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ALMOST AUTOMORPHIC DYNAMICS OF GENERALIZED LIÉNARD EQUATION*

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Abstract In this study, we focus on the solutions of the Liénard equation being bounded in the future and characterize the almost automorphic, asymptotically almost automorphic, and weighted pseudo almost automorphic dynamics. An example is presented to illustrate the main findings.

Keywords Almost automorophy, asymptotically almost automorphy, weighted pseudo almost automorphy, Liénard equation.

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1. Introduction

The notation of almost automorphy, introduced by Bochner [3] is related to and more general than almost periodicity. For more details about this topic, we refer to the recent books [24, 29], where the authors expound a complete background and recent developments on almost automorphic functions and almost automorphic dynamics. A well know extension of almost automorphy is the notion of asymptotical almost automorphy [23] and weighted pseudo almost automorphy [2]. More details about these concepts and the applications to various types of differential equations and dynamic systems can been found in [24, 25].

This study treats a theme which has been deemed to be of interested by many authors. It is a descendants of a well-known theorem of Massera established in 1950 [19], which states the existence of a periodic solution of a scalar periodic ODE when it is known that a bounded solution exists. The Massera theorem was followed by examples of Opial [26], Zhikov-Levitan (1977) [33], and Johnson [14]. In these examples, a scalar ODE with almost periodic time-dependence admits a bounded solution but does not admit an almost periodic solution; on the other hand it does admit an almost automorphic solution. The Opial example is nonlinear, while the Zhikov-Levitan and Johnson examples are linear and nonhomogeneous. These examples seem to have stimulated the interest of later authors in the topic of the existence of almost automorphic solutions to ODEs whose coefficients have

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almost periodic or almost automorphic time dependence and which admit bounded solutions [29].

It is well-known that, since the pioneer work Massera, this topic has found much progress and has been developed from several different viewpoints. On one hand, there has been an increasing interest in extending this results from periodic functions to functions of various more general classes such as anti-periodic [6], quasi-periodic [27], almost periodic [21, 28], almost automorphic [17]. Different from the periodic systems, these systems need additional assumptions to guarantee the existence of the corresponding type solutions. For example, for almost periodic differential equations, if all the solutions are bounded, the almost periodic solutions may not exist, so additional conditions should be considered, such as separation condition or some stability conditions [31]. Also, Ortega and Tarallo [28] discuss the Zhikov-Levitan and Johnson examples, and point out how they illuminate the classical Favard theory.

On the other hand, the classical Massera periodic theorem has been extended from scalar ODEs to high dimensional differential equations. For two dimensional periodic ordinary differential equations, Massera periodic theorem may not true. Massera [19] prove that for two dimensional equation, if all its solutions exists in the future and if one of them is bounded in the future, then periodic solution exists. This result generalizes one of Levinson [16] where the uniform boundedness is assumed. Since then, many authors extended the results of Massera for two dimensional differential equations, for more details one can see [22, 31] .

Liénard equation is one of the important two order differential equations, has attracted a great deal of attention of many mathematicians due to the significance and applications in physics, mechanics, engineering fields, and so on. The behavior of the solutions are one of the attracting topics in the context of Liénard equation, such as limit cycles [10, 32], stability [13], oscillation [12], boundedness [15], periodicity [4, 18], almost periodicity [1, 5, 7], almost automorphy [9]. Particularly, some Massera type criteria are derived for Liénard equation. For example, in the periodic case, the dynamics of Liénard equation was intensively investigated by Martínez-Amores and Torres [18], Campos and Torres [4]; Cieutot and his co-workers have published several papers concerning the existence of almost periodic solution and almost automorphic solution to differential equations in both finite and infinite dimensional space [7]. However, literatures concerning asymptotic almost automorphic solutions, weighted pseudo almost automorphic solutions are very few [5].

In this paper, we characterize the almost automorphic, asymptotically almost automorphic, and weighted pseudo almost automorphic dynamics of a Liénard equation being bounded in the future. The paper is organized as follows. In Section 2, some notations and preliminary results are presented. In section 3, we study the relationship of bounded solution and the existence of almost automorphic, asymptotically almost automorpohic, weighted pseudo almost automorphic solutions of Liénard equation. In section 5, we provide an example to illustrate our main results.

2. Preliminaries

We first introduce some notations, definitions, and preliminary facts that will come into play later on. Let $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ stand for the set of natural numbers, integers, real numbers, and complex numbers, respectively. One denotes by $BC(\mathbb{R}, \mathbb{R})$

the Banach space of bounded continuous functions from $\mathbb{R}$ into $\mathbb{R}$ with the supremum norm. $C(\mathbb{R},\mathbb{R})$ stands for the set of continuous functions from $\mathbb{R}$ into $\mathbb{R}$. Let $C^n(\mathbb{R},\mathbb{R})$ be the space of all continuous functions which have a continuous n-th derivative on $\mathbb{R}$. $L^\infty(\mathbb{R},\mathbb{R})$ denotes the space of essentially bounded measurable functions in $\mathbb{R}$.

Definition 2.1 (Bochner [3]). A function $f \in C(\mathbb{R},\mathbb{R})$ is said to be almost automorphic in Bochner's sense if for every sequence of real numbers $(s_n^{'})_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t+s_n) \tag{2.1}$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t-s_n) = f(t) \tag{2.2}$$

for each $t \in \mathbb{R}$. Denote by $AA(\mathbb{R},\mathbb{R})$ the set of such functions.

From the pointwise convergence, it follows that the function $g \in L^\infty(\mathbb{R},\mathbb{R})$, but it is not necessarily continuous. If the convergence in (2.1) and (2.2) are uniform on any compact subset $K \subset \mathbb{R}$, then f is said to be compact almost automorphic (denoted by $AA_c(\mathbb{R},\mathbb{R})$). It is not difficult to show that $AA(\mathbb{R},\mathbb{R})$ and $AA_c(\mathbb{R},\mathbb{R})$ constitute Banach spaces when endowed with the sup norm.

Definition 2.2 ([11]). A function $f \in C^n(\mathbb{R},\mathbb{R})$ is said to be $C^{(n)}$-almost automorphic for $n \geq 1$, if the i-th derivative f^i of f is almost automorphic for $i = 1,\dots,n$. Denote by $AA^{(n)}(\mathbb{R},\mathbb{R})$ the space of all $C^{(n)}$-almost automorphic functions.

Definition 2.3 ([11]). A function $f \in C^n(\mathbb{R},\mathbb{R})$ is said to be $C^{(n)}$-compact almost automorphic for $n \geq 1$ if, for $i = 1,\dots,n$, the i-th derivative f^i of f is compact almost automorphic. Denote by $AA_c^{(n)}(\mathbb{R},\mathbb{R})$ the space of all $C^{(n)}$-compact almost automorphic functions.

Let U be the set of all functions $\rho : \mathbb{R} \to (0,\infty)$ being positive and locally integrable over $\mathbb{R}$, and there exists a constant $M > 0$ such that $\rho(t) \leq M$. For a given $T > 0$ and each $\rho \in U$, define

$$\mu(T,\rho) := \int_{-T}^{T} \rho(t)\mathrm{d}t \quad \text{and} \quad U_\infty := \{\rho \in U : \lim_{T\to\infty} \mu(T,\rho) = \infty\}.$$

Moreover, for $\rho \in U_\infty$, define

$$C_0(\mathbb{R},\mathbb{R}) := \left\{ f \in BC(\mathbb{R},\mathbb{R}) : \lim_{|t|\to+\infty} |f(t)| = 0 \right\},$$

and

$$WPAA_0(\mathbb{R},\mathbb{R}) := \left\{ f \in BC(\mathbb{R},\mathbb{R}) : \lim_{T\to+\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t)|f(t)|dt = 0 \right\}.$$

Definition 2.4 ([24]). A function $f \in C(\mathbb{R},\mathbb{R})$ is said to be asymptotically almost automorphic if it can be written as $f = g+\varphi$, where $g \in AA(\mathbb{R},\mathbb{R})$ and $\varphi \in C_0(\mathbb{R},\mathbb{R})$. Denote by $AAA(\mathbb{R},\mathbb{R})$ the set of such functions.

Definition 2.5 ([20]). A function $f \in C^n(\mathbb{R},\mathbb{R})$ is said to be asymptotically $C^{(n)}$-almost automorphic if it admits a decomposition $f = g + \varphi$, where $g \in AA^{(n)}(\mathbb{R},\mathbb{R})$ and $\varphi \in C^n(\mathbb{R},\mathbb{R})$ with $\varphi^i \in C_0(\mathbb{R},\mathbb{R})$ for $i = 1, \ldots, n$. Denote by $AAA^{(n)}(\mathbb{R},\mathbb{R})$ the space of all asymptotically $C^{(n)}$-almost automorphic functions.

Definition 2.6. A function $f \in C^n(\mathbb{R},\mathbb{R})$ is said to be asymptotically $C^{(n)}$-compact almost automorphic if it admits a decomposition $f = g + \varphi$, where $g \in AA_c^{(n)}(\mathbb{R},\mathbb{R})$ and $\varphi \in C^n(\mathbb{R},\mathbb{R})$ with $\varphi^i \in C_0(\mathbb{R},\mathbb{R})$ for $i = 1, \ldots, n$. Denote by $AAA_c^{(n)}(\mathbb{R},\mathbb{R})$ the space of all asymptotically $C^{(n)}$-compact almost automorphic functions.

Next, we introduce the definition of weighted pseudo $C^{(n)}$-compact almost automorphic functions.

Definition 2.7 ([2]). A function $f \in C(\mathbb{R},\mathbb{R})$ is said to be weighted pseudo almost automorphic if it can be decomposed as $f = g + \varphi$, where $g \in AA(\mathbb{R},\mathbb{R})$ and $\varphi \in WPAA_0(\mathbb{R},\mathbb{R})$. The function g is called the almost automorphic part of the function f. Denote by $WPAA(\mathbb{R},\mathbb{R})$ the set of such functions.

Definition 2.8. A function $f \in C^n(\mathbb{R},\mathbb{R})$ is said to be weighted pseudo $C^{(n)}$-compact almost automorphic if it admits a decomposition $f = g + \varphi$, where $g \in AA_c^{(n)}(\mathbb{R},\mathbb{R})$ and $\varphi \in C^n(\mathbb{R},\mathbb{R})$ with $\varphi^i \in WPAA_0(\mathbb{R},\mathbb{R})$ for $i = 1, \ldots, n$. Denote by $WPAA_c^{(n)}(\mathbb{R},\mathbb{R})$ the space of all weighted pseudo $C^{(n)}$-compact almost automorphic functions.

Lemma 2.1 ([2]). *Let f be a weighted pseudo almost automorphic function such that $f = g + \varphi$, where $g \in AA(\mathbb{R},\mathbb{R})$ and $\varphi \in WPAA_0(\mathbb{R},\mathbb{R})$, then $\{g(t) : t \in \mathbb{R}\} \subset \overline{\{f(t) : t \in \mathbb{R}\}}$.*

3. Almost Automorphic Dynamics

Consider the following generalized Liénard equation

$$u'' + f(u)u' + e(t)g(u) = p(t), \tag{3.1}$$

where $e, p : \mathbb{R} \to \mathbb{R}$ are bounded and continuous, and $f, g : (a, b) \to \mathbb{R}$ $(-\infty \le a < b \le \infty)$ are continuous.

In this section, we assume that

(H_1) f and $g : (a, b) \to \mathbb{R}$ are locally Lipschitz continuous,

(H_2) g is strictly decreasing,

(H_3) $f(u) \ge 0$ for all $u \in (a, b)$,

(H_4) $e(t) \in AA^{(1)}(\mathbb{R},\mathbb{R}^+)$ with $\inf\limits_{t\in\mathbb{R}} e(t) = \delta > 0$.

Definition 3.1 ([7]). A function $u : (c, +\infty) \to \mathbb{R}$ (with $-\infty \le c < +\infty$) is bounded in the future if there exist r, s and $t_0 > c$ such that

$$a < r \le u(t) \le s < b, \quad \text{for all } \ t > t_0.$$

A function $u : \mathbb{R} \to \mathbb{R}$ is bounded on $\mathbb{R}$ if there exist r and s such that

$$a < r \le u(t) \le s < b, \quad \text{for all } \ t \in \mathbb{R}.$$

By carrying out arguments similar to those in [7], we establish the following for (3.1).

Lemma 3.1. *Let $I = (t_0, +\infty)$ with $t_0 = -\infty$ or $t_0 \in \mathbb{R}$. If u is a solution of (3.1) which is bounded in the future (resp. bounded on $\mathbb{R}$), then u', u'' are bounded in the future (resp. bounded on $\mathbb{R}$), i,e., $\sup_{t\in I} |u'(t)| \leq c_1 < +\infty$, $\sup_{t\in I} |u''(t)| \leq c_2 < +\infty$, where*

$$c_0 := \max(|r|, |s|), \tag{3.2}$$

$$c_1 := \frac{1}{2}\left(\sup_{t\in\mathbb{R}} |p(t)| + \sup_{r\leq z\leq s} |g(z)| \cdot \sup_{t\in\mathbb{R}} e(t)\right) + 2c_0 + 4c_0 \sup_{r\leq z\leq s} |f(z)|, \tag{3.3}$$

$$c_2 := \left(\sup_{t\in\mathbb{R}} |p(t)| + \sup_{r\leq z\leq s} |g(z)| \cdot \sup_{t\in\mathbb{R}} e(t)\right) + c_1 \sup_{r\leq z\leq s} |f(z)|. \tag{3.4}$$

Lemma 3.2. *Let u_1 and u_2 be two different solutions of (3.1) being bounded in the future. Then*

(i) the function $t \to |u_1(t) - u_2(t)|$ is strictly decreasing, i.e., $(u_1(t)-u_2(t))(u_1'(t)-u_2'(t)) < 0$, for every t where both solutions are defined.

(ii) $\lim_{|t|\to+\infty} (|u_1(t) - u_2(t)| + |u_1'(t) - u_2'(t)|) = 0$.

Lemma 3.3. *(3.1) has at most one bounded solution on $\mathbb{R}$.*

3.1. Almost Automorphic Solutions

Consider the following generalized Liénard equation

$$u'' + f(u)u' + e(t)g(u) = p_1(t). \tag{3.5}$$

Definition 3.2 ([9]). For $e, p_1 \in L^\infty(\mathbb{R}, \mathbb{R})$, u is a weak solution on $\mathbb{R}$ of (3.5), if $u \in C^1(\mathbb{R}, \mathbb{R})$ and satisfies

$$u'(t) + \int_\sigma^t \{f(u(\tau))u'(\tau) + e(\tau)g(u(\tau))\}d\tau = u'(\sigma) + \int_\sigma^t p_1(\tau)d\tau,$$

for each σ and $t \in \mathbb{R}$ such that $\sigma \leq t$.

Definition 3.3 ([9]). For $e, p_1 \in C(\mathbb{R}, \mathbb{R})$, u is a (classical) solution on $\mathbb{R}$ of (3.5) if $u \in C^2(\mathbb{R}, \mathbb{R})$ and $u(t)$ satisfies (3.5) for $t \in \mathbb{R}$.

Obviously, a classical solution is obviously a weak solution, and, for e, p_1 being continuous, the notions of weak solution and classical solution are equivalent.

Lemma 3.4. *Assume that $e, p_1 \in L^\infty(\mathbb{R}, \mathbb{R})$, u is a weak solution bounded on $\mathbb{R}$ of (3.5) and $u' \in L^\infty(\mathbb{R}, \mathbb{R})$ is Lipschitzian on $\mathbb{R}$. If there exists a sequence of real numbers $(t_n')_{n\in\mathbb{N}}$ and $e^*(t), p_1^*(t)$ such that*

$$\lim_{n\to+\infty} e(t + t_n) = e^*(t), \qquad \lim_{n\to+\infty} p_1(t + t_n) = p_1^*(t), \quad t \in \mathbb{R},$$

then there exists a subsequence $(t_n)_{n\in\mathbb{N}} \subset (t_n')_{n\in\mathbb{N}}$ and $v(t)$ such that

$$\lim_{n\to+\infty} u(t + t_n) = v(t), \qquad \lim_{n\to+\infty} u'(t + t_n) = v'(t),$$

uniformly on each compact subset of $\mathbb{R}$*, where* v *is a weak solution bounded on* $\mathbb{R}$ *of*

$$v'' + f(v)v' + e^*(t)g(v) = p_1^*(t), \tag{3.6}$$

and $v \in L^\infty(\mathbb{R}, \mathbb{R})$ *is Lipschitzian on* $\mathbb{R}$*.*

Proof. Let u be a weak solution of (3.5) being bounded on $\mathbb{R}$, then there exist $r, s \in \mathbb{R}$ such that

$$a < r \le u(t) \le s < b, \quad t \in \mathbb{R}.$$

Whence, for $t \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$a < r \le u(t + t'_n) \le s < b. \tag{3.7}$$

Since $u' \in L^\infty(\mathbb{R}, \mathbb{R})$, for $t \in \mathbb{R}$, one has

$$|u'(t + t'_n)| \le c := \sup_{t \in \mathbb{R}} |u'(t)| < +\infty, \tag{3.8}$$

then

$$|u(t + t'_n) - u(s + t'_n)| \le c|t - s|, \quad t, s \in \mathbb{R}, \quad n \in \mathbb{N}.$$

By (3.7) and using Arezlà-Ascoli theorem, there exists a subsequence of $(t_n)_{n \in \mathbb{N}} \subset (t'_n)_{n \in \mathbb{N}}$ such that

$$u(t + t_n) \to v(t), \quad n \to \infty$$

uniformly on each compact subset of $\mathbb{R}$. Moreover, since u' is Lipschitzian on $\mathbb{R}$, there exists a $k > 0$ such that

$$|u'(t + t_n) - u'(s + t_n)| \le k|t - s|, \quad t, s \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{3.9}$$

By (3.8), (3.9) and using Arezlà-Ascoli theorem again, there exist $\xi \in C(\mathbb{R}, \mathbb{R})$ and a subsequence of $(t_n)_{n \in \mathbb{N}}$ (which is also denoted by the same notation) such that

$$u'(t + t_n) \to \xi(t), \quad n \to \infty$$

uniformly on each compact subset of $\mathbb{R}$. By the uniform convergence of $\{u'(t + t_n)\}_{n \in \mathbb{N}}$, one has

$$\lim_{n \to +\infty} (u'(t + t_n)) = \left(\lim_{n \to +\infty} u(t + t_n) \right)',$$

then $\xi(t) = v'(t)$ and $v(t)$ is bounded on $\mathbb{R}$ from (3.7).

It remains to prove that v is a weak solution of (3.6). Since u is a weak solution of (3.5), for each σ and $t \in \mathbb{R}$ with $\sigma \le t$, one has

$$u'(t) + \int_\sigma^t \{f(u(\tau))u'(\tau) + e(\tau)g(u(\tau))\}d\tau = u'(\sigma) + \int_\sigma^t p_1(\tau)d\tau.$$

Then

$$\begin{aligned} &u'(t + t_n) + \int_\sigma^t \{f(u(\tau + t_n))u'(\tau + t_n) + e(\tau + t_n)g(u(\tau + t_n))\}d\tau \\ &= u'(\sigma + t_n) + \int_\sigma^t p_1(\tau + t_n)d\tau. \end{aligned}$$

Since

$$e(\tau+t_n) \le \sup_{t\in\mathbb{R}} e(t) < +\infty, \quad |p_1(\tau+t_n)| \le \sup_{t\in\mathbb{R}} |p_1(t)| < +\infty,$$

by Lebesgue dominated convergence theorem, one has

$$v'(t) + \int_\sigma^t \{f(v(\tau))v'(\tau) + e^*(\tau)g(v(\tau))\}d\tau = v'(\sigma) + \int_\sigma^t p_1^*(\tau)d\tau,$$

that is v is a weak solution of (3.6). It is not difficult to show that $v \in L^\infty(\mathbb{R},\mathbb{R})$ and it is Lipschitzian on $\mathbb{R}$. □

Theorem 3.1. *Assume that* (H_1)-(H_4) *hold and* $p_1 \in AA(\mathbb{R},\mathbb{R})$.

(i) *If* (3.5) *has one solution* u *being bounded in the future, then* (3.5) *has a unique solution* ϕ *being bounded on* $\mathbb{R}$. *Moreover,* $\phi \in AA_c^{(1)}(\mathbb{R},\mathbb{R})$.

(ii) *For any solution* u *being bounded in the future of* (3.5) *, one has*

$$\lim_{|t|\to+\infty}(|u(t)-\phi(t)| + |u'(t)-\phi'(t)|) = 0 \tag{3.10}$$

and $u \in AAA_c^{(1)}(\mathbb{R},\mathbb{R})$.

Proof. (i) Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that

$$\lim_{n\to+\infty} t_n = +\infty. \tag{3.11}$$

Since $e \in AA(\mathbb{R},\mathbb{R}^+), p_1 \in AA(\mathbb{R},\mathbb{R})$, there exists a subsequence of $(t_n)_{n\in\mathbb{N}}$ (which is still denoted by the same symbol) such that

$$\lim_{n\to+\infty} e(t+t_n) = e^*(t), \quad \lim_{n\to+\infty} p_1(t+t_n) = p_1^*(t), \quad t\in\mathbb{R}, \tag{3.12}$$

$$\lim_{n\to+\infty} e^*(t-t_n) = e(t), \quad \lim_{n\to+\infty} p_1^*(t-t_n) = p_1(t), \quad t\in\mathbb{R}. \tag{3.13}$$

Since u is a solution of (3.5) being bounded in the future, there exist r, s and $t_0 \in \mathbb{R}$ such that

$$a < r \le u(t) \le s < b, \quad t > t_0,$$

and, for σ and $t \in \mathbb{R}$ with $t_0 < \sigma \le t$, one has

$$u'(t) + \int_\sigma^t \{f(u(\tau))u'(\tau) + e(\tau)g(u(\tau))\}d\tau = u'(\sigma) + \int_\sigma^t p_1(\tau)d\tau.$$

By Lemma 3.1, we obtain

$$\sup_{t>t_0}|u'(t)| \le c_1, \quad \sup_{t>t_0}|u''(t)| \le c_2,$$

where c_1, c_2 are defined by (3.3), (3.4), respectively. Let $u(t+t_n)$ be the solution of

$$u'' + f(u)u' + e(t+t_n)g(u) = p_1(t+t_n)$$

on $(t_0-t_n,+\infty)$ and $T_0 \in \mathbb{R}$, for each $n \in \mathbb{N}$ such that $T_0 + t_n \ge t_0$, one has

$$a < r \le u(t+t_n) \le s < b, \quad t \in (T_0,+\infty),$$

and

$$|u'(t+t_n)| \leq c_1, \quad |u''(t+t_n)| \leq c_2, \quad t \in (T_0, +\infty).$$

By Arezlà-Ascoli theorem, we claim that there exists a subsequence of $(t_n)_{n\in\mathbb{N}}$ and a $u_*(t)$ such that

$$u(t+t_n) \to u_*(t), \quad u'(t+t_n) \to u_*'(t), \quad n \to \infty, \tag{3.14}$$

uniformly on any compact subset of $(T_0, +\infty)$. Since this is fulfilled for all $T_0 \in \mathbb{R}$, one deduces that (3.14) are satisfied uniformly on each compact subset of $\mathbb{R}$. For $\sigma \leq t$ and $n \in \mathbb{N}$ being sufficiently large, one has

$$\begin{aligned} &u'(t+t_n) + \textstyle\int_\sigma^t \{f(u(\tau+t_n))u'(\tau+t_n) + e(\tau+t_n)g(u(\tau+t_n))\}d\tau \\ &= u'(\sigma+t_n) + \textstyle\int_\sigma^t p_1(\tau+t_n)d\tau. \end{aligned}$$

By (3.12) and (3.14), using Lebesgue dominated convergence theorem, u_* is a weak solution on $\mathbb{R}$ of

$$u_*'' + f(u_*)u_*' + e^*(t)g(u_*) = p_1^*(t).$$

By (3.14), it is not difficult to see that u_* is bounded on $\mathbb{R}$ and $u_*' \in L^\infty(\mathbb{R},\mathbb{R})$ is Lipschitzian on $\mathbb{R}$. By (3.13) and using Lemma 3.4 with $u = u_*, e = e_*, p_1 = p_1^*$ and the sequence $(-t_n)_{n\in\mathbb{N}}$, we obtain the existence of a weak solution ϕ of (3.5) that is bounded on $\mathbb{R}$, and $\phi \in L^\infty(\mathbb{R},\mathbb{R})$ is Lipschitzian on $\mathbb{R}$. Since e, p_1 are continuous functions, then ϕ is a solution of (3.5) that is bounded on $\mathbb{R}$. The uniqueness of the bounded solution of (3.5) follows from Lemma 3.3.

Next, we check that $\phi \in AA_c^{(1)}(\mathbb{R},\mathbb{R})$, i.e., $\phi \in AA_c(\mathbb{R},\mathbb{R})$ and $\phi' \in AA_c(\mathbb{R},\mathbb{R})$. Since ϕ is a weak solution bounded on $\mathbb{R}$ of (3.5), and $\phi \in L^\infty(\mathbb{R},\mathbb{R})$ is Lipschitzian on $\mathbb{R}$, by (3.12) and using Lemma 3.4, there exists a subsequence $(t_n)_{n\in\mathbb{N}}$ and $\phi_*(t)$ such that

$$\lim_{n\to+\infty} \phi(t+t_n) = \phi_*(t), \quad \lim_{n\to+\infty} \phi'(t+t_n) = \phi_*'(t), \tag{3.15}$$

uniformly on each compact subset of $\mathbb{R}$, where $\phi_*(t)$ is a weak bounded solution on $\mathbb{R}$ of

$$\phi_*'' + f(\phi_*)\phi_*' + e^*(t)g(\phi_*) = p_1^*(t),$$

and $\phi_* \in L^\infty(\mathbb{R},\mathbb{R})$ is Lipschitzian on $\mathbb{R}$. By (3.13), using Lemma 3.4 with $u = \phi_*, e = e_*, p_1 = p_1^*$ and the sequence $(-t_n)_{n\in\mathbb{N}}$, one has

$$\lim_{n\to+\infty} \phi_*(t-t_n) = \psi(t), \qquad \lim_{n\to+\infty} \phi_*'(t-t_n) = \psi'(t), \tag{3.16}$$

uniformly on each compact subset of $\mathbb{R}$, where ψ is a weak solution on $\mathbb{R}$ of (3.5). Since e, p_1 are continuous functions, then ψ is a solution of (3.5) that is bounded on $\mathbb{R}$. By the uniqueness of the solution being bounded on $\mathbb{R}$ of (3.5), we have $\phi = \psi$. Therefore, by (3.15), (3.16), $\phi \in AA_c^{(1)}(\mathbb{R},\mathbb{R})$.

(ii) By Lemma 3.2,

$$\lim_{|t|\to+\infty}(|u(t)-\phi(t)|) = \lim_{|t|\to+\infty}(|u'(t)-\phi'(t)|) = 0,$$

then $u = \phi + u - \phi \in AAA_c(\mathbb{R},\mathbb{R})$ and $u' = \phi' + u' - \phi' \in AAA_c(\mathbb{R},\mathbb{R})$, i.e., $u \in AAA_c^{(1)}(\mathbb{R},\mathbb{R})$. ☐

Lemma 3.5 ([26]). *If the function $f(t,u,v)$ satisfies*

(i) *$f \in C(\mathbb{R}^3,\mathbb{R})$ and exist two numbers a,b $(a<b)$ such that $f(t,a,0)\le 0$ and $f(t,b,0)\ge 0$ for each $t\in\mathbb{R}$.*

(ii) *V and T are nonnegative continuous functions on $\mathbb{R}$ such that*

$$\int_0^{\infty}\frac{v}{V(v)}dv=+\infty,\quad V(-v)=V(v),\quad V(v)\ge 1,\quad v\in\mathbb{R}.$$

(iii) *$|f(t,u,v)|\le T(t)V(v)$ for each $a\le u\le b$, $t,v\in\mathbb{R}$.*
Then $u''=f(t,u,u')$ admits at least a solution u such that $a\le u\le b$.

Corollary 3.1. *Assume that (H_1)-(H_4) hold. If $\inf\limits_{t\in\mathbb{R}}\dfrac{p_1(t)}{e(t)}$ and $\sup\limits_{t\in\mathbb{R}}\dfrac{p_1(t)}{e(t)}$ are in the range of $g(a,b)$, then (3.5) has a unique bounded solution u on $\mathbb{R}$ and $u\in AA_c^{(1)}$.*

Proof. (i) If $\inf\limits_{t\in\mathbb{R}}\frac{p_1(t)}{e(t)}=\sup\limits_{t\in\mathbb{R}}\frac{p_1(t)}{e(t)}$, i.e. $\frac{p_1(t)}{e(t)}=m_0$ for each $t\in\mathbb{R}$, where m_0 is a constant, then there exists $u_0\in(a,b)$ such that $g(u_0)=m_0$ for $t\in\mathbb{R}$. Therefore, $u(t)=u_0$ is a bounded solution on $\mathbb{R}$. By Theorem 3.1, $u\in AA_c^{(1)}$.

(ii) If $\inf\limits_{t\in\mathbb{R}}\frac{p_1(t)}{e(t)}<\sup\limits_{t\in\mathbb{R}}\frac{p_1(t)}{e(t)}$, there exist $r,s\in\mathbb{R}, a<r<s<b$ such that $g(r)=\sup\limits_{t\in\mathbb{R}}\frac{p_1(t)}{e(t)}$, $g(s)=\inf\limits_{t\in\mathbb{R}}\frac{p_1(t)}{e(t)}$. Let $\widetilde{f}$ and $\widetilde{g}$ be the extensions of $f|_{[r,s]}$ and $g|_{[r,s]}$. The extension $\widetilde{f}:\mathbb{R}\to\mathbb{R}$ is defined by

$$\widetilde{f}(u)=\begin{cases} f(u) & \text{if } r\le u\le s,\\ f(r) & \text{if } u<r,\\ f(s) & \text{if } u>s.\end{cases}$$

It is not difficult to check that $\widetilde{f}$ is continuous. The extension of $\widetilde{g}$ can be similarly defined. Set

$$F(t,u,v):=p_1(t)-\widetilde{f}(u)v-e(t)\widetilde{g}(u),\qquad V(v):=2+|v|,$$

$$T(t):=\max\left\{|p_1(t)|,\ \sup_{r\le u\le s}|f(u)|,\ \sup_{r\le u\le s}|g(u)|\cdot\sup_{t\in\mathbb{R}}e(t)\right\}.$$

By Lemma 3.5, the equation $u''=F(t,u,u')$ has at least one solution u such that $r\le u(t)\le s$ for $t\in\mathbb{R}$. Therefore, u is a solution of (3.5) being bounded on $\mathbb{R}$. By Theorem 3.1, $u\in AA_c^{(1)}$. □

3.2. Asymptotically Almost Automorphic Solutions

Theorem 3.2. *Assume that (H_1)-(H_4) hold and $p=p_1+p_2\in AAA(\mathbb{R},\mathbb{R})$, where $p_1\in AA(\mathbb{R},\mathbb{R})$, $p_2\in C_0(\mathbb{R},\mathbb{R})$.*

(i) *If (3.1) has at least one solution u being bounded in the future, then (3.5) has a unique solution ϕ being bounded on $\mathbb{R}$. Moreover, $\phi\in AA_c^{(1)}(\mathbb{R},\mathbb{R})$.*

(ii) *Every solution u being bounded in the future of (3.1) satisfies*

$$\lim_{|t|\to+\infty}(|u(t)-\phi(t)|+|u'(t)-\phi'(t)|)=0,$$

and $u\in AAA_c^{(1)}(\mathbb{R},\mathbb{R})$.

Proof. (*i*) Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that

$$\lim_{n\to+\infty} t_n = +\infty.$$

Since $e \in AA(\mathbb{R},\mathbb{R}^+), p_1 \in AA(\mathbb{R},\mathbb{R})$, there exists a subsequence of $(t_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to+\infty} e(t+t_n) = e^*(t), \quad \lim_{n\to+\infty} p_1(t+t_n) = p_1^*(t) \text{ for each } t\in\mathbb{R},$$
$$\lim_{n\to+\infty} e^*(t-t_n) = e(t), \quad \lim_{n\to+\infty} p_1^*(t-t_n) = p_1(t) \text{ for each } t\in\mathbb{R}.$$

Let u be a solution of (3.1) being bounded in the future, there exist r, s and $t_0\in\mathbb{R}$ such that

$$a < r \le u(t) \le s < b, \quad \text{for all } t > t_0,$$

and

$$\sup_{t>t_0} |u'(t)| \le c_1, \quad \sup_{t>t_0} |u''(t)| \le c_2.$$

Given any interval $(T_0, +\infty)$, for $n\in\mathbb{N}$ such that $T_0 + t_n \ge t_0$, one has

$$a < r \le u(t+t_n) \le s < b, \quad |u'(t+t_n)| \le c_1, \quad |u''(t+t_n)| \le c_2, \quad t\in(T_0,+\infty).$$

Taking T_0 as a sequence going to $-\infty$ and using Arezl$\grave{a}$-Ascoli theorem, there exists a subsequence of $(t_n)_{n\in\mathbb{N}}$ and $u_*(t)$ such that

$$u(t+t_n)\to u_*(t), \quad u'(t+t_n)\to u_*'(t), \quad n\to+\infty,$$

uniformly on each compact subset of $\mathbb{R}$. Since u is a solution of (3.1), for each $\sigma\le t$ and $n\in\mathbb{N}$ sufficiently large such that $\sigma + t_n \ge t_0$, one has

$$u'(t+t_n) + \int_\sigma^t \{f(u(\tau+t_n))u'(\tau+t_n) + e(\tau+t_n)g(u(\tau+t_n))\}d\tau$$
$$= u'(\sigma+t_n) + \int_\sigma^t p(\tau+t_n)d\tau,$$

i.e.,

$$u'(t+t_n) + \int_\sigma^t \{f(u(\tau+t_n))u'(\tau+t_n) + e(\tau+t_n)g(u(\tau+t_n))\}d\tau$$
$$= u'(\sigma+t_n) + \int_\sigma^t p_1(\tau+t_n)d\tau + \int_\sigma^t p_2(\tau+t_n)d\tau. \tag{3.17}$$

Moreover,

$$\left|\int_\sigma^t p_2(\tau+t_n)d\tau\right| \le \int_\sigma^t |p_2(\tau+t_n)|\,d\tau = \int_{m_n}^{m_n+l} |p_2(\tau)|d\tau,$$

where $m_n = \sigma + t_n, l = t - \sigma$. Since $p_2 \in C_0(\mathbb{R},\mathbb{R})$,

$$\lim_{n\to+\infty} \int_n^{n+1} |p_2(t)|dt = 0, \quad n\in\mathbb{N},$$

then

$$0 \leq \int\limits_{m_n}^{m_n+l} |p_2(\tau)|d\tau \leq \int\limits_{[m_n]}^{[m_n]+[l]+2} |p_2(\tau)|d\tau$$
$$= \sum_{k=[m_n]}^{[m_n]+[l]+1} \int\limits_{k}^{k+1} |p_2(\tau)|d\tau \to 0, \quad n \to +\infty,$$

where $[\cdot]$ is the integer part function. So

$$\int_{\sigma}^{t} p_2(\tau + t_n)d\tau \to 0, \quad n \to +\infty.$$

By (3.17), one has

$$u_*'(t) + \int_{\sigma}^{t} \{f(u_*(\tau))u_*'(\tau) + e^*(\tau)g(u_*(\tau))\}d\tau = u_*'(\sigma) + \int_{\sigma}^{t} p_1^*(\tau)d\tau.$$

It is not difficult to see that u_* is bounded on $\mathbb{R}$ and and $u_*' \in L^\infty(\mathbb{R}, \mathbb{R})$ is c_2-Lipschitzian on $\mathbb{R}$. Then

$$r \leq u_*(t - t_n') \leq s, \quad |u_*'(t - t_n')| \leq c_1, \quad |u_*'(t - t_n) - u_*'(s - t_n)| \leq c_2|t - s|, \quad t \in \mathbb{R}.$$

By Arezl*à*-Ascoli theorem, there exists a subsequence of $(t_n)_{n\in\mathbb{N}}$ and $\nu(t)$ such that

$$u_*(t - t_n) \to \nu(t), \qquad u_*'(t - t_n) \to \nu'(t), \quad n \to \infty,$$

uniformly on any compact subset of $\mathbb{R}$. For σ and $t \in \mathbb{R}$ with $\sigma \leq t$, we have

$$u_*'(t - t_n) + \int\limits_{\sigma}^{t} \{f(u_*(\tau - t_n))u_*'(\tau - t_n) + e^*(\tau - t_n)g(u_*(\tau - t_n))\}d\tau$$
$$= u_*'(\sigma - t_n) + \int\limits_{\sigma}^{t} p_1^*(\tau - t_n)d\tau.$$

Then

$$\nu'(t) + \int_{\sigma}^{t} \{f(\nu(\tau))\nu'(\tau) + e(\tau)g(\nu(\tau))\}d\tau = \nu'(\sigma) + \int_{\sigma}^{t} p_1(\tau)d\tau.$$

Since $e, p_1 \in C(\mathbb{R}, \mathbb{R})$, ν is a solution of (3.5) that is bounded on $\mathbb{R}$. By Theorem 3.1, we deduce the uniqueness of the bounded solution ϕ on $\mathbb{R}$ and $\phi \in AA_c^{(1)}(\mathbb{R}, \mathbb{R})$.

(ii) By Lemma 3.2, one has

$$\lim_{|t|\to+\infty} (|u(t) - \phi(t)|) = \lim_{|t|\to+\infty} (|u'(t) - \phi'(t)|) = 0,$$

then $u, u' \in AAA_c(\mathbb{R}, \mathbb{R})$, i.e., $u \in AAA_c^{(1)}(\mathbb{R}, \mathbb{R})$. □

3.3. Weighted Pseudo Almost Automorphic Solutions

Lemma 3.6 ([30]). *Suppose g is continuous and strictly increasing on $[r,s]$. Then for every $\varepsilon \in (0, s-r)$, there exists $k>0$ such that*

$$g(u) - g(v) \geq k(u - v)$$

for all $u, v \in [r,s]$ and $u - v \geq \varepsilon$.

Theorem 3.3. *Assume that (H_1)-(H_4) hold and $p = p_1 + p_2$, where $p_1 \in AA(\mathbb{R},\mathbb{R})$, $p_2 \in WPAA_0(\mathbb{R},\mathbb{R})$. If $\inf\limits_{t\in\mathbb{R}} \frac{p(t)}{e(t)}$ and $\sup\limits_{t\in\mathbb{R}} \frac{p(t)}{e(t)}$ are in the range of $g(a,b)$, then (3.1) has a unique bounded solution u on $\mathbb{R}$ and $u \in WPAA_c^{(1)}(\mathbb{R},\mathbb{R})$. Furthermore, if we denote by p_1 (respectively u_1) the almost automorphic part of p (respectively u), then u_1 is the almost automorphic solution of (3.5).*

Proof. First, we prove the existence and uniqueness of the solution being bounded on $\mathbb{R}$ of (3.1). In fact, if $\frac{p(t)}{e(t)} = m_0$ for $t \in \mathbb{R}$, then there exists a $u_0 \in (a,b)$ such that $g(u_0) = m_0$ for $t \in \mathbb{R}$. Therefore, $u(t) = u_0$ is a solution being bounded on $\mathbb{R}$. If $\inf\limits_{t\in\mathbb{R}} \frac{p(t)}{e(t)} < \sup\limits_{t\in\mathbb{R}} \frac{p(t)}{e(t)}$, then there exist $r, s \in \mathbb{R}, a < r < s < b$ such that $g(r) = \sup\limits_{t\in\mathbb{R}} \frac{p(t)}{e(t)}$, $g(s) = \inf\limits_{t\in\mathbb{R}} \frac{p(t)}{e(t)}$. The extension $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ is defined by

$$\widetilde{f}(u) = \begin{cases} f(u) & \text{if } r \leq u \leq s, \\ f(r) & \text{if } u < r, \\ f(s) & \text{if } u > s, \end{cases}$$

and $\widetilde{f}$ is continuous. Similarly, we define the extension of $\widetilde{g}$. Let

$$F(t,u,v) := p(t) - \widetilde{f}(u)v - e(t)\widetilde{g}(u), \quad V(v) := 2 + |v|,$$

$$T(t) := \max\left\{ |p(t)|, \sup_{r\leq u\leq s} |f(u)|, \sup_{r\leq u\leq s} |g(u)| \cdot \sup_{t\in\mathbb{R}} e(t) \right\}.$$

By Lemma 3.5, the equation $u'' = F(t,u,u')$ has at least one solution u such that $r \leq u(t) \leq s$ for $t \in \mathbb{R}$. Therefore, u is a solution being bounded on $\mathbb{R}$ of (3.1). The uniqueness of a solution being bounded on $\mathbb{R}$ follows from Lemma 3.3.

Next, we prove the existence and uniqueness of almost automorphic solution of (3.5). By Lemma 2.1, one has $p_1(\mathbb{R}) \subset p(\mathbb{R})$. Therefore, $\inf\limits_{t\in\mathbb{R}} \frac{p_1(t)}{e(t)}$ and $\sup\limits_{t\in\mathbb{R}} \frac{p_1(t)}{e(t)}$ are in the range of $g(a,b)$. By Corollary 3.1, (3.5) has a unique bounded solution u_1 and $u_1 \in AA_c^{(1)}(\mathbb{R},\mathbb{R})$.

Since u is the bounded solution of (3.1), u_1 is the almost automorphic solution of (3.5). Define

$$h := u - u_1,$$

then $h \in C(\mathbb{R},\mathbb{R})$ and $\|h\|_\infty := \sup\limits_{t\in\mathbb{R}} |h(t)| < +\infty$. We only need to show that $h \in WPAA_0(\mathbb{R},\mathbb{R})$, $h' \in WPAA_0(\mathbb{R},\mathbb{R})$, i.e.,

$$\lim_{T\to+\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t)|h(t)|dt = 0, \quad \lim_{T\to+\infty} \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t)|h'(t)|dt = 0.$$

Since u, u_1 are bounded solution on $\mathbb{R}$ of (3.1) and (3.5), respectively, there exist $r, s \in \mathbb{R}$ such that

$$a < r \leq u(t) \leq s < b, \quad a < r \leq u_1(t) \leq s < b, \quad t \in \mathbb{R}, \tag{3.18}$$

and

$$\begin{aligned} &u''(t) + f(u(t))u'(t) + e(t)g(u(t)) = p(t), \\ &u_1''(t) + f(u_1(t))u_1'(t) + e(t)g(u_1(t)) = p_1(t), \end{aligned}$$

then

$$\begin{aligned} &h''(t) + f(u(t))u'(t) - f(u_1(t))u_1'(t) + e(t)g(u(t)) - e(t)g(u_1(t)) \\ =&p(t) - p_1(t) = p_2(t). \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t)(h''(t) + f(u(t))u'(t) - f(u_1(t))u_1'(t))\chi_{[h>0]}(t)dt \\ =&\frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t)p_2(t)\chi_{[h>0]}(t)dt \\ &+ \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t)e(t)\left(g(u_1(t)) - g(u(t))\right)\chi_{[h>0]}(t)dt, \end{aligned} \tag{3.19}$$

where $[h > 0] := \{t \in \mathbb{R} : h(t) > 0\}$ and

$$\chi_{[h>0]}(t) = \begin{cases} 1 & t \in [h > 0], \\ 0 & t \notin [h > 0]. \end{cases}$$

On the other hand, since $p_2 \in WPAA_0(\mathbb{R}, \mathbb{R})$, for any $\varepsilon > 0$, there exists a $T_0 > 0$ such that, for any $T > T_0$,

$$\frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t)|p_2(t)|dt < \varepsilon.$$

By (H_4), one has

$$e(t)(g(u_1(t)) - g(u(t)))\chi_{[h>0]}(t) \geq 0.$$

Then, for $T > T_0$,

$$\begin{aligned} &\frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t)e(t)(g(u_1(t)) - g(u(t)))\chi_{[h\geq\varepsilon]}(t)dt \\ \leq&\frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t)e(t)(g(u_1(t)) - g(u(t)))\chi_{[h>0]}(t)dt, \end{aligned}$$

where $[h \geq \varepsilon] := \{t \in \mathbb{R} : h(t) \geq \varepsilon\}$ and

$$\chi_{[h\geq\varepsilon]}(t) = \begin{cases} 1 & t \in [h \geq \varepsilon], \\ 0 & t \notin [h \geq \varepsilon]. \end{cases}$$

By (H_4) and Lemma 3.6, if $h(t) \geq \varepsilon$, then there exists a $k > 0$ such that

$$e(t)(g(u_1(t)) - g(u(t))) \geq k\delta h(t).$$

Therefore, for $T > T_0$, by (3.19), one has

$$\begin{aligned}
&\frac{k\delta}{\mu(T,\rho)} \int_{-T}^{T} \rho(t)h(t)\chi_{[h\geq\varepsilon]}(t)dt \\
\leq& \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t)e(t)(g(u_1(t)) - g(u(t)))\chi_{[h\geq\varepsilon]}(t)dt \\
\leq& \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t)e(t)(g(u_1(t)) - g(u(t)))\chi_{[h>0]}(t)dt \\
\leq& \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t)(h''(t) + f(u(t))u'(t) - f(u_1(t))u_1'(t))\chi_{[h>0]}(t)dt \\
&+ \left| \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t)p_2(t)\chi_{[h>0]}(t)dt \right| \\
\leq& \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t)(h''(t) + f(u(t))u'(t) - f(u_1(t))u_1'(t))\chi_{[h>0]}(t)dt \\
&+ \frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t)|p_2(t)|dt. \qquad (3.20)
\end{aligned}$$

We claim that

$$\frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t)(h''(t) + f(u(t))u'(t) - f(u_1(t))u_1'(t))\chi_{[h>0]}(t)dt \leq \frac{2Mc_3}{\mu(T,\rho)}, \quad (3.21)$$

where $c_3 = 2c_1 + 2c_0 \sup_{r\leq z\leq s} |f(z)|$. Denote by O_T the open subset of $(-T,T)$ defined by $O_T := \{t \in (-T,T) : h(t) > 0\}$. The components of O_T are open intervals ω_i $(i \in I)$ included in $(-T,T)$, where the set I is countable. Let $m_i := \inf_{i\in\omega_i} t, M_i := \sup_{i\in\omega_i} t$.

If $O_T = (-T,T)$, i.e., there is one component in $O_T = (-T,T)$, then $h(t) > 0$ for each $t \in (-T,T)$,

$$\begin{aligned}
&\frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t)(h''(t) + f(u(t))u'(t) - f(u_1(t))u_1'(t))\chi_{[h>0]}(t)dt \\
=&\frac{1}{\mu(T,\rho)} \int_{-T}^{T} \rho(t)(h''(t) + f(u(t))u'(t) - f(u_1(t))u_1'(t))dt \\
\leq&\frac{M}{\mu(T,\rho)} \int_{-T}^{T} V'(t)dt = \frac{M}{\mu(T,\rho)}(V(T) - V(-T)),
\end{aligned}$$

where $V(t) = h'(t) + \int_{u_1(t)}^{u(t)} f(z)dz$, $t \in \mathbb{R}$. By Lemma 3.1, for any $t \in \mathbb{R}$,

$$V(t) \leq \left| h'(t) + \int_{u_1(t)}^{u(t)} f(z)dz \right| \leq |u'(t) - u_1'(t)| + |u(t) - u_1(t)| \sup_{r\leq z\leq s} |f(z)| \leq c_3,$$

so (3.21) holds.

If O_T has several components, then one has $h(t) > 0$ for $t \in \omega_i$ and

$$\begin{aligned}&\frac{1}{\mu(T,\rho)}\int_{\omega_i}\rho(t)(h''(t)+f(u(t))u'(t)-f(u_1(t))u_1'(t))dt\\ \le\ &\frac{M}{\mu(T,\rho)}\int_{\omega_i}V'(t)dt=\frac{M}{\mu(T,\rho)}(V(M_i)-V(m_i)).\end{aligned}$$

(i) If ω_i satisfies $-T < m_i < M_i < T$, then $h(m_i) = h(M_i) = 0$, i.e., $u(m_i) = u_1(m_i), u(M_i) = u_1(M_i)$. Since $h(t) > 0$ for each $t \in \omega_i$, $h'(m_i) \geq 0$ and $h'(M_i) \leq 0$, then

$$V(M_i)=h'(M_i)+\int_{u_1(M_i)}^{u(M_i)}f(z)dz\le 0,\qquad V(m_i)=h'(m_i)+\int_{u_1(m_i)}^{u(m_i)}f(z)dz\ge 0.$$

Whence,

$$\frac{1}{\mu(T,\rho)}\int_{\omega_i}\rho(t)(h''(t)+f(u(t))u'(t)-f(u_1(t))u_1'(t))dt\le 0.$$

(ii) If ω_i satisfies $-T = m_i < M_i < T$ (there is at most one such component), then $h(M_i) = 0$, $u(M_i) = u_1(M_i)$, $h'(M_i) \leq 0$, $V(M_i) \leq 0$. Whence,

$$\frac{1}{\mu(T,\rho)}\int_{\omega_i}\rho(t)(h''(t)+f(u(t))u'(t)-f(u_1(t))u_1'(t))dt\le -\frac{M}{\mu(T,\rho)}V(m_i)\le\frac{Mc_3}{\mu(T,\rho)}.$$

(iii) If ω_i satisfies $-T < m_i < M_i = T$ (there is at most one such component), then $h(m_i) = 0$, $u(m_i) = u_1(m_i)$, $h'(m_i) \geq 0$, $V(m_i) \geq 0$. Whence,

$$\frac{1}{\mu(T,\rho)}\int_{\omega_i}\rho(t)(h''(t)+f(u(t))u'(t)-f(u_1(t))u_1'(t))dt\le\frac{M}{\mu(T,\rho)}V(M_i)\le\frac{Mc_3}{\mu(T,\rho)}.$$

Since

$$\begin{aligned}&\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)(h''(t)+f(u(t))u'(t)-f(u_1(t))u_1'(t))\chi_{[h>0]}(t)dt\\ =&\sum_{i\in I}\frac{1}{\mu(T,\rho)}\int_{\omega_i}\rho(t)(h''(t)+f(u(t))u'(t)-f(u_1(t))u_1'(t))dt,\end{aligned}$$

using the fact that there exists at most a component satisfying $-T = m_i < M_i < T$ and $-T < m_i < M_i = T$, by (i)-(iii), (3.21) holds. For $T > T_0$, from (3.20), it follows that

$$\frac{k\delta}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)h(t)\chi_{[h\ge\varepsilon]}(t)dt\le\frac{2Mc_3}{\mu(T,\rho)}+\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)|p_2(t)|dt.$$

Then

$$\lim_{T\to+\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)h(t)\chi_{[h\ge\varepsilon]}(t)dt=0. \tag{3.22}$$

By carrying out similar arguments to the proof of (3.22), one has

$$\lim_{T\to+\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)h(t)\chi_{[h\le-\varepsilon]}(t)dt=0.$$

Then

$$\lim_{T\to+\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)|h(t)|\chi_{[|h|\geq\varepsilon]}(t)dt=0. \tag{3.23}$$

Note that, for $T>T_0$,

$$\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)|h(t)|\chi_{[|h|<\varepsilon]}(t)dt<\varepsilon,$$

then

$$\lim_{T\to+\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)|h(t)|\chi_{[|h|<\varepsilon]}(t)dt=0. \tag{3.24}$$

By (3.23) and (3.24), one has

$$\lim_{T\to+\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)|h(t)|dt=0,$$

that is, $h\in WPAA_0(\mathbb{R},\mathbb{R})$. Then $u=u_1+h\in WPAA_c(\mathbb{R},\mathbb{R})$.

Next, we show that $h'\in WPAA_0(\mathbb{R},\mathbb{R})$. Since $h(t)$ is bounded on $\mathbb{R}$, there exists a constant $\widetilde{M}>0$ such that $|h(t)|\leq\widetilde{M}$ for $t\in\mathbb{R}$. From $\lim\limits_{T\to+\infty}2M\widetilde{M}/\mu(T,\rho)=0$, it follows that, for any $\varepsilon>0$, there exists a $T_1>0$ such that $2M\widetilde{M}/\mu(T,\rho)<\varepsilon/2$ for $T>T_1$. For $T>T_1$, we have

$$\begin{aligned}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)h'(t)\chi_{[h'\geq\varepsilon]}(t)dt&\leq\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)h'(t)dt\\&\leq\frac{M(h(T)-h(-T))}{\mu(T,\rho)}\leq\frac{2M\widetilde{M}}{\mu(T,\rho)}<\frac{\varepsilon}{2},\end{aligned}$$

and

$$\begin{aligned}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)h'(t)\chi_{[h'\leq-\varepsilon]}(t)dt&\leq\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)h'(t)dt\\&\leq\frac{M(h(T)-h(-T))}{\mu(T,\rho)}\leq\frac{2M\widetilde{M}}{\mu(T,\rho)}<\frac{\varepsilon}{2}.\end{aligned}$$

Then

$$\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)|h'(t)|\chi_{[|h'|\geq\varepsilon]}(t)dt<\varepsilon.$$

In addition, one has

$$\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)|h'(t)|\chi_{[|h'|<\varepsilon]}(t)dt<\varepsilon.$$

Hence,

$$\lim_{T\to+\infty}\frac{1}{\mu(T,\rho)}\int_{-T}^{T}\rho(t)|h'(t)|dt=0,$$

that is, $h'\in WPAA_0(\mathbb{R},\mathbb{R})$. Therefore, $u'=u_1'+h'\in WPAA_c(\mathbb{R},\mathbb{R})$ and $u\in WPAA_c^{(1)}(\mathbb{R},\mathbb{R})$. □

4. Example

Consider the following Liénard equation

$$u'' + u^2 u' + e(t)\frac{1}{u^\alpha} = p(t), \quad t \in \mathbb{R}^+, \tag{4.1}$$

where $\alpha > 0, p(t) \in WPAA(\mathbb{R}^+, \mathbb{R}),\ e(t) \in AA^{(1)}(\mathbb{R}^+, \mathbb{R}^+)$ with $\inf_{t\in\mathbb{R}} e(t) > 0$. In (4.1), $f(u) = u^2, g(u) = \frac{1}{u^\alpha}$, it is not difficult to show that (H_1)-(H_4) hold. Since $g(0, +\infty) = (0, +\infty)$, if $\inf_{t\in\mathbb{R}} \frac{p(t)}{e(t)} > 0$ and $\sup_{t\in\mathbb{R}} \frac{p(t)}{e(t)} < +\infty$, then $\inf_{t\in\mathbb{R}} \frac{p(t)}{e(t)}$ and $\sup_{t\in\mathbb{R}} \frac{p(t)}{e(t)}$ are in the range of $g(0, +\infty)$. Hence, by Theorem 3.3, we reach the following claim.

Theorem 4.1. *If* $\inf_{t\in\mathbb{R}} \frac{p(t)}{e(t)} > 0$ *and* $\sup_{t\in\mathbb{R}} \frac{p(t)}{e(t)} < +\infty$*, then* (4.1) *admits a unique bounded solution* $u \in WPAA_c^{(1)}(\mathbb{R}, \mathbb{R})$.

Finally, we end this paper with numerical simulations illustrating the claim in Theorem 4.1. Fig. 1 illustrates the unique $WPAA_c^{(1)}$ solution of (4.1), where

$$e(t) = \sin\frac{1}{2 + \cos t + \cos\sqrt{2}t} + 5.1, \quad \rho(t) = 1, \quad \alpha = 1$$
$$p(t) = 1.6 + 1.5\sin\frac{1}{2 + \cos t + \cos\pi t} + \frac{8e^{-t^2}\cos^2 t}{(1+t^2)^4}.$$

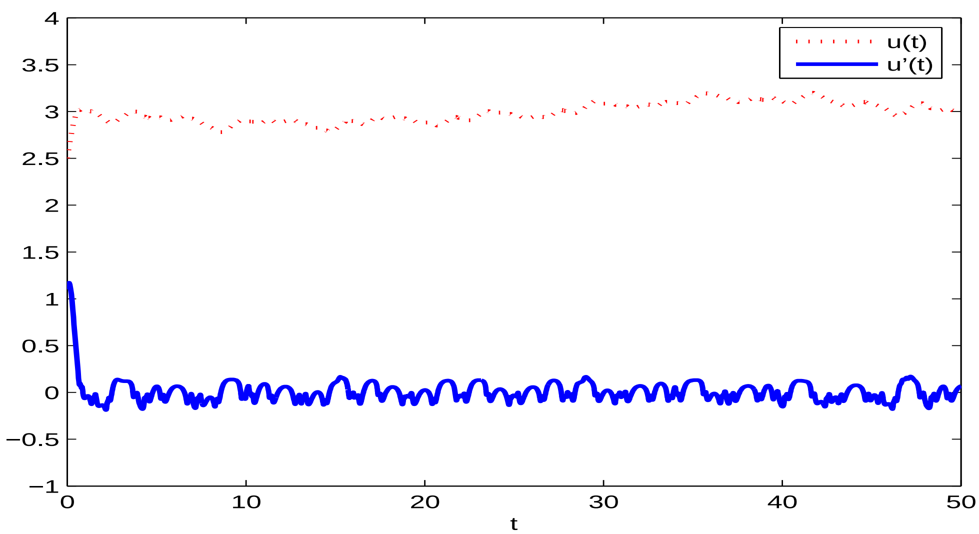

Figure 1. (4.1) admits a unique $WPAA_c^{(1)}$ solution.

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