ACCESSIBILITY OF PARTIALLY HYPERBOLIC ENDOMORPHISMS WITH 1D CENTER-BUNDLES*

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Abstract We prove that partially hyperbolic endomorphisms with one dimensional center-bundles and non-trivial unstable bundles are stably accessible. And there is residual subset $\mathcal{R}$ of partially hyperbolic volume preserving endomorphisms with one dimensional center-bundles such that every $f \in \mathcal{R}$ is stably accessible. In the end, we prove the accessibility of Gan's example.

Keywords Accessibility, partially hyperbolic endomorphisms, unstable manifolds, orbit space, universal covering space.

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1. Introduction

Our work is motivated by Gan-Shaobo's example and problem:

Problem 1.1. *Let f be endomorphism on 2-torus $\mathbb{R}^2/\mathbb{Z}^2$:*

$$f(x,y) = (2x, y + \lambda \sin 2\pi x), \ \lambda \neq 0.$$

Is f stably ergodic?

Here, $f \in \mathrm{Diff}_m^2(M)$ is said to be stably ergodic, if there exists C^1 neighborhood $\mathcal{U} \ni f$ such that every $g \in \mathcal{U} \cap \mathrm{Diff}_m^2(M)$ is ergodic. The above example is a partially hyperbolic endomorphism. For stable ergodicity of partially hyperbolic diffeomorphisms, there are many important progress in recent years [4, 6, 8]. It is mostly motivated by Pugh-Shub's famous stable ergodicity conjecture:

Conjecture 1.1 (Pugh-Shub [11]). *On any compact manifold, ergodicity holds for an open and dense set of C^2 volume preserving partially hyperbolic diffeomorphisms.*

In the same paper, Pugh-Shub posed a programme of the conjecture:

Conjecture 1.2 (Pugh-Shub). *Accessibility holds for an open and dense subset of C^2 partially hyperbolic diffeomorphisms, volume preserving or not.*

Conjecture 1.3 (Pugh-Shub). *A partially hyperbolic C^2 volume preserving diffeomorphism wth the essential accessibility property is ergodic.*

For a partially hyperbolic endomorphism f, it is said to be accessible if any two points can be connected by a path consists of stable manifolds and unstable

manifolds. In partially hyperbolic diffeomorphisms with one dimensional center, Didier [5] proves the openness of accessibility.

Theorem 1.1 ([5])**.** *Accessibility is C^1-open among partially hyperbolic diffeomorphisms with one dimensional center bundle.*

In partially hyperbolic volume-preserving diffeomorphisms with one dimensional center bundle, Hertz-Hertz-Ures's [8] prove the density of accessibility and consequently solve the stable ergodicity conjecture.

Theorem 1.2 ([8])**.** *Stable accessibility/ergodicity is C^r-dense among volume preserving partially hyperbolic diffeomorphisms with one dimensional center bundle, for all $r \geq 1/r \geq 2$.*

In much important works [3, 4, 6, 8, 12]on stable ergodicity conjecture, the accessibility of dynamics is the key tool to prove the ergodicity of dynamics.

In this paper, we mainly generalize Didier's [5] and Hertz-Hertz-Ures's [8] partial results to endomorphisms. In partially hyperbolic endomorphism with one dimensional center, we characteristic the openness and density of accessibility. In fact, we want to completely generalize Hertz-Hertz-Ures's work [8] to partially hyperbolic endomorphisms with 1-dimensional center-bundles. Now, we are preparing the work on how to prove ergodicity by the accessibility [7].

2. definitions and notations

2.1. orbit spaces

We firstly recall the orbit spaces of endomorphisms. It is formerly said to be inverse limits Let $f : M \hookleftarrow$ be a regular($\det(Df) \neq 0$) endomorphism. The orbit space $\tilde{M}$ of f is the set consists with every orbit of f:

$$\tilde{M} = \{\tilde{x} = (x_i)_{-\infty}^{+\infty}, f(x_i) = x_{i+1}\}.$$

The left-shift map $\tilde{f}$ is naturally induced by f, which satisfying $\pi_0 \tilde{f} = f\pi_0$. Here, π_0 is the projection of 0-coordinate from $\tilde{M}$ to M.

Let $\hat{M}$ be the universal covering of M, and $\pi : \hat{M} \to M$ be the covering map. The corresponding lifting map $\hat{f}$ satisfying $\pi\hat{f} = f\pi$.

By the commutative diagram, one has that an orbit of $\hat{x}$ in $\hat{M}$ induces an orbit of x in M. For simplicity, the induced orbit is still denoted by $\hat{x}$. Then, we have this natural map

$$\varphi : \hat{M} \to \tilde{M}, \varphi(\hat{x}) = \tilde{x}.$$

Generally speaking, φ is not injective. For example f_A, $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

The following lemma is easy to check.

Lemma 2.1. *$\varphi(\hat{M})$ is a dense subset of $\tilde{M}$.*

2.2. partially hyperbolic endomorphisms

Now, we introduce the definition of partially hyperbolic endomorphisms, which is very similar with diffeomorphisms'.

Definition 2.1. f is said to be partially hyperbolic, if $T\hat{M}$ has an invariant sub-bundle splitting $T\hat{M} = E^s \oplus E^c \oplus E^u$ satisfying that there exists $l \in \mathbb{N}$ such that for any $\hat{x} \in \hat{M}$, any triple of unit vectors $\theta^s \in E^s(\hat{x})$, $\theta^c \in E^c(\hat{x})$, and $\theta^u \in E^u(\hat{x})$, one has that

- $||D\hat{f}^l(\hat{x})(\theta^s)|| < \frac{1}{2}$.
- $||D\hat{f}^l(\hat{x})(\theta^u)|| > 2$.
- $||D\hat{f}^l(\hat{x})(\theta^s)|| < \frac{1}{2}||D\hat{f}^l(\hat{x})(\theta^c)|| < \frac{1}{4}||D\hat{f}^l(\hat{x})(\theta^u)||$.

Also, we introduce the definition of partially hyperbolic endomorphisms under orbit spaces. Formally, we give a bundle structure $T\tilde{M}$ on the orbit space $\tilde{M}$: for any $\tilde{x} = (x_i)_{-\infty}^{+\infty} \in \tilde{M}$, $T_{\tilde{x}}M = T_{x_0}M$. It is not difficult to show that the above definition is equivalent with the following.

Definition 2.2. f is said to be partially hyperbolic, if $T\tilde{M}$ has an invariant sub-bundle splitting $T\tilde{M} = E^s \oplus E^c \oplus E^u$ satisfying that there exists $l \in \mathbb{N}$ such that for any $\tilde{x} \in \tilde{M}$, any triple of unit vectors $\theta^s \in E^s(\tilde{x})$, $\theta^c \in E^c(\tilde{x})$, and $\theta^u \in E^u(\tilde{x})$, one has that

- $||Df^l(x)(\theta^s)|| < \frac{1}{2}$.
- $||Df^l(x)(\theta^u)|| > 2$.
- $||Df^l(x)(\theta^s)|| < \frac{1}{2}||Df^l(x)(\theta^c)|| < \frac{1}{4}||Df^l(x)(\theta^u)||$.

It is not difficult to show that for any $x \in M$, $E^s(x)$ does not depends on the choice of orbits of x. But generally speaking, E^c and E^u both depends on the choice of orbits. For instance, Gan's example in our paper. It is standard argument to show that

Lemma 2.2. *For any partially hyperbolic endomorphisms f, it has the following properties:*

- *The partially hyperbolic splitting is unique.*
- *The splitting has uniform transversality: the angles between E^s, E^c, and E^u are uniformly bounded from zero.*
- *The splitting is continuous: $E^\sigma(\tilde{x})$ depends continuously on the orbit $\tilde{x}$, $\sigma = s, c, u$.*
- *Partially hyperbolic is persistent: there exists open neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$, g is partially hyperbolic.*

The proof is essentially same with diffeomorphisms' [2, Appendix B], just by replacing points by orbits. So, we omit its' proof.

For partially hyperbolic systems, we introduce the strong stable and unstable manifolds on orbit space $\tilde{M}$, M, and $\hat{M}$. For any orbit $\tilde{x} \in \tilde{M}$, the local strong unstable manifolds $W_\delta^u(\tilde{f}, \tilde{x})$ in orbit space $\tilde{M}$ is the set

$$\{\tilde{y} : d(\tilde{f}^{-n}(\tilde{y}), \tilde{f}^{-n}(\tilde{x})) < \delta, \text{ and } \exists N \ s.t. \ \frac{d(\tilde{f}^{-n}(\tilde{y}), \tilde{f}^{-n}(\tilde{x}))}{||D\tilde{f}^{-n}|_{E^c(\tilde{x})}||} < \frac{1}{2}, \forall n > N\};$$

the local strong unstable manifolds $W_\delta^u(f, \tilde{x})$ in M is the set

$$\{y : \exists \tilde{y} \ s.t. \ d(\tilde{f}^{-n}(\tilde{y}), \tilde{f}^{-n}(\tilde{x})) < \delta, \text{ and } \exists N \ s.t. \ \frac{d(\tilde{f}^{-n}(\tilde{y}), \tilde{f}^{-n}(\tilde{x}))}{||D\tilde{f}^{-n}|_{E^c(\tilde{x})}||} < \frac{1}{2}, \forall n > N\}.$$

For a point $\hat{x}$ in the universal $\hat{M}$ of M, its' local strong unstable manifolds $W^u_\delta(\hat{f}, \hat{x})$ (in the universal $\hat{M}$ of M) is the set

$$\{\hat{y} : d(\hat{f}^{-n}(\hat{y}), \hat{f}^{-n}(\hat{x})) < \delta, \text{ and } \exists N \ s.t. \ \frac{d(\hat{f}^{-n}(\hat{y}), \hat{f}^{-n}(\hat{x}))}{||D\hat{f}^{-n}|_{E^c(\hat{x})}||} < \frac{1}{2}, \forall n > N\}.$$

By these definitions, we can see that

$$W^u_\delta(f, \tilde{x}) = \pi_0(W^u_\delta(\tilde{f}, \tilde{x})).$$

Also, it is not difficult to deduce that for any $\hat{x} \in \hat{M} \subset \tilde{M}$,

$$W^u_\delta(f, \hat{x}) = \pi(W^u_\delta(\hat{f}, \hat{x})).$$

Similarily, we can define local strong stable manifolds $W^s_\delta(\tilde{f}, \tilde{x})$ in orbit space $\tilde{M}$, $W^s_\delta(f, x)$ in space M, or $W^s_\delta(\hat{f}, \hat{x})$ in the universal $\hat{M}$ of M. Also, we have that

$$W^s_\delta(f, x) = \pi_0(W^s_\delta(\tilde{f}, \tilde{x})) = \pi(W^s_\delta(\hat{f}, \hat{x})).$$

In the end, we recall the classic stable and unstable manifolds theory:

Theorem 2.1 ([9, 10]). *Let f be a C^1 endomorphism on compact manifold M. Then, there exists δ_1 such that for any $\delta \leq \delta_1$, the local strong stable $W^s_\delta(\tilde{f}, \tilde{x})$ $(W^s_\delta(f, x), W^s_\delta(\hat{f}, \hat{x}))$ and the local strong unstable manifold $W^u_\delta(\tilde{f}, \tilde{x})$ $(W^u_\delta(f, \tilde{x}), W^u_\delta(\hat{f}, \hat{x}))$ are tangent with E^s and E^u respectively, and vary continuously with respect to the orbit or point.*

2.3. the accessibility classes

The accessibility is a powerful tool to check the ergodicity.

Definition 2.3. For a partially hyperbolic endomorphisms f, the accessibility class $A_f(x)$ is the set consists with the points which have su-paths from these points to the point x. Here the su-path is a concatenation of finitely many sub-paths, each of which lies entirely in a local stable/unstable manifold.

Definition 2.4. For a partially hyperbolic endomorphisms f, f is said to be accessible if f has only one accessibility class.

Note that the strong stable manifolds don't depend on the choice of orbits. Then if a partially hyperbolic endomorphism has no unstable bundles, it can't be accessible. However, we can use the diverse orbits of the point for non-inverse systems to get the accessibility. So, $\dim E^s$ might be zero.

Let $\mathrm{PH}^{1,1}(M)$ be the set of all non-inverse partially hyperbolic endomorphism on M with $\dim E^c \leq 1$ and $\dim E^u \geq 1$.

Let $r \geq 1, \mathrm{PH}^{r,1}_m(M)$ be the set of all C^r partially hyperbolic volume preserving endomorphism on M with $\dim E^c \leq 1$ and $\dim E^u \geq 1$.

And let $A_f(x, \delta, l)$ is the set consists with the points which have su-orbits at l-steps with δ-length, from these to the point x. For any subset $X \subset M$, $A_f(X, \delta, l) \doteq \bigcup_{x \in X} A_f(x, \delta, l)$.

2.4. Other notations

It is not difficult to show that there exists δ_f such that for any $\hat{x} \in B(\hat{x}, \delta)$, π is diffeomorphic on $B(\hat{x}, \delta_f)$. Let $\delta = \min\{\delta_1, \delta_f\}$, here δ_1 is the number in theorem 2.1.

Through the paper, let $W^c_\delta(\hat{f}, \hat{x})$ be a smooth curve tangent with $E^c(\hat{M})$ and centered at $\hat{x}$. For any $\hat{X} \subset \hat{M}$, let

$$W^\sigma_\delta(\hat{f}, \hat{X}) = \bigcup_{\hat{x} \in \hat{X}} W^\sigma_\delta(\hat{f}, \hat{x}), \ \sigma = s, u.$$

For simplicity, let

$$W^{cu}_\delta(\hat{f}, \hat{x}) = W^u_\delta(\hat{f}, W^c_\delta(\hat{f}, \hat{x})).$$

Similarly, we have $W^{us}_\delta(\hat{f}, \hat{x})$.

Let $W^\sigma_\delta(f, \hat{x}) = \pi(W^\sigma_\delta(\hat{f}, \hat{x}))$, $\sigma = cu, us, c, s, u$.

In the end, we emphasize that

- The defined sub-manifolds in this section belonging to which space$(M, \hat{M}, \tilde{M})$ is determined by the map$(f, \hat{f}, \tilde{f})$ in the notation.

- $\hat{x}$ in the sub-manifolds of $\hat{f}$ stands for a point in the universal space $\hat{M}$ of M. And $\hat{x}$ in the sub-manifolds of f stands for an orbit in $\varphi(\hat{M}) \subset \tilde{M}$. In other unaccounted cases, $\hat{f}$ is a point of the universal space $\hat{M}$ of M.

- x is a point in M, $\hat{x}$ is a point in $\hat{M}$ or an orbit in $\varphi(\hat{M}) \subset \tilde{M}$, and $\tilde{x}$ is a point in $\tilde{M}$.

3. main results

Partially hyperbolic endomorphisms f is said to be stably accessible, if there is a C^1-neighborhood $\mathcal{U}$ such that every $g \in \mathcal{U}$ is accessible.

Theorem 3.1. *Let $f \in \mathrm{PH}^{1,1}(M)$. If f is accessible, then f is stably accessible.*

The next result characteristics the density of accessibility in $\mathrm{PH}^{r,1}_m(M)$.

Theorem 3.2. *There is a residual subset $\mathcal{R}$ of $\mathrm{PH}^{r,1}_m(M)$ such that for any $f \in \mathcal{R}$, f is accessible, $r > 1$.*

For Gan's example, We can prove that it is stably accessible.

Proposition 3.1. *Let f be a endomorphism on 2-torus $\mathbb{R}^2/\mathbb{Z}^2$:*

$$f(x, y) = (2x, y + \lambda \sin 2\pi x), \ \lambda \neq 0.$$

Then, f is stably accessible.

4. the structure of accessibility classes

In the section, we focus endomorphisms in $\mathrm{PH}^{1,1}(M)$. The following is a basic and important lemma in the characteristic of accessibility classes. It happens in a small neighborhood. So its' proof is the same with diffeomorphisms' [8, Proposition A.4.].

Lemma 4.1. *Let $A_f(x)$ be a accessibility class of f, the following conditions are equivalent:*

- *$A_f(x)$ is open.*

- *$A_f(x)$ contains a curve transversal with E^s and E^u.*

- *Interior$(A_f(x)) \neq \emptyset$.*

Then, for any open set $U \subset M$, $A_f(U, r, l)$ is open.

Let $U(f)$ be the union of open accessibility classes, and $\mathrm{T}(f) = M \backslash U(f)$. Now, we characteristic the accessibility classes:

Lemma 4.2. *Let $A_f(a)$ be a accessibility class of f. Then,*

1. *$A_f(a) \subset \mathrm{T}(f)$ iff there exists ε such that for any $x \in A_f(a)$, any orbit $\hat{x} \in \hat{M}$, any $y \in W_\varepsilon^u(f, \hat{x})$, $z \in W_\varepsilon^s(f, x)$, and any orbit $\hat{z}'$ of z, $W_\delta^s(f, y) \cap W_\delta^u(f, \hat{z}') \neq \emptyset$. Particularly for the case $\dim E^s = 0$, $A_f(a) \subset \mathrm{T}(f)$ iff there exists ε such that for any $x \in A_f(a)$, any orbit $\hat{x}, \hat{x}' \in \hat{M}$, $W_\varepsilon^u(f, \hat{x}') \subset W_\delta^u(f, \hat{x})$.*

2. *If there exists two orbits $\hat{x}$ and $\hat{x}'$ of x satisfying $E^s(x) + E^u(\hat{x}') + E^u(\hat{x}) = T_x M$, then $A_f(x)$ is open.*

3. *If $\dim(E^s) = 0$, then $A_f(a) \subset \mathrm{T}(f)$ iff for any $x \in A_f(a)$, the unstable space of y does not depend on the choice of orbits.*

The first characteristic is paralled with the case of diffeomorphisms [8].
Proof.   1. Note that $\varphi \hat{M}$ is dense in $\tilde{M}$, and the continuity of stable/unstalbe manifolds. Then "$\Longleftarrow$" is obvious. Now we deduce the left half. Let ε satisfying that for any $W_\delta^{cu}(f, \hat{x})$ and any $y \in B(x, 2\varepsilon)$, $W_\delta^{cu}(f, \hat{x}) \cap W_\delta^s(f, y) \neq \emptyset$. On the contrary, we suppose there exists $x \in A_f(a)$, $y \in W_\varepsilon^u(f, \hat{x})$, $z \in W_\varepsilon^s(f, x)$, and an orbit $\hat{z}'$ of z such that, $W_\delta^s(f, y) \cap W_\delta^u(f, \hat{z}') = \emptyset$. Take a curve $W_\delta^c(f, \hat{z}')$ tangent with $E^c(\hat{f})$, and $W_\delta^{cu}(\hat{z}')$. Let $w = W_\delta^{cu}(f, \hat{z}') \cap W_\delta^s(f, y)$. Consider $\gamma = W_\delta^{cu}(f, \hat{z}') \cap W_\delta^s(W_\varepsilon^u(f, \hat{x}))$. Since the stable manifolds vary continuously, γ is a continuous curve. Note that $\{z, w\} \subset \gamma \subset A_f(x)$. Then, by the continuity of unstable manifolds,

$$A_f(x) \supset W_\delta^u(\gamma) \supset \text{open sub-interval of } W_\delta^c(f, \hat{z}').$$

Then by Lemma 4.1, $A_f(x, \delta, 4)$ is open. Contradiction!
In the above proof, for the case $\dim E^s = 0$, $W_\delta^s(f, y) = y, W_\delta^s(f, x) = x = z, W_\delta^s(W_\varepsilon^u(f, \hat{x})) = W_\varepsilon^u(f, \hat{x})$.
2. It is not difficult to be deduced by the above item.
3. By the first characteristic of accessibility, we can see that $A_f(a) \subset \mathrm{T}(f)$ iff there exists ε such that, for any $x \in A_f(a)$, and any two orbits $\hat{x}'$ and $\hat{x}$, $W_\varepsilon^u(f, \hat{x}') \subset W_\delta^u(f, \hat{x})$. Then, for $A_f(a) \subset \mathrm{T}(f)$, we have that for any $x \in A_f(a)$, the unstable space of x does not depend on the choice of orbits.
On the other hand, suppose that for any $x \in A_f(a)$, the unstable space of x does not depend on the choice of orbits. Then for any $x \in A_f(a)$ and any orbit $\{x_i\}$ of x, the unstable space of x_i also does not depend on the choice of orbits, $i < 0$. On the contrary, we suppose that there exists $x \in A_f(a)$ and two orbits $\hat{x}'$ and $\hat{x}$, $W_\varepsilon^u(f, \hat{x}') \not\subset W_\delta^u(f, \hat{x})$. Let $\varphi_i : B(x_i, \delta) \to M$ satisfying $f\varphi_i = id$ and $\varphi_i(x_i) = x_{i-1}$. Take a non-trivial E^c-curve $I^c = [y, z]$ satisfying $y \in W_\varepsilon^u(f, \hat{x}')$ and $z \in W_\delta^u(f, \hat{x})$. Take the corresponding triangle $\Delta(x, y, z)$, which consists with the

segment of $W_\delta^u(f, \hat{x})$, the segment of $W_\delta^u(f, \hat{x})'$, and the segment $I^c = [y, z]$. By the contracting on the segment of $W_\delta^u(f, \hat{x})$ and the segment of $W_\delta^u(f, \hat{x})'$,

$$\varphi_{-1}(\Delta(x, y, z)) \subset B(x_{-1}, \delta).$$

Inductively, we have that

$$\varphi_{-n} \cdots \varphi_{-1}(\Delta(x, y, z)) \subset B(x_{-n}, \delta), \ n > 0.$$

But, by the last property of partial hyperbolicity (dominated splitting), we see that

$$\ell(\varphi_{-n} \cdots \varphi_{-1}(I^c)) >> \ell(\varphi_{-n} \cdots \varphi_{-1}(W_\delta^u(f, \hat{x})')) + \ell(\varphi_{-n} \cdots \varphi_{-1}(W_\delta^u(f, \hat{x}))), n \to \infty.$$

By this contradiction, we get that if for any $y \in A_f(a)$, the unstable space of y does not depend on the choice of orbits, then $A_a(f) \subset \mathrm{T}(f)$.   □

Remark 4.1.   • Obviously the condition in the second argument is robust. We believe that it is also necessary.

- For the case dim $E^s > 0$, the latter condition in the third argument is not necessary. For example, $f \times Id$, here f is non-special Anosov endomorphism, and Id is identity.

For any $\hat{x}$, let the unstable manifold $W^u(\hat{f}, \hat{x})$ of $\hat{x}$ is the set

$$\bigcup_{n>0} \hat{f}^n(W_\delta^u(\hat{f}, \hat{f}^{-n}\hat{x})).$$

For any $\hat{x}$, let the stable manifold $W^s(\hat{f}, \hat{x})$ of $\hat{x}$ is the set

$$\bigcup_{n<0} \hat{f}^n(W_\delta^s(\hat{f}, \hat{f}^{-n}\hat{x})).$$

And let $W^{us}(f, \hat{x}) = \pi(W^{us}(\hat{f}, \hat{x}))$.

The following is a directed consequence of the above lemma.

Corollary 4.1. *For any $x \in \mathrm{T}(f)$ and any $\hat{x} \in \hat{M}$, we have that $A_f(x) = W^{us}(f, \hat{x})$.*

Remark 4.2. Similarily, we can define stable manifold on M. But generally speaking, such stable manifold is not connected! In fact, for any $\hat{x} \in \hat{M}$, $W^s(f, \hat{x})$ is exactly the connected component of $W^s(f, x)$.

By the continuity of stable/unstable manifolds and the dominated splitting, we can see that the phenonema of 4-legs in the proof of above lemma is robust. Then by the lemma4.2, we get the following corollary.

Corollary 4.2. *If $A_f(a)$ is open, then there exists open neighborhood $\mathcal{U}$, open subset $U \subset M$ and $x \in A_f(a)$ such that for any $g \in \mathcal{U}$, $A_g(x, \delta, 4) \supset U$.*

5. C^r openness and density of accessibility

Based on the characteristic of accessibility classes in above section, we prove the openness and density of accessibility. At first, we give its' openness.

Theorem 5.1. *If f is accessible, then f is stable accessibility.*

Under the above charateristic of accessibility classes, the proof is exactly same with diffeomorphisms' [5]. For completion, we give the proof.

Proof.   By Corollary 4.2, there exists open neighborhood $\mathcal{U}$ of f, $x \in M$, and open subset U of M such that for any $g \in \mathcal{U}$, $A_g(x, r, 4) \supset U$. By the accessibility of f, obviously $\bigcup_{n>0} A_f(x, r, n) = M$. Naively, $\bigcup_{n>0} A_f(U, r, n) = M$. By Lemma 4.1, $A_f(U, r, n)$ is an open subset for any n. So, there exists N such that $A_f(x, r, N) \supseteq A_f(U, r, N-4) = M$. By the continuity of stable/unstable manifolds, there exists neighborhood $\mathcal{U}_1$ of f such that for any $g \in \mathcal{U}_1$, $A_g(x, r, N+4) \supset A_g(U, r, N) = M$.   □

Among these systems, the accessibility is also a C^r-density property.

Theorem 5.2. *There is a residual subset $\mathcal{R}$ of $\mathrm{PH}_m^{r,1}(M)$ such that for any $f \in \mathcal{R}$, f is accessible, $r \geq 1$.*

From now on, we focus endomorphism $f \in \mathrm{PH}_m^{1,1}(M)$. By the volume-preserving, it is not difficult to see that $|det(Df_x)| \geq 1$ for any $x \in M$. Then for the case $\dim E^u = 0$, one has that E^c is uniformly expanding, i.e., f is Anosov. For Anosov volume-preserving endomorphisms, they are born accessible and ergodic [1, 13, 14]. So, we suppose $\dim E^u > 0$. Then, we should use the diverse orbits of the point for non-inverse systems.

To prove the density, we modify the strategy in [8]:

- Firstly prove that there is no periodical point in T_f by a perturbation lemma. In our non-diffeomorphic case, the perturbation is totally different.

- In our non-diffeomorphic case, $\mathrm{T}(f)$ is not totally invariant subset, we find another totally invariant subset $\mathrm{T}_*(f) \subseteq \mathrm{T}(f)$ replacing it. Then analyze on the boundary of $\mathrm{T}_*(f)$ to find an invariant accessible class $W^{us}(x)$; and by the hyperbolicity on $W^{us}(x)$, we find a periodical point. The analysis on the boundary of $\mathrm{T}(f)$ is a local argument in [8]. In our non-diffeomorphic case, we projective the local center/stable/unstable manifold structure of $\hat{f}$ into (M, f), and follow the idea of [8] to analyze the dynamics in a small neighborhood in M.

We first give the key lemma.

Lemma 5.1. *There is residual subset $\mathcal{R}$ of $\mathrm{PH}_m^{r,1}(M)$ such that for any $f \in \mathcal{R}$ and any periodical point p of f, $A_f(p)$ is open.*

The following is its perturbation form.

Lemma 5.2. *Let p be a hyperbolic periodical point of f. Then, there is an open subset $\mathcal{U} \subset \mathrm{PH}_m^{r,1}(M)$ such that $f \in \overline{\mathcal{U}}$ and for any $g \in \mathcal{U}$, $A_g(p_g)$ is open.*

Proof.   In [8], they deal with the case of diffeomorphisms. For the case of non-diffeomorphism, our strategy is to make perturbation in arbitraily small neighborhood of the periodic orbit such that, it preserves this periodic orbit and there is another orbit having different unstable space from the periodic orbit's.

If $A_f(p)$ is open, then there is an open subset $\mathcal{U}$ such that for any $g \in \mathcal{U}$, $A_g(p_g)$ is open, by Corollary 4.2. Suppose $A_f(p) \subset \mathrm{T}(f)$. Let $\hat{p}$ be the periodical orbit of p. By Lemma 4.2, one has that for any $\tilde{p}$ of p,

$$E^{su}(f, p) \doteq E^s(f, p) \oplus E^u(f, \hat{p}) = E^s(f, p) \oplus E^u(f, \tilde{p}).$$

Take $p_{-1} \in f^{-1}(p)$ satisfying $p_{-1} \notin \hat{p}$. Suppose $\hat{p} \cap B(x, \varepsilon_0) = \emptyset$. For any small $\theta > 0$ satisfying $\theta^r < \max\{\delta_f, \varepsilon_0\}$, let $\varepsilon = \theta^r$. Then, there exists an orbit $\{p_i\}_{-\infty}^{+\infty}$ of p satisfying $p_i \notin B(p_{-1}, \varepsilon), i \neq -1$. Then by the lemma xx, there exists $\theta\text{-}C^r$-perturbation g of f such that

- $g(x) = f(x)$ outside $B(p_{-1}, \varepsilon) - p_{-1}$.
- under the basis of $\{E^s(f, \tilde{f}^{-1}(\tilde{p})), E^c(f, \tilde{f}^{-1}(\tilde{p})), E^u(f, \tilde{f}^{-1}(\tilde{p}))\}$,

$$
D_{p_{-1}}g = \begin{pmatrix} I^s & 0 & 0 \\ 0 & I^c & \theta \\ 0 & 0 & I^u \end{pmatrix} D_{p_{-1}}f,
$$

here I^σ stands the identity on the space $E^\sigma(f, \tilde{f}^{-1}(\tilde{p}))$, $\sigma = s, c, u$, and θ in the matrix is the vector $(\theta, \theta, \cdots, \theta)$.

Obviously g preserve the orbits $\hat{p}$ and $\tilde{p}$ of f. Since $p_i \notin B(p_{-1}, \varepsilon), i \neq -1$, $E^{su}(f, \hat{p}) = E^{su}(g, \hat{p})$, and $E^u(f, \tilde{f}^{-1}(\tilde{p})) = E^u(g, \tilde{f}^{-1}(\tilde{p}))$. By the derivative Dg on the point p_{-1}, we have that

$$
E^u(g, \tilde{p}) \not\subset E^{su}(f, \tilde{p}) = E^{su}(f, \hat{p}) = E^{su}(g, \hat{p}).
$$

Then, $E^s(g, p) + E^u(g, \hat{p}) + E^u(g, \tilde{p}) = T_pM$. By Lemma 4.2, one has that $A_g(p) = A_g(p_g)$ is open.   □

The proof of Lemma 5.1 is a common generic argument. For completion, we give the proof.

Proof of Lemma 5.1. Let $\text{Per}^k(f)$ be the union of $k-$perodical points of f, $\mathcal{K}(M)$ be the set consists with all compact subset of M, endowed with Hausdorff distance. Define the map $\Psi : \text{PH}_m^{r,1}(M) \to \text{K}(M)$, $\Psi(f) = \overline{\text{Per}(f)}$. Note that Ψ is lower-continuous. By the classic semi-continuous theorem and Kupa-Smale theorem, we see that there is residual subset $\mathcal{R}_1$ of $\text{PH}_m^{r,1}(M)$ such that for any $f \in \mathcal{R}_1$, f is continuity point of Ψ and $\text{Per}^k(f)$ is hyperbolic.

Take a dense subset $\{f_n\}_{n \geq 1}$ of $\mathcal{R}_1$. Suppose $\text{Per}^k(f_n) = \{p^1, \cdots, p^l\}$. Then, by the above claim and Corollary 4.2, we see that there is an open subset $\mathcal{U}_{n,k,1}$ such that $f_n \in \overline{\mathcal{U}_{n,k,1}}$ and for any $g \in \mathcal{U}_{n,k,1}$, $A_g(p_g^1)$ is open. And by Corollary 4.2 and the above claim again, we deduce that there is an open subset $\mathcal{U}_{n,k,2} \subset \mathcal{U}_{n,k,1}$ such that $f_n \in \overline{\mathcal{U}_{n,k,2}}$ and for any $g \in \mathcal{U}_{n,k,2}$, $A_g(p_g^2)$ is open. Inductively, we have that there exists an open subset $\mathcal{U}_{n,k,l}$ such that $f_n \in \overline{\mathcal{U}_{n,k,l}}$ and for any $g \in \mathcal{U}_{n,k,l}$, $A_g(p_g^i)$ is open, $1 \leq i \leq l$. Let $\mathcal{U}_{n,k} = \mathcal{U}_{n,k,l}$. By the continuity of Ψ in f, we see that for any $g \in \mathcal{U}_{n,k}$ and any periodical points $p_g \in \text{Per}^k(g)$, $A_g(p_g)$ is open. Let $\mathcal{R} = \mathcal{R}_1 \cap (\bigcap_{k \geq 0} (\bigcup_{n \geq 0} \mathcal{U}(g_{n,k})))$. Obviously, for any f in the residual set $\mathcal{R}$ and any periodical point p of f, $A_f(p)$ is open.   □

Now, we prove Theorem 5.2.

Proof of Theorem 5.2. Note $\text{T}(f)$ is not invariant subset. We should find another invariant subset replacing it. By the definition of accessibility class, it is not difficult to show that $f(A(x)) \subset A(fx)$. Then, $f^{-1}(A(fx)) \supset A(x)$. Let

$$
\text{T}_*(f) = \cap_{n \geq 1} f^{-n}(\text{T}(f)), \ U_*(f) = M \backslash \text{T}_*(f).
$$

Since f is local diffeomorphism, $int f^{-1}(A(x)) = \emptyset$ for any $x \subset \mathrm{T}(f)$. By Lemma 4.1, $f^{-1}(\mathrm{T}(f)) \subseteq \mathrm{T}(f)$ and $f^{-1}(\mathrm{T}(f))$ is union of accessibility classes. By the induction, it is not difficult to show that for any positive integer n, $f^{-n}(\mathrm{T}(f)) \subseteq \mathrm{T}(f)$ and $f^{-n}(\mathrm{T}(f))$ is union of accessibility classes. Then, it is not difficult to deduce that,

- f is both negative invariant: $f^{-1}(\mathrm{T}_*(f)) = \mathrm{T}_*(f)$, and positive invariant: $f(\mathrm{T}_*(f)) = \mathrm{T}_*(f)$.
- For any $x \in \mathrm{T}_*(f)$, $A_f(x) \subset \mathrm{T}_*(f)$.
- $\mathrm{T}_*(f) \subseteq \mathrm{T}(f)$ is closed.

By the key lemma 5.1, there exists residual subset $\mathcal{R}$ of $\mathrm{PH}_m^{r,1}(M)$ such that for any $f \in \mathcal{R}$, $\mathrm{T}(f) \cap \mathrm{Per}(f) = \emptyset$. To the contrary, we suppose that there exists $f \in \mathcal{R}$ such that $\mathrm{T}(f) \neq M$. Since f is surjection, $\mathrm{T}_*(f) \neq \emptyset$. Then, it is not difficult to take a closed curve $(\hat{a}, \hat{b}) = \hat{I} \subset W_\eta^c(\hat{f}, \hat{b})$ and let $I = (a, b) = \pi(\hat{I})$ satisfying that

$$a \in \mathrm{T}_*(f), (a, b] \subset U_*(f), \text{ and } \eta < \frac{\delta}{2}.$$

In the following, we projective the local center/stable/unstable manifold structure of $\hat{f}$ onto (M, f), and analyze the dynamics in a small neighborhood in M. Let

$$W_\delta^{us}(\hat{f}, \hat{I}) = \bigcup_{\hat{x} \in \hat{I}} W_\delta^{us}(\hat{f}, \hat{x}).$$

And, let $W_\delta^{us}(f, \hat{I}) = \pi(W_\delta^{us}(\hat{f}, \hat{I}))$. By Lemma 4.1, we can see that $W_\delta^{us}(f, \hat{a}) \subset \partial(W_\delta^{us}(f, \hat{I}))$. By the transversality, it is not difficult to show that the following easy fact:

Fact. There exists ε such that for any $\hat{w} \in B(\hat{a}, \varepsilon)$, $W_\eta^c(f, \hat{w}) \cap W_\delta^{us}(f, \hat{a}) \neq \emptyset$.

Note that $\Omega(f) = M$. Then there exists $\hat{x} \in \subset B(\hat{a}, \varepsilon)$ and let $x = \pi(\hat{x})$ satisfying that

$$\{x, f^k x, f^{k+l} x\} \subset B(a, \varepsilon) \cap W_\delta^{us}(f, \hat{I}), \quad k, l > 0.$$

Let $0 \neq v \in T_x M$. Among $v, Df^k(v), Df^{k+l}(v)$, obviously there are two vectors satisfying that the angle between them is smaller than $\frac{\pi}{2}$. Without loss generation, suppose $\angle(v, Df^k(v)) < \frac{\pi}{2}$. Take $\gamma = [x, y] \subset W_\delta^c(f, \hat{x})$, $y \in W_\delta^{us}(f, \hat{a})$. Now, we check that $f^k y \subset W_\delta^{us}(f, \hat{a})$. Otherwise, we have the following two cases.

Case 1: $f^k(\gamma) \cap W_\delta^{us}(f, \hat{a}) = \emptyset$.

By the above fact, we see that $\ell(f^k(\gamma)) \leq \eta$. Then by the fact $W_\delta^{us}(f, \hat{a}) \subset \partial(W_\delta^{us}(f, \hat{I}))$, we deduce that $f^k(\gamma) \subset W_\delta^{us}(f, (\hat{a}, \hat{b}))$. So, $f^k y \in U_*(f)$ which contradicts the positive invariance of $\mathrm{T}_*(f)$.

Case 2: there exists $z \in (x, y)$ such that $f^k z \in W_\delta^{us}(f, \hat{a})$.

Since $(x, y) \subset U_*(f)$, it contradicts the negative invariance of $\mathrm{T}_*(f)$.

So, $f^k y \in W_\delta^{us}(f, \hat{a})$. Duing to the hyperbolicity on $W^{us}(f, \hat{a})$, we have the following shadowing property:

Claim 5.1. There exists ε such that if $f^n(x) \in W_\varepsilon^{us}(f, \hat{x}) \subset T(f), n > 0$, then there exists a periodical point $p \in W_\delta^{us}(f, \hat{x})$.

By the above claim, there is a periodical point contained in $\mathrm{T}(f)$, which contradicts with $\mathrm{T}(f) \cap \mathrm{Per}(f) = \emptyset$. $\qquad \square$

For the general shadowing property of endomorphisms, Bowen's argument should be still effective on the manifold W^{us}. For completion, we give a trick proof of the above claim.

Proof of Claim 5.1. For simplicity, for any $z \in T(f)$, $W^\sigma(f, \hat{z})$ is denoted by $W^\sigma(z), \sigma = us, u, s$. Suppose x is not periodical. By the uniform expanding on the unstable manifold, we can take ε small enough such that n is big enough and satisfying the following properties:

- For any $z \in M$, $W_\delta^u(z) \subset f^n(W_{\frac{\delta}{2}}^u(f^n z))$.

- Let $\varphi : W_\delta^u(f^n x) \to W_\delta^u(x)$ satisfying $f^n \varphi = id.$ for any $y \in W_\delta^u(f^n x)$, $W_\delta^s(\varphi(y))$ have exactly one point with $W_\delta^u(f^n y)$.

Then, we define a continuous endomorphism h on $W_\delta^u(f^n x)$:

$$h(y) = W_\delta^s(\varphi(y)) \cap W_\delta^u(f^n x).$$

Then, by the classic Brouwer fixed-point theorem , h has a fixed point y, i.e, $\varphi(y) \in W_\delta^s(y)$. By the uniform contraction and the coherence of stable manifolds, we see that $\{f^{mn}(y)\}_{m>0} \subset W_{\text{loc}}^s(y)$ is a Cauchy sequence. Obviously $\lim_{m>1} f^{mn}(y) = p \in W_\delta^{us}(x)$ be a periodical point of f. For the case $\dim E^s = 0$, we can get the periodical point of f by the last deduction. $\square$

6. Gan's example

Now, we check the accessibility of Gan's example.

Proposition 6.1. *Let f be a endomorphism on 2-torus $\mathbb{R}^2/\mathbb{Z}^2$:*

$$f(x, y) = (2x, y + \lambda \sin 2\pi x), \ \lambda \neq 0.$$

Then, f is stably accessible.

Proof. Let $\varphi(x) = \lambda \sin 2\pi x$ be the endomorphism on $\mathbb{T}$. For any $\tilde{z} = (z_n)_{-\infty}^{+\infty} \in \widetilde{\mathbb{T}^2}$, $z_n = (x_n, y_n), n \in \mathbb{Z}$. For any $\tilde{z} \in \widetilde{\mathbb{T}^2}$, we let $E^c(\tilde{z}) = \{(0, t) : t \in \mathbb{R}\}$, and $\mathcal{C}_1^u(\tilde{z}) = \{(t, t\eta) : t \in \mathbb{R}, |\eta| \leq 1\}$. Then for any $\tilde{z} \in \widetilde{\mathbb{T}^2}$, we have that

- $Df(E^c(\tilde{z})) = E^c(\tilde{z})$, and $||Df|_{E^c(\tilde{z})}|| = 1$.
- $Df(\mathcal{C}_1^u(\tilde{z})) \subset Int(\mathcal{C}_1^u(\tilde{z}))$, and $m(Df|_{\mathcal{C}_1^u(\tilde{z})}) > 1$.

Then $E^u(\tilde{z}) = \cap_{n \geq 0} Df^n(\mathcal{C}_1^u)(\tilde{f}^{-n}(\tilde{z}))$. It is not difficult to calculate that for any $\tilde{z} \in \widetilde{\mathbb{T}^2}$,

$$(1, \eta(\tilde{z})) \in E^u(\tilde{z}), \ \eta(\tilde{z}) = \sum_{i=1}^{+\infty} \frac{\varphi'(x_{-i})}{2^i}.$$

Take two orbits $\widehat{(0, t)}$ and $\widetilde{(0, t)}$ of $(0, t)$:

$$\widehat{(0, t)}_n = (0, y_n), \ \widetilde{(0, 0)}_n = (\frac{1}{2^n}, y_n'), \ n < 0.$$

Note that $\varphi'(x) = 2\pi\lambda \cos(2\pi x)$ gets the maximal value at 0. Then, $E^u(\widehat{(0, t)}) \neq E^u(\widetilde{(0, t)})$ for any t. So, by Lemma 4.1, one has that for any t, $A((0, t))$ is open. For

any $(x, y) \in \mathbb{T}^2$ and any orbit $\widehat{(x, y)}$, $W^u(\hat{f}, \widehat{(x, y)})$ is tangent to $\mathcal{C}_1^u(\widehat{(x, y)})$. Then, there exists t such that $(0, t) \in W^u(\hat{f}, (x, y))$. So, for any $(x, y) \in \mathbb{T}^2$, $A((x, y))$ contains $(0, t)$, and then it is open. Then, f is accessible, and stably accessible by Theorem 5.1.

$\square$

Acknowledgments

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