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DIVERGENT SOLUTION TO THE NONLINEAR SCHRÖDINGER EQUATION WITH THE COMBINED POWER-TYPE NONLINEARITIES*

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Abstract In this paper, we consider the Cauchy problem for the nonlinear Schrödinger equation with combined power-type nonlinearities, which is mass-critical/supercr-itical, and energy-subcritical. Combing Du, Wu and Zhang' argument with the variational method, we prove that if the energy of the initial data is negative (or under some more general condition), then the H^1-norm of the solution to the Cauchy problem will go to infinity in some finite time or infinite time.

Keywords Nonlinear Schrödinger equation, combined power-type nonlinearities, blow-up.

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1. Introduction

In this paper, we study the blow-up phenomenon for the following nonlinear Schrödinger equation (NSE) with combined power-type nonlinearities,

$$i\partial_t u + \Delta u = \mu_1 |u|^{p_1} u + \mu_2 |u|^{p_2} u, \tag{1.1}$$

and the initial data

$$u(0,x) = u_0(x) \in H^1(R^N). \tag{1.2}$$

Here $(t,x) \in R^N$, $\mu_1 \in R, \mu_2 < 0$ and the powers p_1, p_2 satisfy that

$$0 < p_1 < p_2, \frac{4}{N} < p_2 < \frac{4}{N-2}.$$

It is well-known that equation (1.1) has the energy conservation

$$\begin{aligned} E(u(t)) := &\int |\nabla u(t,x)|^2 \, dx + \frac{2\mu_1}{p_1+2} \int |u(t,x)|^{p_1+2} dx \\ &+ \frac{2\mu_2}{p_2+2} \int |u(t,x)|^{p_2+2} dx = E(u_0), \end{aligned}$$

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and the mass conservation

$$M(u(t)) := \int |u(t,x)|^2\, dx = M(u_0).$$

Under some suitable conditions of the initial data, these two conservation laws give the uniform bound of the H^1-solution which is dependent on the parameters μ_1, μ_2 and p_1, p_2.

The problem (1.1)–(1.2) was studied by several authors. In particular, by virtue of fixed point argument together with the Strichartz estimates for the linear propagator, Cazenave [5] obtained the local well-posedness of the solution. In [22], Tao etc proved the global well-posedness and scattering for general H^1 initial data, when $\mu_2 > 0$ (the defocusing case) and $0 < p_1 < p_2 \leq 1 + \frac{4}{N-2}$. In the focusing case, some additional restrictions on the initial data should be enforced to obtain the global theory and scattering, see for instances [16].

Throughout this paper, we focus on the blow-up solution of the problem (1.1)–(1.2). For the nonlinear schrödinger equation with single nonlinearity

$$i\partial_t u + \Delta u = -|u|^p u, \quad \frac{4}{N} < p < \frac{4}{N-2},$$

it can be seen that for the finite variance data ($xu_0 \in L^2(R^N)$), if further the energy of the initial data is negative, then the solution must blow up in finite time. This was proved by Glassey in [10]. Reducing the finite variance condition, Ogawa and Tsutsumi [17] gained the similar results for the radial initial data. But for general H^1 initial data, the results still remain open now. However, some weaken version of the corresponding results were considered by researchers. Glangetas and Merle [9] proved that for general H^1 data with negative energy, the corresponding solution must blow up in finite or infinite time, in the cases of mass-critical/mass-supercritical and energy subcritical (that is, $\frac{4}{N} \leq p < \frac{4}{N-2}$). According to the concentration-compactness method developed by Kenig and Merle [14], Holmer and Roudenko [13] generalized the results in [9]. They considered the special case of $p = 2, N = 2$, and showed that if the initial data satisfied

$$E(u_0)M(u_0) < E(R)M(R) \quad \text{and} \quad ||\nabla u_0||_{L^2}||u_0||_{L^2} < ||\nabla R||_{L^2}||R||_{L^2},$$

where R was the corresponding ground state of the equation, then the solution must blow up in finite or infinite time. See some others but similar results in [3,12]. Recently, Holmer and Roudenko's proof was simplied by Du etc [6], which also contains some results on energy-supercritical case.

In this paper, we use the argument in [6] to study the finite or infinite time blow-up results for the nonlinear schrödinger equation with combined power-type nonlinearities (1.1). Equation (1.1) is more complicated than the equation with single nonlinearity, since the scaling invariance is not valid for equation (1.1). Now, we first introduce some notations that we need in the following.

Denote

$$J_\omega(u) := \int |\nabla u|^2 dx + \omega \int |u|^2 dx + \frac{2\mu_1}{p_1+2} \int |u|^{p_1+2} dx + \frac{2\mu_2}{p_2+2} \int |u|^{p_2+2} dx, \tag{1.3}$$

$$K(u) := 8 \int |\nabla u|^2 dx + \frac{4\mu_1 p_1 N}{p_1+2} \int |u|^{p_1+2} dx + \frac{4\mu_2 p_2 N}{p_2+2} \int |u|^{p_2+2} dx. \tag{1.4}$$

Then one may find that

$$J_\omega(u) = E(u) + \omega M(u).$$

So for the solution u of (1.1), $J_\omega(u(t))$ is conserved along the time.

Moreover, let Q_ω be the radial ground state of the following elliptic equation

$$-\Delta\psi + \omega\psi + \mu_1\psi^{p_1+1} + \mu_2\psi^{p_2+1} = 0, \qquad \omega > 0. \tag{1.5}$$

Usually, we define the ground state of (1.5) to be the solution of (1.5), Q_ω, which satisfies

$$J_\omega(Q_\omega) \le J_\omega(\psi), \qquad \text{for any solution } \psi \in H^1(\mathcal{R}^N)\backslash\{0\} \text{ of (1.5).}$$

It can be asserted that the ground state Q_ω always exists when $\mu_2 < 0$, we can refer to [1, 21]. Thus (1.1) has the solitary solution $e^{i\omega t}Q_\omega$.

Now, we state our main theorem in this paper.

Theorem 1.1. *Let $\mu_2 < 0$, $\frac{4}{N} < p_2 < \frac{4}{N-2}$, moreover, μ_1 and p_1 satisfy one of the following conditions,*

(a) $\mu_1 \ge 0$ and $0 < p_1 \le \frac{4}{N}$;

(b) $\mu_1 < 0$ and $\frac{4}{N} < p_1 < p_2$.

Suppose that $u_0 \in H^1(R^N)$ and

$$J_\omega(u_0) < J_\omega(Q_\omega), K(u_0) < 0. \tag{1.6}$$

In addition, assume that the interval $(-T_*, T_*)$ (T_* may be infinity) be the maximal lifespan of the corresponding solution u. Then the solution u of (1.1) must blow up in finite or infinite time, that is,

$$\sup_{t\in(-T_*,T_*)} ||u(t)||_{H^1} = +\infty.$$

The conditions (a), (b) in Theorem 1.1 are employed in the variation argument. They are not shown to be sharp for blow-up solution in this paper. However, it is more general than the exciting results. Now we will try to show some rationality on these conditions. For this purpose, let us have a look at the stable and unstable theories of the standing wave solution $e^{i\omega t}Q_\omega$, which are much related to the blow-up theories. In particular, in the case of $0 < p_1 \le \frac{4}{N} < p_2 < \frac{4}{N-2}$, it was proved by Ohta [19] that the standing wave $e^{i\omega t}Q_\omega$ was unstable for any $\omega > 0$ in one dimensional case when $\mu_1 \ge 0$, $\mu_2 < 0$ (that is, condition (a)); however it was concluded in [7, 18–20] that the standing wave $e^{i\omega t}Q_\omega$ was stable when ω was small enough (unstable when ω was large enough) when $\mu_1 < 0, \mu_2 < 0$. On the other hand, in the case of $0 < \frac{4}{N} < p_1 < p_2 < \frac{4}{N-2}$, it was shown by Berestycki and Cazenave [2] that the standing wave $e^{i\omega t}Q_\omega$ was unstable for any $\omega > 0$ when $\mu_1 < 0$, $\mu_2 < 0$ (that is, condition (b)), see Ohta [19] for the results in one dimensional case when $\mu_1 > 0$, $\mu_2 < 0$.

Under some suitable assumptions on p_1, p_2 and

$$J_\omega(u_0) < J_\omega(Q_\omega), K(u_0) > 0,$$

one may assert that the solution is global existence in time, and scattering at least when the initial data is radial, see [8, 11, 15, 16, 23] for some special cases.

Furthermore, with the help of the virial identities, the authors in [16, 22] established some finite time blow-up results for the combined power-type nonlinear Schödinger equation when the initial data were radial. While, for the general data without radial assumption, there is another difficulty which can not be solved by a simple usage of the virial identities. As mentioned above, such finite time blow-up results also remain open for single nonlinearity. In [6], the authors observed that if the L^2-norm of the initial data u_0 was small enough in the exterior ball B_R, then in a long time L, the L^2-norm of the corresponding solution $u(t)$ was also sufficient small in an exterior ball with a slightly expansion of the radii, where L had a similar size as R. In other word, the small L^2-estimate in the exterior ball kept being in a long time compared with the radii of the ball. Using this observation, Glassey's argument can lead to the contradiction when the radii goes to be large. According to this result in our discussion and some further variation argument, we can prove our theorem.

The structure of this paper is as follows. First, we show that under the condition of (1.6), the quantity $K(u(t))$ is strictly away from zero along the flow. Second, we give the specific expression of the local virial identities. Using these identities, we prove the L^2-estimate in the exterior ball. Then we apply this smallness estimate to another virial identities and give the intensive analysis that each remainder term is small enough when the time lies in a suitable long region. Finally, our main results follow by the standard Glassey's argument.

2. A variational lemma

In this section, we recall some variational results. Let

$$J_\omega(\varphi) = E(\varphi) + \omega M(\varphi).$$

We have the following rigidity results of Q_ω.

Proposition 2.1. *Under the conditions of Theorem 1.1, the following holds:*

$$J_\omega(Q_\omega) = \min\{J_\omega(\phi) : \phi \in H^1(R^N) \setminus \{0\}, K(\phi) = 0\}.$$

Proof. Let the manifold

$$\mathcal{M}_\omega = \{\phi \in H^1(\mathcal{R}^N)\backslash\{0\} : J_\omega(\phi) = d, K(\phi) = 0\},$$

where

$$d = \min\{J_\omega(\phi) : \phi \in H^1(\mathcal{R}^N)\backslash\{0\} : K(\phi) = 0\}.$$

We will prove this proposition in two steps.

Step 1: $\mathcal{M}_\omega$ is non-empty.

It is sufficient to prove that for the minimizer sequence $\{\phi_n\}$ which satisfies

$$J_\omega(\phi_n) \to d, \quad K(\phi_n) = 0, \tag{2.1}$$

there exists $x_n \in \mathcal{R}^N$, $\phi \in \mathcal{M}_\omega$ such that

$$\phi_n \to \phi \quad \text{in} \quad H^1(\mathcal{R}^N).$$

To get the assertion, we need the following lemmas.

Lemma 2.1. *Under the condition (a) or (b),*

$$J_\omega(\phi) - \frac{1}{8}K(\phi) \geq 0.$$

Proof. By the definitions of $J_\omega(\phi)$ and $K(\phi)$ in (1.3) and (1.4), we have

$$J_\omega(\phi) - \frac{1}{8}K(\phi) = \omega \int |\phi|^2 dx + \frac{2\mu_1}{p_1+2}\big(1 - \frac{p_1 N}{4}\big) \int |u|^{p_1+2} dx + \frac{2\mu_2}{p_2+2}\big(1 - \frac{p_2 N}{4}\big) \int |u|^{p_2+2} dx. \tag{2.2}$$

Note that no matter in Condition (a) or Condition (b), we have

$$\frac{2\mu_1}{p_1+2}\big(1 - \frac{p_1 N}{4}\big) \geq 0. \tag{2.3}$$

Moreover, since $p_2 > \frac{4}{N}$ and $\mu_2 < 0$, it follows that

$$\frac{2\mu_2}{p_2+2}\big(1 - \frac{p_2 N}{4}\big) > 0. \tag{2.4}$$

Hence $J_\omega(\phi) - \frac{1}{8}K(\phi) \geq 0.$ □

Lemma 2.2. *Suppose that $K(\phi) < 0$, then*

$$J_\omega(\phi) - \frac{1}{8}K(\phi) > d.$$

Proof. From (1.4), we have

$$K(\lambda\phi) = \lambda^2 \Big[8 \int |\nabla u|^2 dx + \lambda^{p_1} \frac{4\mu_1 p_1 N}{p_1+2} \int |u|^{p_1+2} dx + \lambda^{p_2} \frac{4\mu_2 p_2 N}{p_2+2} \int |u|^{p_2+2} dx \Big].$$

Note that $1 < p_1 < p_2$, it can be concluded that when $\lambda > 0$ and λ is sufficient small, $K(\lambda\phi) > 0$. So if $K(\phi) < 0$, there exists $\lambda < 1$, such that

$$K(\lambda\phi) = 0.$$

Thanks to the definition of d, it yields

$$J_\omega(\lambda\phi) \geq d,$$

and thus

$$J_\omega(\lambda\phi) - \frac{1}{8}K(\lambda\phi) \geq d.$$

On the other hand, by Lemma 2.1 and $\lambda < 1$, we have

$$\begin{aligned} J_\omega(\lambda\phi) - \frac{1}{8}K(\lambda\phi) &= \lambda^2 \int |\phi|^2 dx + \lambda^{p_1+2} \cdot \frac{2\mu_1}{p_1+2}\big(1 - \frac{p_1 N}{4}\big) \int |u|^{p_1+2} dx \\ &\quad + \lambda^{p_2+2} \cdot \frac{2\mu_2}{p_2+2}\big(1 - \frac{p_2 N}{4}\big) \int |u|^{p_2+2} dx \\ &< J_\omega(\phi) - \frac{1}{8}K(\phi), \end{aligned}$$

thus,

$$J_\omega(\phi) - K(\phi) > d,$$

which completes the proof. □

Lemma 2.3. *(i) $d \neq 0$; (ii) For the minimizer sequence $\{\phi_n\}$ satisfying* (2.1)*, there exists $q \in (2, p_2+2)$ such that* $\sup_n ||\phi_n||_{L^q} > 0$.

Proof. (i) If not, there exists $\{\phi_n\} \in H^1(R^N)\backslash\{0\}$, such that

$$J_\omega(\phi_n) \to 0, K(\phi_n) = 0.$$

Then

$$J_\omega(\phi_n) - \frac{1}{8}K(\phi_n) \to 0 \quad \text{as} \quad n \to \infty,$$

and thus from Lemma 2.1, we obtain for any $q \in [2, p_2+2]$,

$$||\phi_n||_{L^q} \to 0 \quad \text{as} \quad n \to \infty.$$

If $p_1 \leq \frac{4}{N}$, then $\mu_1 \geq 0$. Note that $2 + \frac{4}{N} \leq p_2 + 2 < 2^* = \frac{2N}{N-2}$, so there exist $q \in (2, p_2+2)$ and a constant $C_{N,p_2} > 0$, such that

$$\int |\phi_n|^{p_2+2}dx \leq C_{N,p_2}||\phi_n||_{L^q}^{p_2}||\nabla\phi_n||_{L^2}^2.$$

Hence for large n, we have

$$\begin{aligned} K(\phi_n) &\geq 8\int |\nabla\phi_n|^2 dx + \frac{4\mu_2 p_2 N}{p_2+2}\int |\phi_n|^{p_2+2}dx \\ &\geq 8||\nabla\phi_n||_{L^2}^2 - C_{N,p_2}||\phi_n||_{L^q}^{p_2}||\nabla\phi_n||_{L^2}^2 \\ &\geq 4||\nabla\phi_n||_{L^2}. \end{aligned}$$

Else if $p_1 > \frac{4}{N}$, then $\mu_1 < 0$. Along the same line of the above process, we can get

$$\begin{aligned} K(\phi_n) \geq& 8\int |\nabla\phi_n|^2 dx + \frac{4\mu_1 p_1 N}{p_1+2}\int |\phi_n|^{p_1+2}dx + \frac{4\mu_2 p_2 N}{p_2+2}\int |\phi_n|^{p_2+2}dx \\ \geq& 8||\nabla\phi_n||_{L^2}^2 - C_{N,p_1}||\phi_n||_{L^{q_1}}^{p_1}||\nabla\phi_n||_{L^2}^2 - C_{N,p_2}||\phi_n||_{L^{q_2}}^{p_2}||\nabla\phi_n||_{L^2}^2 \\ \geq& 4||\nabla\phi_n||_{L^2}, \end{aligned}$$

for some $q_1, q_2 \in (2, p_2+2)$.

Combining with $K(\phi_n) = 0$, it tells us that

$$\nabla\phi_n \equiv 0,$$

and thus $\phi_n \equiv 0$, which is a contradiction. Therefore, $d \neq 0$.

(ii) We prove this by contradiction argument.

If for any $q \in (2, p_2+2)$,

$$||\phi_n||_{L^q} \to 0 \quad \text{as} \quad n \to \infty.$$

Then the similar argument as in (i) also yields that

$$K(\phi_n) = 0$$

and thus $\phi_n = 0$. This is a contradiction. □

Now we return to prove the conclusion in *Step 1.*

Since $\sup_n ||\phi_n||_{L^q} > 0$ and ϕ_n is bounded in $H^1(R^N)$, there exist $\{y_n\}$ and $\phi \in H^1(R^n)\backslash\{0\}$ such that

$$\phi_n(-y_n) \rightharpoonup \phi \quad \text{in} \quad H^1(R^n).$$

Moreover, by Fotou's estimate, when $n \to \infty$, we have

$$||\phi_n||_{L^r}^r - ||\phi_n - \phi||_{L^r}^r \to ||\phi||_{L^r}^r, \quad \text{for} \quad r = 2,\ p_1 + 2,\ p_2 + 2,$$

and

$$||\nabla\phi_n||_{L^2}^2 - ||\nabla\phi_n - \nabla\phi||_{L^2}^2 \to ||\nabla\phi||_{L^2}^2.$$

Therefore,

$$J_\omega(\phi_n) - J_\omega(\phi_n - \phi) \to J_\omega(\phi), \tag{2.5}$$

$$K(\phi_n) - K(\phi_n - \phi) \to K(\phi). \tag{2.6}$$

Combing (2.5) with (2.6), it follows that

$$P_\omega(\phi_n) - P_\omega(\phi_n - \phi) \to P_\omega(\phi), \tag{2.7}$$

where $P_\omega(f) = J_\omega(f) - \frac{1}{8}K(f)$.

Since $P_\omega(\phi_n) \to d$, and from Lemma 2.1, $P_\omega(\phi_n - \phi) \geq 0, P_\omega(\phi) > 0$ (since $\phi \neq 0$), we have for large n,

$$P_\omega(\phi) \leq d \text{ and } P_\omega(\phi_n - \phi) \leq d. \tag{2.8}$$

Thus by Lemma 2.2, we know

$$K(\phi_n - \phi) \geq 0 \text{ and } K(\phi) \geq 0.$$

But $K(\phi_n) = 0$. Hence, from (2.2), we get

$$K(\phi) = 0 \text{ and } K(\phi_n - \phi) \to 0. \tag{2.9}$$

By the definition of d, we obtain

$$J_\omega(\phi) \geq d.$$

However, from (2.8), $P_\omega(\phi) \leq d$, we have

$$J_\omega(\phi) = P_\omega(\phi) + \frac{1}{8}K(\phi) \leq d.$$

These yield that $J_\omega(\phi) = d$. Thus, from (2.5), $J_\omega(\phi_n - \phi) \to 0$. Therefore, together with (2.9), we get

$$P_\omega(\phi_n - \phi) \to 0 \quad \text{as} \quad n \to \infty.$$

Combining with (2.2), it implies that

$$||\phi_n - \phi||_{L^2} + ||\phi_n - \phi||_{L^{p_1+2}} + ||\phi_n - \phi||_{L^{p_2+2}} \to 0 \quad \text{as} \quad n \to \infty.$$

Again, with the help of $J_\omega(\phi_n - \phi) \to 0$, we further have

$$||\phi_n - \phi||_{H^1} \to 0 \quad \text{as} \quad n \to \infty.$$

This proves that $\mathcal{M}_\omega$ is nonempty.

Step 2. $d = J_\omega(Q_\omega)$.

For any $\phi \in \mathcal{M}$, by the Larange multiplier theory, there exists $\lambda \in R$, such that

$$J'_\omega(\phi) = \lambda K'_\omega(\phi).$$

Let $\varphi = 4N\phi + 8x \cdot \nabla\phi$, then we take L^2 product with φ on the both two sides and get

$$\big\langle J'_\omega(\phi), \varphi\big\rangle = \lambda\big\langle K'(\phi), \varphi\big\rangle. \tag{2.10}$$

Here $\langle f, g\rangle = \int_{R^N} f(x)\bar{g}(x)\,dx$. On one hand,

$$\big\langle J'_\omega(\phi), \varphi\big\rangle = K(\phi) = 0.$$

On the other hand, direct calculation also gives

$$\big\langle K'(\phi), \varphi\big\rangle = 16\int |\nabla\phi|\,dx + \frac{2N^2p_1^2}{p_1+2}\int |\phi|^{p_1+2}\,dx + \frac{2N^2p_2^2}{p_2+2}\int |\phi|^{p_2+2}\,dx.$$

Note that $K(\phi) = 0$, we further get

$$\big\langle K'(\phi), \varphi\big\rangle = \frac{2\mu_1}{p_1+2}\Big(1 - \frac{4}{Np_1}\Big)\int |\phi|^{p_1+2}\,dx + \frac{2\mu_2}{p_2+2}\Big(1 - \frac{4}{Np_2}\Big)\int |\phi|^{p_2+2}\,dx.$$

Employing (2.3) and (2.4), we have

$$\big\langle K'(\phi), \varphi\big\rangle < 0.$$

Now together with (2.10), we obtain that $\lambda = 0$. Hence, $J'_\omega(\phi) = 0$. That is, ϕ obeys (1.5). So by the definition of ground state, we have $d \geq J_\omega(Q_\omega)$. But on the other hand, for any function ϕ solves (1.5), we have $K(\phi) = 0$. Thus by the definition of d again, $d \leq J_\omega(Q_\omega)$. Therefore, $d = J_\omega(Q_\omega)$. So we complete the proof of the proposition. □

Denote

$$\mathcal{A} = \{\varphi \in H^1(\mathcal{R}^d) : J_\omega(\varphi) < J_\omega(Q_\omega), K(\varphi) < 0\}.$$

Then by Proposition 2.1, we have the following lemma. Here we use the argument in [6] to prove the lemma.

Lemma 2.4. *If $u_0 \in \mathcal{A}$, then there exists $\beta_0 = \beta_0(u_0) > 0$, such that*

$$\sup_{t\in(-T_*,T^*)} K(u(t)) < -\beta_0. \tag{2.11}$$

Proof. We argue for contradiction, then by continuity, there exists $\{t_n\} \subset (-T_*, T^*)$, s.t.

$$K(u(t_n)) \uparrow 0 \quad \text{as } n \to \infty.$$

One may find that there exists $\lambda_n \downarrow 1$, such that

$$K(\lambda_n u(t_n)) = 0.$$

With the help of Lemma 2.1, $J_\omega(Q_\omega) = \min\{J_\omega(\varphi) : \varphi \in H^1(\mathcal{R}^n \backslash \{0\}), K(\varphi) = 0\}$. Thus we have

$$J_\omega(\lambda_n u(t_n)) \geq J_\omega(Q_\omega). \tag{2.12}$$

Note that by mass and energy conservation laws, it asserts that

$$\begin{aligned} J_\omega(\lambda_n u(t_n)) = \lambda_n^2 \Big[& J_\omega(u(t_n)) + (\lambda_n^{p_1} - 1)\frac{2\mu_1}{p_1+2}\int |u(t_n)|^{p_1+2}dx \\ & + (\lambda_n^{p_2} - 1)\frac{2\mu_2}{p_2+2}\int |u(t_n)|^{p_2+2}dx\Big] \\ = & \lambda_n^2 J_\omega(u(t_n)) + \lambda_n^2 A_n \\ = & \lambda_n^2 J_\omega(u_0) + \lambda_n^2 A_n, \end{aligned}$$

where

$$A_n = (\lambda_n^{p_1} - 1)\frac{2\mu_1}{p_1+2}\int |u(t_n)|^{p_1+2}dx + (\lambda_n^{p_2} - 1)\frac{2\mu_2}{p_2+2}\int |u(t_n)|^{p_2+2}dx.$$

Since $\lambda_n \downarrow 1$, and $K(u(t_n)) \leq 0$, we know that the second term of A_n is negative and strictly larger in absolute value than the first term when n is large enough. Therefore, for sufficient large n, we have $A_n \leq 0$. This implies that

$$J_\omega(\lambda_n u(t_n)) \leq \lambda_n^2 J_\omega(u_0) \to J_\omega(u_0) = J_\omega(Q_\omega) + \big(J_\omega(u_0) - J_\omega(Q_\omega)\big),$$

which contradicts with (2.12) since $J_\omega(u_0) - J_\omega(Q_\omega) < 0$. □

3. Local virial identity

In this section, we give the local virial identities of the problem (1.1)–(1.2). This is the key point of the proof for our main theorem. In this paper, as in [6] we need the specific expression of them.

3.1. Local virial identity

Let ϕ be a smooth, radial function. Denote

$$I(t) = \int \phi(x)|u(t,x)|^2 dx.$$

Then we have the following virial identities.

Lemma 3.1. *Let u be the solution of* (1.1)*, then*

$$I'(t) = 2Im\int \nabla\phi\bar{u}\cdot\nabla u dx; \tag{3.1}$$

$$\begin{aligned} I''(t) = & 4\int \frac{\phi'}{r}|\nabla u|^2 dx + 4\int (\frac{\phi''}{r^2} - \frac{\phi'}{r^3})|x\cdot\nabla u|^2 dx \\ & - \frac{2\mu_1 p_1}{p_1+2}\int (\phi'' + (N-1)\frac{\phi'}{r})|u|^{p_1+2}dx \\ & - \frac{2\mu_2 p_2}{p_2+2}\int (\phi'' + (N-1)\frac{\phi'}{r})|u|^{p_2+2}dx - \int \Delta^2\phi|u|^2 dx. \end{aligned} \tag{3.2}$$

Proof. From (1.1), we have

$$u_t = i(\Delta u - \mu_1 |u|^{p_1} u - \mu_2 |u|^{p_2} u).$$

Thus,

$$\begin{aligned} I'(t) =& 2Re \int \phi(x)\overline{u(t,x)} u_t(t,x) dx, \\ =& 2Re \int \phi(x)\overline{u(t,x)} i(\Delta u - \mu_1 |u|^{p_1} u - \mu_2 |u|^{p_2} u) dx \\ =& -2Im \int \phi(x)\overline{u(t,x)} (\Delta u - \mu_1 |u|^{p_1} u - \mu_2 |u|^{p_2} u) dx \\ =& 2Im \int \nabla\phi \overline{u(t,x)} \cdot \nabla u(t,x) dx. \end{aligned}$$

Moreover, when ϕ is a radial function,

$$\begin{aligned} I''(t) =& 2Im \int \nabla\phi \bar{u}_t \cdot \nabla u dx + 2Im \int \nabla\phi \bar{u} \cdot \nabla u_t dx \\ =& 4Im \int \nabla\phi \bar{u}_t \cdot \nabla u dx - 2Im \int \Delta\phi \bar{u} \cdot u_t dx \\ =& -4Im \int \nabla\phi \cdot \nabla u \, i(\Delta u - \mu_1 |u|^{p_1} u - \mu_2 |u|^{p_2} u) dx - 2Im \int \Delta\phi \bar{u} \cdot u_t dx \\ =& -4Re \int \nabla\phi \cdot \nabla u \Delta u dx - \frac{4\mu_1}{p_1+2} \int \Delta\phi |u|^{p_1+2} dx - \frac{4\mu_2}{p_2+2} \int \Delta\phi |u|^{p_2+2} dx \\ & - 2Im \int \Delta\phi \bar{u} \, i(\Delta u - \mu_1 |u|^{p_1} u - \mu_2 |u|^{p_2} u) dx \\ =& 4Re \int \partial_j \partial_k \phi \, \partial_j u \partial_k \bar{u} dx - \frac{2\mu_1 p_1}{p_1+2} \int \Delta\phi |u|^{p_1+2} dx - \frac{2\mu_2 p_2}{p_2+2} \int \Delta\phi |u|^{p_2+2} dx \\ & - \int \Delta^2 \phi |u|^2 dx \\ =& 4 \int \frac{\phi'}{r} |\nabla u|^2 dx + 4 \int (\frac{\phi''}{r^2} - \frac{\phi'}{r^3}) |x \cdot \nabla u|^2 dx - \frac{2\mu_1 p_1}{p_1+2} \int (\phi'' + (N-1)\frac{\phi'}{r}) \cdot \\ & |u|^{p_1+2} dx - \frac{2\mu_2 p_2}{p_2+2} \int (\phi'' + (N-1)\frac{\phi'}{r}) |u|^{p_2+2} dx - \int \Delta^2 \phi |u|^2 dx, \end{aligned}$$

which completes the proof. □

Remark 3.1. If u is a radial function, we have

$$\frac{x \cdot \nabla u}{r} = |\nabla u|^2.$$

This gives very powerful control of the terms which are supported outside of the ball, see [17] for example. But for general u without radial, we can not use this benefit.

3.2. Blow-up for $\Sigma-$data

The finite time blow-up for $\Sigma-$data ($\Sigma = \{\psi : x\psi \in L^2(R^N)\}$) can be established by the standard Glassey's argument [10] and the virial identities above. In fact, when $xu_0 \in L^2(R^N)$, then choosing $\phi = |x|^2$ in Lemma 3.1 we have

$$I''(t) = 8\int |\nabla u|^2 dx + \frac{2\mu_1 p_1 N}{p_1+2}\int |u|^{p_1+2}dx + \frac{2\mu_2 p_2 N}{p_2+2}\int |u|^{p_2+2}dx = K(u).$$

Suppose that there exists a positive constant β_0, such that

$$\sup_{t\in(-T_*,T^*)} K(u(t)) \leq -\beta_0, \tag{3.3}$$

then

$$I''(t) \leq -\beta_0. \tag{3.4}$$

Note that according to Lemma 2.4, the condition is valid if the initial data u_0 satisfies $J_\omega(u_0) < J_\omega(Q_\omega)$ and $K(u_0) < 0$. Now (3.4) implies that for any $T \in (-T_*, T^*)$,

$$I(T) \leq -\beta_0 T^2 + I'(0)|T|.$$

Since $I'(0) \leq CR$ for some constant $C > 0$, it can be asserted that

$$I(T) \leq -\beta_0 T^2 + CR|T|.$$

Therefore, if $T^* = \infty$, let $T \to \infty$, we get the contradiction since $I(T) \geq 0$. Moreover, from the above inequality, we know

$$I(t) = O(T^* - t), \text{ for any } t \uparrow T^*.$$

Hence, by Hardy's inequality, we get

$$M(u) \leq C||xu_0||_2 \cdot ||\nabla u||_{L^2} \leq (T^*-t)||\nabla u||_{L^2}. \tag{3.5}$$

This gives us the estimate,

$$||\nabla u(t)||_2 \geq \frac{||u_0||^2_{L^2}}{T^*-t}, \ \forall t \in [0, T^*). \tag{3.6}$$

Some similar results also hold in negative time.

4. L^2-estimate in the exterior ball

Let ϕ be the smooth and radial function, such that

$$\phi = \begin{cases} 0, & 0 \leq r \leq R/2, \\ 1, & r \geq R, \end{cases} \tag{4.1}$$

and there exists $C_k > 0$, such that $0 \leq \phi \leq 1, \phi^{(k)} \leq C_k R^{-k}$, for any integer $k \geq 0$.

In this section, we prove the following L^2-estimate in the exterior ball. Let η_0 be a small positive constant which will be decided later.

Lemma 4.1. *Suppose that*

$$A_0 \triangleq \sup_{t\in R^+} ||\nabla u(t)||_{L^2}$$

and let $T_\triangle = \eta_0 R/(2C_1 m_0 A_0)$. *Then for any* $t \leq T_\triangle$,

$$\int_{|x|>R} |u(t,x)|^2 dx \leq \eta_0 + O_R(1).$$

Proof. By using Lemma 3.1, we have

$$\frac{d}{dt}\int \phi(x)|u(t,x)|^2dx = 2Im\int \nabla\phi\cdot\nabla u\bar{u}dx.$$

Thus,

$$\int \phi(x)|u(t,x)|^2dx = \int \phi(x)|u_0|^2dx + 2\int_0^t Im\int \nabla\phi\cdot\nabla u\bar{u}dxds.$$

Since

$$\begin{aligned}|\int_0^t Im\int \nabla\phi\cdot\nabla u\bar{u}dxds| &\le \int_0^t ||\nabla\phi||_\infty||\nabla u||_2||u||_2ds\\ &\le C_1\frac{t\sqrt{m_0}}{R}A_0\\ &\le \eta_0,\end{aligned}$$

for any $t \le T_\triangle$. Moreover,

$$\int \phi(x)|u_0|^2dx \le \int_{|x|\ge R/2}|u_0|^2dx = O_R(1),$$

and

$$\int \phi(x)|u(t,x)|^2dx \ge \int_{|x|\ge R}|u(t,x)|^2dx.$$

Therefore, thanks to the two estimates above, we obtain

$$\int_{|x|\ge R}|u(t,x)|^2dx \le O_R(1)+\eta_0. \tag{4.2}$$

□

4.1. Proof of Theorem 1.1

Proof. Now we define a new smooth, radial weight function ψ, such that

$$\psi = \begin{cases} r, & 0\le r\le R,\\ 0, & r\ge 2R,\end{cases} \tag{4.3}$$

and

$$0\le\psi\le r^2, \psi^{''}\le 2, \psi^{(4)}\le\frac{4}{R^2}.$$

Then by the virial identities in Lemma 3.1 (with ϕ being replaced by ψ),

$$\begin{aligned}I^{''}(t) =& 8K(u(t))\\ &+4\int(\frac{\psi'}{r}-2)|\nabla u|^2\,dx + 4\int\Big(\frac{\psi''}{r^2}-\frac{\psi'}{r^3}\Big)|x\cdot\nabla u|^2\,dx \qquad (4.4)\\ &-\int\Big[\psi''+(N-1)\frac{\psi'}{r}-2N\Big]\Big(\frac{2p_1}{p_1+2}|u|^{p_1+2}+\frac{2p_2}{p_2+2}|u|^{p_2+2}\Big)\,dx \qquad (4.5)\\ &-\int\Delta^2\psi|u|^2\,dx. \qquad (4.6)\end{aligned}$$

Now we first consider (4.4). The situation here is almost the same as the one in [6]. We find that

$$(4.4) \leq 0. \tag{4.7}$$

Indeed, if $\frac{\psi''}{r^2} - \frac{\psi'}{r^3} \leq 0$, we have the claim since $\frac{\psi'}{r} - 2 \leq 0$.

Else if $\frac{\psi''}{r^2} - \frac{\psi'}{r^3} \geq 0$, then

$$\begin{aligned}(4.4) &\leq 4\int \big(\frac{\psi'}{r} - 2 + \psi'' - \frac{\psi'}{r}\big)|\nabla u|^2 dx \\ &= 4\int (\psi'' - 2)|\nabla u|^2 dx \leq 0.\end{aligned}$$

We turn to consider (4.5). Since

$$\begin{aligned}(4.5) &\leq C\int_{|x|\geq R} |u|^{p_1+2} dx + C\int_{|x|\geq R} |u|^{p_2+2} dx \\ &\leq C||\nabla u||_2^{(p_1+2)(1-\alpha_1)} ||u||_{L^2(|x|>R)}^{(p_1+2)\alpha_1} + C||\nabla u||_2^{(p_2+2)(1-\alpha_2)} ||u||_{L^2(|x|>R)}^{(p_2+2)\alpha_2},\end{aligned}$$

where

$$\alpha_1 = \frac{N}{p_1+2} - \frac{N}{2} + 1,\ \alpha_2 = \frac{N}{p_2+2} - \frac{N}{2} + 1.$$

Then by the boundedness $||\nabla u(t)||_{L^2}$, we obtain

$$(4.5) \leq \tilde{C}\big(||u||_{L^2(|x|>R)}^{(p_1+2)\alpha_1} + ||u||_{L^2(|x|>R)}^{(p_2+2)\alpha_2}\big)$$

for some $\tilde{C} = \tilde{C}(A_0, N, p_1, p_2) > 0$. Thanks to Lemma 4.1, by choosing η small enough and R large enough, there exists $\alpha > 0$, such that

$$(4.5) \leq \tilde{C}\eta_0^\alpha + O_R(1). \tag{4.8}$$

At last, we consider (4.6). Since

$$|\Delta^2 \psi| \lesssim \frac{1}{R^2}.$$

We have

$$(4.6) \leq \frac{1}{R^2}||u_0||_{L^2} = O_R(1). \tag{4.9}$$

Combing the estimates (4.7)–(4.9), it follows that

$$I''(t) \leq 8K(u(t)) + \tilde{C}\eta_0^\alpha + O_R(1).$$

Therefore, according to Lemma 2.4, we have

$$I''(t) \leq -8\beta_0 + \tilde{C}\eta_0^\alpha + O_R(1).$$

Choosing η_0, such that

$$\tilde{C}\eta_0^\alpha = 2\beta_0,$$

and choosing large R_0, such that for $R \geq R_0, O_R(1) \leq 2\beta_0$, one has

$$I''(t) \leq -4\beta_0.$$

Henceforth, we obtain the same estimate as (3.4). Then by the same process in Section 3.2, we can complete the proof. □

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