'})\frac{\partial T_n}{\partial z}]|T_n|^2T_n.\end{aligned}\tag{3.11}$$

By Hölder inequality and Young inequality,

$$|\int_\Omega Q|T_n|^2T_n|\leq c|Q|_4^4+\varepsilon|T_n|_4^4. \tag{3.12}$$

With integration by parts and (2.14)–(2.16), we have

$$-\int_\Omega[(v_n\cdot\nabla)T_n-(\int_{-1}^z\operatorname{div}v_ndz^{'})\frac{\partial T_n}{\partial z}]|T_n|^2T_n=0. \tag{3.13}$$

Since $T_n^4(x,y,z)=-\int_z^0\frac{\partial T_n^4}{\partial z^{'}}dz^{'}+T_n^4|_{z=0}$, by using Hölder inequality and Cauchy-Schwarz inequality,

$$\begin{aligned}&|T_n|_4^4\\\leq&c\int_M[(\int_{-1}^0|T_n|^2|\frac{\partial T_n}{\partial z}|^2dz)^{\frac{1}{2}}(\int_{-1}^0T_n^4dz)^{\frac{1}{2}}]+|T_n|_{z=0}|_4^4\\\leq&c\int_\Omega|T_n|^2|\frac{\partial T_n}{\partial z}|^2+\frac{1}{2}\int_\Omega T_n^4+|T_n|_{z=0}|_4^4.\end{aligned}\tag{3.14}$$

Choosing ε small enough, we derive from (3.11)–(3.14)

$$\begin{aligned}&\frac{d|T_n|_4^4}{dt}+3\int_\Omega|\nabla T_n|^2|T_n|^2+3\int_\Omega|\frac{\partial T_n}{\partial z}|^2|T_n|^2+\alpha_s\int_M|T_n|_{z=0}|^4\\\leq&c|Q|_4^4.\end{aligned}\tag{3.15}$$

By Gronwall inequality, from (3.14) and (3.15), we have

$$\begin{aligned}|T_n(t)|_4^4&\leq e^{-c_3t}|T_{n0}|_4^4+c|Q|_4^4\\&\leq E_3,\end{aligned}\tag{3.16}$$

where $t \geq 0,\ c_3,\ E_3$ are positive constants independent of n. From (3.15) and (3.16), we get

$$c_1 \int_t^{t+r} |T_n|_{z=0}|_4^4 \leq 2E_3, \text{ for any } t \geq 0. \tag{3.17}$$

L^3 estimates about $\tilde{v}_n$ We take the inner product of equation (2.20) with $|\tilde{v}_n|\tilde{v}_n$ in $L^2(\Omega) \times L^2(\Omega)$, and obtain

$$\begin{aligned}
&\frac{1}{3}\frac{d|\tilde{v}_n|_3^3}{dt} + \int_\Omega (|\nabla \tilde{v}_n|^2 |\tilde{v}_n| + \frac{4}{9}|\nabla |\tilde{v}_n|^{\frac{3}{2}}|^2) + \int_\Omega (|\partial_z \tilde{v}_n|^2 |\tilde{v}_n| + \frac{4}{9}|\partial_z |\tilde{v}_n|^{\frac{3}{2}}|^2) \\
=& -\int_\Omega [(\tilde{v}_n \cdot \nabla)\tilde{v}_n - (\int_{-1}^z \mathrm{div}\tilde{v}_n dz^{'})\frac{\partial \tilde{v}_n}{\partial z}] \cdot |\tilde{v}_n|\tilde{v}_n - \int_\Omega [(\bar{v}_n \cdot \nabla)\tilde{v}_n] \cdot |\tilde{v}_n|\tilde{v}_n \\
&- \int_\Omega |\tilde{v}_n|\tilde{v}_n \cdot [(\tilde{v}_n \cdot \nabla)\bar{v}_n] + \int_\Omega (\int_{-1}^z \nabla T_n dz^{'} - \int_{-1}^0 \int_{-1}^z \nabla T_n dz^{'} dz) \cdot |\tilde{v}_n|\tilde{v}_n \\
&+ \int_\Omega \overline{[\tilde{v}_n \mathrm{div}\tilde{v}_n + (\tilde{v}_n \cdot \nabla)\tilde{v}_n]} \cdot |\tilde{v}_n|\tilde{v}_n - \int_\Omega (fk \times \tilde{v}_n) \cdot |\tilde{v}_n|\tilde{v}_n.
\end{aligned} \tag{3.18}$$

By integration by parts and (2.18), we get

$$\begin{aligned}
\int_\Omega [(\bar{v}_n \cdot \nabla)\tilde{v}_n] \cdot |\tilde{v}_n|\tilde{v}_n &= \frac{1}{3}\int_\Omega (\bar{v}_n \cdot \nabla)|\tilde{v}_n|^3 \\
&= -\frac{1}{3}\int_\Omega |\tilde{v}_n|^3 \mathrm{div}\bar{v}_n \\
&= 0.
\end{aligned} \tag{3.19}$$

Since

$$\begin{aligned}
&\int_\Omega \mathrm{div}(|\tilde{v}_n|\tilde{v}_n \cdot \bar{v}_n)\tilde{v}_n \\
=&\int_\Omega [|\tilde{v}_n|\tilde{v}_n \cdot (\tilde{v}_n \cdot \nabla)\bar{v}_n + \bar{v}_n \cdot (\tilde{v}_n \cdot \nabla)|\tilde{v}_n|\tilde{v}_n] + \int_\Omega |\tilde{v}_n|\tilde{v}_n \cdot \bar{v}_n \mathrm{div}\tilde{v}_n \\
=&0,
\end{aligned} \tag{3.20}$$

$$-\int_\Omega |\tilde{v}_n|\tilde{v}_n \cdot (\tilde{v}_n \cdot \nabla)\bar{v}_n = \int_\Omega \bar{v}_n \cdot (\tilde{v}_n \cdot \nabla)|\tilde{v}_n|\tilde{v}_n + \int_\Omega |\tilde{v}_n|\tilde{v}_n \cdot \bar{v}_n \mathrm{div}\tilde{v}_n. \tag{3.21}$$

Using integration by parts, we obtain

$$\begin{aligned}
&\int_\Omega [\int_{-1}^0 (\tilde{v}_n \mathrm{div}\tilde{v}_n + (\tilde{v}_n \cdot \nabla)\tilde{v}_n) dz] \cdot |\tilde{v}_n|\tilde{v}_n \\
=& \int_\Omega (\int_{-1}^0 \tilde{v}_{nx}\tilde{v}_n dz) \cdot \partial_x(|\tilde{v}_n|\tilde{v}_n) + \int_\Omega (\int_{-1}^0 \tilde{v}_{ny}\tilde{v}_n dz) \cdot \partial_y(|\tilde{v}_n|\tilde{v}_n).
\end{aligned} \tag{3.22}$$

From (3.18) to (3.22),

$$\begin{aligned}
&\frac{1}{3}\frac{d|\tilde{v}_n|_3^3}{dt} + \int_\Omega (|\nabla \tilde{v}_n|^2 |\tilde{v}_n| + \frac{4}{9}|\nabla |\tilde{v}_n|^{\frac{3}{2}}|^2) + \int_\Omega (|\partial_z \tilde{v}_n|^2 |\tilde{v}_n| + \frac{4}{9}|\partial_z |\tilde{v}_n|^{\frac{3}{2}}|^2) \\
=& \int_\Omega [(\int_{-1}^0 \tilde{v}_{nx}\tilde{v}_n dz) \cdot \partial_x(|\tilde{v}_n|\tilde{v}_n) + (\int_{-1}^0 \tilde{v}_{ny}\tilde{v}_n dz) \cdot \partial_{ny}(|\tilde{v}_n|\tilde{v}_n)]
\end{aligned}$$

$$+\int_\Omega |\tilde v_n|\tilde v_n\cdot\bar v_n \mathrm{div}\tilde v_n+\int_\Omega \bar v_n\cdot(\tilde v_n\cdot\nabla)|\tilde v_n|\tilde v_n$$
$$-\int_\Omega (\int_{-1}^z T_n dz' -\int_{-1}^0\int_{-1}^z T_n dz' dz)\mathrm{div}(|\tilde v_n|\tilde v_n). \tag{3.23}$$

By Hölder inequality, we derive from (3.23)

$$\frac{1}{3}\frac{d|\tilde v_n|_3^3}{dt}+\int_\Omega(|\tilde v_n||\nabla\tilde v_n|^2+\frac{4}{9}|\nabla|\tilde v_n|^{\frac{3}{2}}|^2)+\int_\Omega(|\partial_z\tilde v_n|^2|\tilde v_n|+\frac{4}{9}|\partial_z|\tilde v_n|^{\frac{3}{2}}|^2)$$
$$\leq c|\bar v_n|_4(\int_\Omega|\tilde v_n||\nabla\tilde v_n|^2)^{\frac{1}{2}}[\int_M(\int_{-1}^0|\tilde v_n|^3dz)^2]^{\frac{1}{4}}+c|\overline{|T_n|}|_4|\tilde v_n|_2^{\frac{1}{2}}(\int_\Omega|\tilde v_n||\nabla\tilde v_n|^2)^{\frac{1}{2}}$$
$$+c(\int_\Omega|\tilde v_n||\nabla\tilde v_n|^2)^{\frac{1}{2}}\cdot[\int_M(\int_{-1}^0|\tilde v_n|^2dz)^{\frac{5}{2}}]^{\frac{1}{2}}. \tag{3.24}$$

By Minkowski inequality, the interpolation inequalities and Hölder inequality, we have

$$\begin{aligned}[\int_M(\int_{-1}^0|\tilde v_n|^3dz)^2]^{\frac{1}{2}} &\leq \int_{-1}^0[\int_M(|\tilde v_n|^{\frac{3}{2}})^4]^{\frac{1}{2}}dz\\ &\leq \int_{-1}^0\||\tilde v_n|^{\frac{3}{2}}\|_{L^2(M)}(\|\nabla|\tilde v_n|^{\frac{3}{2}}\|_{L^2(M)}^2+\||\tilde v_n|^{\frac{3}{2}}\|_{L^2(M)}^2)^{\frac{1}{2}}dz\\ &\leq c|\tilde v_n|_3^{\frac{3}{2}}[\int_{-1}^0(\|\nabla|\tilde v_n|^{\frac{3}{2}}\|_{L^2(M)}^2+\||\tilde v_n|^{\frac{3}{2}}\|_{L^2(M)}^2)dz]^{\frac{1}{2}}. \end{aligned}\tag{3.25}$$

By Minkowski inequality, Hölder inequality and $\|u\|_{L^5(M)}\leq c\|u\|_{L^3(M)}^{\frac{3}{5}}\|u\|_{H^1(M)}^{\frac{2}{5}}$ for any $u\in H^1(M)$, we get

$$\int_M(\int_{-1}^0|\tilde v_n|^2dz)^{\frac{5}{2}}\leq[\int_{-1}^0(\int_M|\tilde v_n|^5)^{\frac{2}{5}}dz]^{\frac{5}{2}}\leq c\|\tilde v_n\|^2|\tilde v_n|_3^3. \tag{3.26}$$

By Minkowski inequality,

$$\|\overline{|T_n|}\|_{L^4(M)}=(\int_M(\int_{-1}^0|T_n|dz)^4)^{\frac{1}{4}}\leq|T_n|_4. \tag{3.27}$$

By Young inequality and the interpolation inequalities, we derive from (3.24)–(3.27)

$$\frac{d|\tilde v_n|_3^3}{dt}+\int_\Omega(|\nabla\tilde v_n|^2|\tilde v_n|+\frac{4}{9}|\nabla|\tilde v_n|^{\frac{3}{2}}|^2)+\int_\Omega(|\partial_z\tilde v_n|^2|\tilde v_n|+\frac{4}{9}|\partial_z|\tilde v_n|^{\frac{3}{2}}|^2)$$
$$\leq c(\|\bar v_n\|_{L^2(M)}^2\|\bar v_n\|_{H^1(M)}^2+\|\bar v_n\|_{H^1(M)}^2+\|\tilde v_n\|^2)|\tilde v_n|_3^3+c|T_n|_4^4+c|\tilde v_n|_2^2. \tag{3.28}$$

By the uniform Gronwall Lemma, (3.8), (3.9), (3.16) and $|\tilde v_n|_3^3\leq|\tilde v_n|_2^{\frac{3}{2}}\|\tilde v_n\|_2^{\frac{3}{2}}$, we obtain

$$|\tilde v_n(t+r)|_3^3\leq E_4, \tag{3.29}$$

where $E_4=E_4(\|U_0\|_X,\|Q\|_1)>0$ independent of n and $t\geq 0$.

L^4 **estimates about** $\tilde{v}_n$ We take the inner product of equation (2.20) with $|\tilde{v}_n|^2\tilde{v}_n$ in $L^2(\Omega)\times L^2(\Omega)$, and obtain

$$\begin{aligned}
&\frac{1}{4}\frac{d|\tilde{v}_n|_4^4}{dt}+\int_\Omega(|\nabla\tilde{v}_n|^2|\tilde{v}_n|^2+\frac{1}{2}|\nabla|\tilde{v}_n|^2|^2)+\int_\Omega(|\partial_z\tilde{v}_n|^2|\tilde{v}_n|^2+\frac{1}{2}|\partial_z|\tilde{v}_n|^2|^2)\\
=&-\int_\Omega[(\tilde{v}_n\cdot\nabla)\tilde{v}_n-(\int_{-1}^z \mathrm{div}\tilde{v}_n dz^{'})\frac{\partial\tilde{v}_n}{\partial z}]\cdot|\tilde{v}_n|^2\tilde{v}_n-\int_\Omega[(\bar{v}_n\cdot\nabla)\tilde{v}_n]\cdot|\tilde{v}_n|^2\tilde{v}_n\\
&-\int_\Omega|\tilde{v}_n|^2\tilde{v}_n\cdot(\tilde{v}_n\cdot\nabla)\bar{v}_n+\int_\Omega(\int_{-1}^z\nabla T_n dz^{'}-\int_{-1}^0\int_{-1}^z\nabla T_n dz^{'}dz)\cdot|\tilde{v}_n|^2\tilde{v}_n\\
&+\int_\Omega\overline{[\tilde{v}_n\mathrm{div}\tilde{v}_n+(\tilde{v}_n\cdot\nabla)\tilde{v}_n]}\cdot|\tilde{v}_n|^2\tilde{v}_n-\int_\Omega(fk\times\tilde{v}_n)\cdot|\tilde{v}_n|^2\tilde{v}_n. \qquad (3.30)
\end{aligned}$$

Similarly to (3.23), we derive from (3.30)

$$\begin{aligned}
&\frac{1}{4}\frac{d|\tilde{v}_n|_4^4}{dt}+\int_\Omega(|\nabla\tilde{v}_n|^2|\tilde{v}_n|^2+\frac{1}{2}|\nabla|\tilde{v}_n|^2|^2)+\int_\Omega(|\partial_z\tilde{v}_n|^2|\tilde{v}_n|^2+\frac{1}{2}|\partial_z|\tilde{v}_n|^2|^2)\\
=&\int_\Omega\bar{v}_n\cdot(\tilde{v}_n\cdot\nabla)|\tilde{v}_n|^2\tilde{v}_n+\int_\Omega|\tilde{v}_n|^2\tilde{v}_n\cdot\bar{v}_n\mathrm{div}\tilde{v}_n\\
&-\int_\Omega(\int_{-1}^z T_n dz^{'}-\int_{-1}^0\int_{-1}^z T_n dz^{'}dz)\mathrm{div}(|\tilde{v}_n|^2\tilde{v}_n)\\
&+\int_\Omega[(\int_{-1}^0\tilde{v}_{nx}\tilde{v}_n dz)\partial_x(|\tilde{v}_n|^2\tilde{v}_n)+(\int_{-1}^0\tilde{v}_{ny}\tilde{v}_n dz)\partial_y(|\tilde{v}_n|^2\tilde{v}_n)]. \qquad (3.31)
\end{aligned}$$

By Hölder inequality, we obtain from the above

$$\begin{aligned}
&\frac{1}{4}\frac{d|\tilde{v}_n|_4^4}{dt}+\int_\Omega(|\nabla\tilde{v}_n|^2|\tilde{v}_n|^2+\frac{1}{2}|\nabla|\tilde{v}_n|^2|^2)+\int_\Omega(|\partial_z\tilde{v}_n|^2|\tilde{v}_n|^2+\frac{1}{2}|\partial_z|\tilde{v}_n|^2|^2)\\
\le& c\|\bar{v}_n\|_{L^4(M)}(\int_\Omega|\tilde{v}_n|^2|\nabla\tilde{v}_n|^2)^{\frac{1}{2}}[\int_M(\int_{-1}^0|\tilde{v}_n|^4dz)^2]^{\frac{1}{4}}\\
&+c\|\overline{|T_n|}\|_{L^4(M)}[\int_M(\int_{-1}^0|\tilde{v}_n|^2dz)^2]^{\frac{1}{4}}(\int_\Omega|\tilde{v}_n|^2|\nabla\tilde{v}_n|^2)^{\frac{1}{2}}\\
&+c(\int_\Omega|\tilde{v}_n|^2|\nabla\tilde{v}_n|^2)^{\frac{1}{2}}[\int_M(\int_{-1}^0|\tilde{v}_n|^2dz)^3]^{\frac{1}{2}}. \qquad (3.32)
\end{aligned}$$

By Minkowski inequality, the interpolation inequalities and Hölder inequality,

$$\begin{aligned}
[\int_M(\int_{-1}^0|\tilde{v}_n|^4dz)^2]^{\frac{1}{2}}&\le\int_{-1}^0[\int_M(|\tilde{v}_n|^2)^4]^{\frac{1}{2}}dz\\
&\le c|\tilde{v}_n|_4^2[\int_{-1}^0(\|\nabla|\tilde{v}_n|^2\|_{L^2(M)}^2+\||\tilde{v}_n|^2\|_{L^2(M)}^2)dz]^{\frac{1}{2}}. \qquad (3.33)
\end{aligned}$$

Similarly to (3.33),

$$\begin{aligned}
\int_M(\int_{-1}^0|\tilde{v}_n|^2dz)^3&\le[\int_{-1}^0(\int_M|\tilde{v}_n|^6)^{\frac{1}{3}}dz]^3\\
&\le[\int_{-1}^0(\|\tilde{v}_n\|_{L^4(M)}^4\|\tilde{v}_n\|_{H^1(M)}^2)^{\frac{1}{3}}dz]^3\\
&\le c\|\tilde{v}_n\|^2|\tilde{v}_n|_4^4. \qquad (3.34)
\end{aligned}$$

By Young inequality, we derive from (3.32)–(3.34)

$$\begin{aligned}&\frac{1}{4}\frac{d|\tilde{v}_n|_4^4}{dt}+\int_\Omega(|\nabla\tilde{v}_n|^2|\tilde{v}_n|^2+\frac{1}{2}|\nabla|\tilde{v}_n|^2|^2)+\int_\Omega(|\partial_z\tilde{v}_n|^2|\tilde{v}_n|^2+\frac{1}{2}|\partial_z|\tilde{v}_n|^2|^2)\\ \leq& c(1+\|\bar{v}_n\|^2_{L^2(M)}\|\bar{v}_n\|^2_{H^1(M)}+\|\bar{v}_n\|^2_{H^1(M)}+\|\tilde{v}_n\|^2)|\tilde{v}_n|_4^4+c|T_n|_4^4. \qquad (3.35)\end{aligned}$$

By the uniform Gronwall Lemma, (3.8), (3.9), (3.16), (3.29) and $|\tilde{v}_n|_4^4\leq|\tilde{v}_n|_3^2\|\tilde{v}_n\|^2$, we get

$$|\tilde{v}_n(t+2r)|_4^4\leq E_5, \qquad (3.36)$$

where $E_5=E_5(\|U_0\|_X,\|Q\|_1)>0$ independent of n and $t\geq0$. From (3.35) and (3.36), we have

$$\begin{aligned}&\int_{t+2r}^{t+3r}[\int_\Omega(|\nabla\tilde{v}_n|^2|\tilde{v}_n|^2+\frac{1}{2}|\nabla|\tilde{v}_n|^2|^2)+\int_\Omega(|\partial_z\tilde{v}_n|^2|\tilde{v}_n|^2+\frac{1}{2}|\partial_z|\tilde{v}_n|^2|^2)]\\ \leq& E_5^2+E_5\\ =&E_6. \qquad (3.37)\end{aligned}$$

By Gronwall inequality, from (3.35),

$$|\tilde{v}_n(t)|_4^4\leq C_1, \qquad (3.38)$$

where $C_1=C_1(\|U_0\|_X,\ \|Q\|_1)>0$ and $0\leq t<2r$.

L^2 estimates about $\partial_z v_n$ Taking the derivative, with respect to z, of equation (2.11), we get

$$\begin{aligned}&\frac{\partial v_{nz}}{\partial t}-\triangle v_{nz}-\frac{\partial^2 v_{nz}}{\partial z^2}+(v_n\cdot\nabla)v_{nz}+W(v_n)\frac{\partial v_{nz}}{\partial z}\\ &+(v_{nz}\cdot\nabla)v_n-(\mathrm{div}v_n)v_{nz}+fk\times v_{nz}-\nabla T_n\\ =&0, \qquad (3.39)\end{aligned}$$

where $v_{nz}=\partial_z v_n$. Taking the inner product of equation (3.39) with v_{nz} in $L^2(\Omega)\times L^2(\Omega)$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d|v_{nz}|_2^2}{dt}+\int_\Omega|\nabla v_{nz}|^2+\int_\Omega|\frac{\partial v_{nz}}{\partial z}|^2\\ =&-\int_\Omega[(v_n\cdot\nabla)v_{nz}+W(v_n)\frac{\partial v_{nz}}{\partial z}]\cdot v_{nz}-\int_\Omega(fk\times v_{nz})\cdot v_{nz}\\ &-\int_\Omega[(v_{nz}\cdot\nabla)v_n-(\mathrm{div}v_n)v_{nz}]\cdot v_{nz}-\int_\Omega\nabla T_n\cdot v_{nz}. \qquad (3.40)\end{aligned}$$

With integration by parts,

$$\int_\Omega[(v_n\cdot\nabla)v_{nz}+W(v_n)\frac{\partial v_{nz}}{\partial z}]\cdot v_{nz}=0. \qquad (3.41)$$

By integration by parts, Hölder inequality, the interpolation inequalities, Minkowski inequality and Young inequality, we have

$$
\begin{aligned}
&-\int_\Omega [(v_{nz}\cdot\nabla)v_n-(\mathrm{div}v_n)v_{nz}]\cdot v_{nz}\\
\le\, & c\int_\Omega |v_n||v_{nz}||\nabla v_{nz}|\\
\le\, & c\int_\Omega (|\bar{v}_n|+|\tilde{v}_n|)|v_{nz}||\nabla v_{nz}|\\
\le\, & c|\tilde{v}_n|_4|v_{nz}|_4(\int_\Omega|\nabla v_{nz}|^2)^{\frac{1}{2}}+c\int_M|\bar{v}_n|(\int_{-1}^0|v_{nz}|^2)^{\frac{1}{2}}(\int_{-1}^0|\nabla v_{nz}|^2)^{\frac{1}{2}}\\
\le\, & c|\tilde{v}_n|_4|v_{nz}|_2^{\frac{1}{4}}\|v_{nz}\|^{\frac{7}{4}}+c(\int_M|\bar{v}_n|^4)^{\frac{1}{4}}[\int_{-1}^0(\int_M|v_{nz}|^4)^{\frac{1}{2}}]^{\frac{1}{2}}|\nabla v_{nz}|_2\\
\le\, & \varepsilon(|\nabla v_{nz}|_2^2+|\frac{\partial v_{nz}}{\partial z}|_2^2)+c(\|\bar{v}_n\|^4_{L^4(M)}+|\tilde{v}_n|_4^8)|v_{nz}|_2^2. \qquad (3.42)
\end{aligned}
$$

By integration by parts, Hölder inequality and Young inequality,

$$-\int_\Omega\nabla T_n\cdot v_{nz}=\int_\Omega T_n\mathrm{div}v_{nz}\le c|T_n|_2^2+\varepsilon|\nabla v_{nz}|_2^2. \qquad (3.43)$$

Choosing ε small enough, we derive from (3.40)–(3.43),

$$
\begin{aligned}
&\frac{d|v_{nz}|_2^2}{dt}+\int_\Omega|\nabla v_{nz}|^2+\int_\Omega|\frac{\partial v_{nz}}{\partial z}|^2\\
\le\, & c(\|\bar{v}_n\|^4_{L^4(M)}+|\tilde{v}_n|_4^8)|v_{nz}|_2^2+c|T_n|_2^2. \qquad (3.44)
\end{aligned}
$$

By the uniform Gronwall Lemma, (3.3), (3.8), (3.10), (3.36) and (3.44), we get

$$|v_{nz}(t+3r)|_2^2\le E_7, \qquad (3.45)$$

where $E_7=E_7(\|U_0\|_X,\|Q\|_1)>0$ independent of n and $t\ge 0$. From (3.44) and (3.45), we have

$$c_1\int_{t+3r}^{t+4r}\|v_{nz}\|^2\le E_7^2+2E_7=E_8. \qquad (3.46)$$

By Gronwall inequality, from (3.44),

$$\int_0^t\|v_{nz}\|^2+|v_{nz}(t)|_2^2\le C_2, \qquad (3.47)$$

where $C_2=C_2(\|U_0\|_X,\ \|Q\|_1)>0$ and $0\le t<3r$.

L^2 **estimates about** $\partial_z T_n$ Taking the derivative, with respect to z, of the equation (2.12), we get

$$
\begin{aligned}
&\frac{\partial T_{nz}}{\partial t}-\triangle T_{nz}-\frac{\partial^2T_{nz}}{\partial z^2}+(v_n\cdot\nabla)T_{nz}+W(v_n)\frac{\partial T_{nz}}{\partial z}+(v_{nz}\cdot\nabla)T_n-(\mathrm{div}v_n)\frac{\partial T_n}{\partial z}\\
=\, & Q_z, \qquad (3.48)
\end{aligned}
$$

where $T_{nz} = \partial_z T_n,\ Q_z = \partial_z Q$. We take the inner product of equation (3.48) with T_{nz} in $L^2(\Omega)$, and obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d|T_{nz}|_2^2}{dt} + \int_\Omega |\nabla T_{nz}|^2 + \int_\Omega |T_{nzz}|^2 - \int_M (T_{nz}|_{z=0} \cdot T_{nzz}|_{z=0})\\
=& - \int_\Omega [(v_n \cdot \nabla) T_{nz} + W(v_n)\frac{\partial T_{nz}}{\partial z}]T_{nz} - \int_\Omega [(v_{nz} \cdot \nabla)T_n - (\mathrm{div} v_n)\frac{\partial T_n}{\partial z}]T_{nz}\\
&+ \int_\Omega Q_z T_{nz}, \qquad (3.49)
\end{aligned}$$

where $T_{nzz} = \frac{\partial^2 T_n}{\partial z^2}$. Similarly to (3.42), by integration by parts, Hölder inequality, the interpolation inequalities, Poincaré inequality and Young inequality, we obtain

$$\begin{aligned}
&|\int_\Omega ((v_{nz} \cdot \nabla)T_n - \mathrm{div} v_n \frac{\partial T_n}{\partial z})T_{nz}|\\
\le& c\int_\Omega [|\nabla v_{nz}||T_n||T_{nz}| + |v_{nz}||T_n||\nabla T_{nz}| + (|\bar{v}_n| + |\tilde{v}_n|)|\nabla T_{nz}||T_{nz}|]\\
\le& c|\nabla v_{nz}|_2^2 + \frac{\varepsilon}{2}|\nabla T_{nz}|_2^2 + c(|T_n|_4^2 + |\tilde{v}_n|_4^2)|T_{nz}|_4^2 + c|v_{nz}|_4^2|T_n|_4^2\\
&+ c\|\bar{v}_n\|_{L^4(M)}^2 \int_{-1}^0 (\int_M |T_{nz}|^4)^{\frac{1}{2}}\\
\le& c|\nabla v_{nz}|_2^2 + \varepsilon|\nabla T_{nz}|_2^2 + c(|T_n|_4^2 + |\tilde{v}_n|_4^2)|T_{nz}|_2^{\frac{1}{2}}(|\nabla T_{nz}|_2^2 + |T_{nzz}|_2^2)^{\frac{3}{4}}\\
&+ c\|\bar{v}_n\|_{L^4(M)}^4 |T_{nz}|_2^2 + c|T_n|_4^2|v_{nz}|_2^{\frac{1}{2}}(|v_{nzz}|_2^2 + |\nabla v_{nz}|^2)^{\frac{3}{4}}\\
\le& \varepsilon(|T_{nzz}|_2^2 + |\nabla T_{nz}|_2^2) + c(|v_{nzz}|_2^2 + |\nabla v_{nz}|_2^2) + c|T_n|_4^8|v_{nz}|_2^2\\
&+ c(|T_n|_4^8 + |\tilde{v}_n|_4^8 + \|\bar{v}_n\|_{L^4(M)}^4)|T_{nz}|_2^2. \qquad (3.50)
\end{aligned}$$

Taking the trace on $z = 0$ of equation (2.12),

$$T_{nzz}|_{z=0} = \frac{\partial T_n|_{z=0}}{\partial t} + [(v_n \cdot \nabla)T_n]|_{z=0} - \triangle T_n|_{z=0} - Q|_{z=0}. \qquad (3.51)$$

From (2.14), (3.51), we get

$$\begin{aligned}
&-\int_M (T_{nz}|_{z=0} T_{nzz}|_{z=0})\\
=& \alpha_s \int_M T_n|_{z=0}[\frac{\partial T_n|_{z=0}}{\partial t} + ((v_n \cdot \nabla)T_n)|_{z=0} - \triangle T_n|_{z=0} - Q|_{z=0}]\\
=& \alpha_s(\frac{1}{2}\frac{d|T_n|_{z=0}|_2^2}{dt} + |\nabla T_n|_{z=0}|_2^2) + \alpha_s \int_M T_n|_{z=0}[((v_n \cdot \nabla)T_n)|_{z=0} - Q|_{z=0}].
\end{aligned} \qquad (3.52)$$

With integration by parts, we have

$$\begin{aligned}
&-\alpha_s \int_M T_n|_{z=0}[((v_n \cdot \nabla)T_n)|_{z=0} - Q|_{z=0}]\\
=& -\frac{\alpha_s}{2}\int_M ((v_n \cdot \nabla)T_n^2)|_{z=0} + \alpha_s \int_M T_n|_{z=0} Q|_{z=0}\\
=& \frac{\alpha_s}{2}\int_M T_n^2|_{z=0} \mathrm{div} v_n|_{z=0} + \alpha_s \int_M T_n|_{z=0} Q|_{z=0}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha_s}{2}\int_{-1}^{0}\int_M T_n^2|_{z=0}\Big(\int_z^0 \mathrm{div} v_{nz}dz' + \mathrm{div} v_n\Big) + \alpha_s\int_M T_n|_{z=0}Q|_{z=0} \\
&\le c|T_n|_{z=0}|_4^4 + c\|v_{nz}\|^2 + c\|v_n\|^2 + c|T_n|_{z=0}|_2^2 + c|Q|_{z=0}|_2^2. \qquad (3.53)
\end{aligned}
$$

By Young inequality and choosing ε small enough, we derive from (3.49)–(3.53)

$$
\begin{aligned}
&\frac{d(|T_{nz}|_2^2 + \alpha_s|T_n|_{z=0}|_2^2)}{dt} + \int_\Omega |\nabla T_{nz}|^2 + \int_\Omega |T_{nzz}|^2 + \alpha_s|\nabla T_n|_{z=0}|_2^2 \\
\le& c(1 + |T_n|_4^8 + |\tilde{v}_n|_4^8 + + \|\bar{v}_n\|_{L^4(M)}^4)|T_{nz}|_2^2 + c\|v_{nz}\|^2 + c\|v_n\|^2 \\
&+ c|T_n|_4^8|v_{nz}|_2^2 + c|T_n|_{z=0}|_4^4 + c|T_n|_{z=0}|_2^2 + c|Q_n|_{z=0}|_2^2 + c|Q_{nz}|_2^2. \qquad (3.54)
\end{aligned}
$$

By the uniform Gronwall Lemma, (3.4), (3.8), (3.10), (3.16), (3.17), (3.36), (3.46), we get

$$|T_{nz}(t+4r)|_2^2 \le E_9, \qquad (3.55)$$

where $E_9 = E_9(\|U_0\|_X, \|Q\|_1) > 0$ independent of n. From (3.54) and (3.55), we have

$$c_1\int_{t+4r}^{t+5r} \|T_{nz}\|^2 \le E_9^2 + 2E_9 = E_{10}. \qquad (3.56)$$

By Gronwall inequality, from (3.54) we obtain

$$\int_0^t \|T_{nz}\|^2 + |T_{nz}(t)|_2^2 \le C_3, \qquad (3.57)$$

where $C_3 = C_3(\|U_0\|_X,\ \|Q\|_1) > 0$ and $0 \le t < 4r$.

3.2. long-time dynamics

From the a priori estimates in subsection 3.1, we can easily obtain the following result.

Proposition 3.1 (Long-time behavior of weakly strong solutions)**.** *If U is a global weakly strong solution to the system (2.11)–(2.17), then U satisfies* $\partial_z v \in L^\infty(0,\infty;(L^2(\Omega))^2)$, $\tilde{v} \in L^\infty(0,\infty;(L^4(\Omega))^2)$, $T \in L^\infty(0,\infty;X_2)$.

If $U_0 = (v_0, T_0) \in V$, we can obtain the following results which are similar to those in [4]. Here we omitted the details of proof.

Proposition 3.2 (Existence of bounded absorbing sets for the dynamical system (2.11)–(2.16))**.** *If $Q \in H^1(\Omega)$, $U_0 = (v_0, T_0) \in V$, Then the global strong solution U of the system (2.11)–(2.17) satisfies $U \in L^\infty(0,\infty;V)$ and*

$$\|U(t)\| \le C(\|U_0\|, \|Q\|_1),$$

where C is a positive constant dependent on $\|U_0\|, \|Q\|_1$ and $0 \le t \le +\infty$. Moreover, the corresponding semigroup $\{S(t)\}_{t\ge0}$ possesses a bounded absorbing set B_ρ in V, i.e., for every bounded set $B \subset V$, there exists $t_0(B) > 0$ big enough such that

$$S(t)B \subset B_\rho, \text{ for any } t \ge t_0,$$

where $B_\rho = \{U; \|U\| \le \rho\}$ and ρ is a positive constant dependent on $\|Q\|_1$.

Proposition 3.3 (Existence of the universal attractor for the system (2.11)–(2.16))**.** *The system (2.11)–(2.16) possesses a (weak) universal attractor* $\mathcal{A} = \cap_{s\geq 0}\overline{\cup_{t\geq s}T(t)B_\rho}$ *that captures all the trajectories, where the closures are taken with respect to* V*-weak topology. The (weak) universal attractor* $\mathcal{A}$ *has the following properties:*

(i) (weak compact) $\mathcal{A}$ *is bounded and weakly closed in* V*;*

(ii) (invariant) for every $t \geq 0$, $S(t)\mathcal{A} = \mathcal{A}$*;*

(ii) (attracting) for every bounded set B *in* V*, the sets* $S(t)B$ *converge to* $\mathcal{A}$ *with respect to* V*-weak topology as* $t \to +\infty$*, i.e.,*

$$\lim_{t\to+\infty} d_V^w(S(t)B,\ \mathcal{A}) = 0,$$

where the distance d_V^w *is induced by the* V*-weak topology.*

4. The uniqueness of weakly strong solutions

Proof of Theorem 1.2. Let (v_1, T_1) and (v_2, T_2) be two weakly strong solutions of (2.11)–(2.17) on the time interval $[0, \mathcal{T}]$ with p_{s_1}, p_{s_2}, and initial data $((v_0)_1, (T_0)_1)$, $((v_0)_2, (T_0)_2)$, respectively.

Define $v = v_1 - v_2$, $T = T_1 - T_2$, $p_s = p_{s_1} - p_{s_2}$. Then v, T, p_s satisfy the following system

$$\frac{\partial v}{\partial t} - \triangle v - \frac{\partial^2 v}{\partial z^2} + (v_1\cdot\nabla)v + (v\cdot\nabla)v_2 + W(v_1)\frac{\partial v}{\partial z} + W(v)\frac{\partial v_2}{\partial z}$$
$$+ fk\times v + \nabla p_s - \int_{-1}^{z}\nabla T dz' = 0, \tag{4.1}$$

$$\frac{\partial T}{\partial t} - \triangle T - \frac{\partial^2 T}{\partial z^2} + (v_1\cdot\nabla)T + (v\cdot\nabla)T_2 + W(v_1)\frac{\partial T}{\partial z} + W(v)\frac{\partial T_2}{\partial z} = 0, \tag{4.2}$$

$$v|_{t=0} = (v_0)_1 - (v_0)_2, \tag{4.3}$$

$$T|_{t=0} = (T_0)_1 - (T_0)_2, \tag{4.4}$$

$$\frac{\partial v}{\partial z} = 0, \quad \frac{\partial T}{\partial z} = -\alpha_s T \ \text{ on } \Gamma_u, \tag{4.5}$$

$$\frac{\partial v}{\partial z} = 0, \quad \frac{\partial T}{\partial z} = 0 \text{ on } \Gamma_b, \tag{4.6}$$

$$v\cdot\vec{n} = 0,\ \frac{\partial v}{\partial \vec{n}}\times\vec{n} = 0,\ \frac{\partial T}{\partial \vec{n}} = 0 \text{ on } \Gamma_l. \tag{4.7}$$

We take the inner product of equation (4.1) with v in $L^2(\Omega)\times L^2(\Omega)$ and obtain

$$\frac{1}{2}\frac{d|v|_2^2}{dt} + \int_\Omega |\nabla v|^2 + \int_\Omega |v_z|^2$$
$$= -\int_\Omega ((v_1\cdot\nabla)v + W(v_1)\frac{\partial v}{\partial z})\cdot v - \int_\Omega (v\cdot\nabla)v_2\cdot v - \int_\Omega W(v)\frac{\partial v_2}{\partial z}\cdot v$$
$$- \int_\Omega (fk\times v + \nabla p_s)\cdot v + \int_\Omega (\int_{-1}^{z}\nabla T dz')\cdot v. \tag{4.8}$$

With integration by parts,

$$\int_\Omega [(v_1\cdot\nabla)v + W(v_1)\frac{\partial v}{\partial z}]\cdot v = 0. \tag{4.9}$$

By integration by parts, Hölder inequality, Young inequality and the interpolation inequalities, we get

$$\begin{aligned}|\int_\Omega (v\cdot\nabla)v_2\cdot v| &= |\int_\Omega [v_2\cdot(v\cdot\nabla)v + v_2\cdot v\mathrm{div}v]|\\ &\le \varepsilon\int_\Omega |\nabla v|^2 + c|\tilde{v}_2|_4^2|v|_2^{\frac{1}{2}}\|v\|^{\frac{3}{2}} + c\|\bar{v}_2\|^2_{L^4(M)}|v|_2\|v\|\\ &\le 2\varepsilon\int_\Omega(|\nabla v|^2+|v_z|^2) + c(\|\bar{v}_2\|^4_{L^4(M)} + |\tilde{v}_2|_4^8)|v|_2^2. \end{aligned}\tag{4.10}$$

By Hölder inequality, Young inequality, Minkowski inequality and the interpolation inequalities, we obtain

$$\begin{aligned}&|\int_\Omega W(v)\frac{\partial v_2}{\partial z}\cdot v|\\ \le&\int_M (\int_{-1}^0 |\nabla v|dz\int_{-1}^0|v_{2z}||v|dz)\\ \le&2\varepsilon\int_\Omega|\nabla v|^2 + c[(|v_{2z}|_2^2+1)\int_\Omega|\nabla v_{2z}|^2 + |v_{2z}|_2^4 + |v_{2z}|_2^2]|v|_2^2. \end{aligned}\tag{4.11}$$

We derive from (4.8)–(4.11)

$$\begin{aligned}&\frac{1}{2}\frac{d|v|_2^2}{dt} + \int_\Omega|\nabla v|^2 + \int_\Omega|v_z|^2\\ \le&4\varepsilon\int_\Omega(|\nabla v|^2+|v_z|^2) + \varepsilon|\nabla T|_2^2\\ &+c[1+\|\bar{v}_2\|^4_{L^4(M)} + |\tilde{v}_2|_4^8 + (|v_{2z}|_2^2+1)|\nabla v_{2z}|_2^2 + |v_{2z}|_2^4]|v|_2^2. \end{aligned}\tag{4.12}$$

By taking the inner product of equation (4.2) with T in $L^2(\Omega)$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d|T|_2^2}{dt} + \int_\Omega|\nabla T|^2 + \int_\Omega|T_z|^2 + \alpha_s|T|_{z=0}|_2^2\\ =&-\int_\Omega[(v_1\cdot\nabla)T + W(v_1)\frac{\partial T}{\partial z}]T - \int_\Omega T(v\cdot\nabla)T_2 - \int_\Omega W(v)\frac{\partial T_2}{\partial z}T. \end{aligned}\tag{4.13}$$

Similarly to (4.12),

$$\begin{aligned}&\frac{1}{2}\frac{d|T|_2^2}{dt} + \int_\Omega|\nabla T|^2 + \int_\Omega|T_z|^2 + \alpha_s|T|_{z=0}|_2^2\\ \le&3\varepsilon\int_\Omega(|\nabla v|^2+|v_z|^2) + 3\varepsilon\int_\Omega(|\nabla T|^2+|T_z|^2) + c(|T_2|_4^8|v|_2^2\\ &+c[|T_2|_4^8 + (|T_{2z}|_2^2+1)|\nabla T_{2z}|_2^2 + |T_{2z}|_2^4 + |T_{2z}|_2^2]|T|_2^2. \end{aligned}\tag{4.14}$$

From (4.12) and (4.14), and choosing ε small enough, we obtain

$$\begin{aligned}&\frac{d(|v|_2^2+|T|_2^2)}{dt} + \int_\Omega|\nabla v|^2 + \int_\Omega|v_z|^2 + \int_\Omega|\nabla T|^2 + \int_\Omega|T_z|^2 + \alpha_s|T|_{z=0}|_2^2\\ \le&c[1+|T_2|_4^8 + \|\bar{v}_2\|^4_{L^4(M)} + |\tilde{v}_2|_4^8 + (|v_{2z}|_2^2+1)|\nabla v_{2z}|_2^2 + |v_{2z}|_2^4]|v|_2^2\\ &+c[|T_2|_4^8 + (|T_{2z}|_2^2+1)|\nabla T_{2z}|_2^2 + |T_{2z}|_2^4 + |T_{2z}|_2^2]|T|_2^2. \end{aligned}\tag{4.15}$$

By Gronwall inequality, Theorem 1.1 and (4.15), we prove Theorem 1.2. □

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