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STOCHASTIC CENTER OF SYSTEMS OF STOCHASTIC DIFFERENTIAL EQUATIONS ON THE PLANE*

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Abstract We study a stochastic analogy of the famous center problem of Dulac for quadratic differential equations in the plane. We introduce the concept of center for systems of stochastic differential equations of Itô's type on the plane, called stochastic center. We derive a criterion for the existence of such a center. We apply it to obtain necessary and sufficient conditions for quadratic stochastic differential equations in dimension 2.

Keywords Center conditions, stochastic differential equations.

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1. Introduction

Let us consider a planar vector field given by the following ordinary differential equations

$$\frac{dx}{dt} = -y + P_2(x, y), \quad \frac{dy}{dt} = x + Q_2(x, y),$$

where P_2, Q_2 are homogeneous quadratic polynomials in x, y. The famous center problem of Dulac consists in finding conditions on the coefficients of P_2 and Q_2 such that the vector field has a center in a neighborhood of the origin (or it is analytically integrable at the origin). Since Dulac, many authors have been interested in the center problem and its generalizations. We refer to [1, 6, 10, 11] for various center conditions or integrability conditions.

Recall that the above differential system has a center at the origin if there exists a neighborhood U of the origin and an analytic function H in U such that for any $(x_0, y_0) \in U$, if $(x(t), y(t))$ is the solution of the differential system verifying the initial condition $x(0) = x_0, y(0) = y_0$, then for any $t \geq 0$, $H(x(t), y(t)) = H(x_0, y_0)$. A common way to obtain necessary conditions is to calculate the successive terms in the Taylor expression of the assumed first integral H. Then, the focal values F_k are the coefficients of the so-called obstacles to its existence:

$$H = x^2 + y^2 + \cdots, \quad \dot{H} = \sum_{k=1}^{\infty} F_k (x^2 + y^2)^{k+1},$$

where F_k are polynomial functions on the coefficients of the initial differential system. Hence $F_k = 0, k = 1, 2, \cdots$ are necessary conditions for the existence of a center at the origin.

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In the present paper, we shall give a stochastic version of the center problem for systems of stochastic differential equations on the plane.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $w(t)$ be a standard one-dimensional Wiener process on Ω. We consider stochastic differential systems of Itô's form

$$\begin{aligned} dx(t) &= f_1(x,y)dt + g_1(x,y)dw(t), \\ dy(t) &= f_2(x,y)dt + g_2(x,y)dw(t), \end{aligned} \tag{1.1}$$

where f_1, f_2, g_1, g_2 are (non random) real polynomails in the variables x, y without constant terms. That is the origin is an equilibrium of the system.

According to [14], for any $(x_0, y_0) \in \mathbb{R}^2$, there exists a unique maximal local solution defined on $[0, \sigma_{(x_0,y_0)})$, where $\sigma_{(x_0,y_0)}$, if finite, is the explosion time.

Definition 1.1. We say that system (1.1) has a stochastic center at the origin (or is stochastically integrable) if there is a non constant analytic function $H(x,y)$ in a neighborhood U of the origin such that for any $(x_0, y_0) \in U$, if $(x(t), y(t))$ is the maximal solution of system (1.1) with the initial conditions $x(0) = x_0, y(0) = y_0$, then for all $0 \le t < \sigma_{(x_0,y_0)}$,

$$H(x(t), y(t)) - H(x_0, y_0) = 0 \quad a.s. \tag{1.2}$$

Observe that if the curve $\{(x,y) \in \mathbb{R}^2 : H(x,y) = H(x_0,y_0)\}$ is closed, then $\sigma_{(x_0,y_0)} = +\infty$.

Example 1.1. We first consider a linear system of the form

$$dx(t) = (-\tfrac{1}{2}x + \lambda y)dt - ydw(t), \quad dy(t) = (-\lambda x - \tfrac{1}{2}y)dt + xdw(t), \tag{1.3}$$

or in the matrix form

$$dX(t) = AX(t)dt + BX(t)dw(t),$$

where $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $A = \begin{pmatrix} -\frac{1}{2} & \lambda \\ -\lambda & -\frac{1}{2} \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, where λ is a real constant.

Since $AB = BA$, one can solve the above system explicitly by using Itô's formula:

$$X(t) = \exp[(A - \frac{1}{2}B^2)t + Bw(t)]X(0) = \exp[B(\lambda t + w(t))]X(0).$$

Then one has, for any $t \ge 0$,

$$\begin{aligned} x(t) &= x_0 \cos(\lambda t + w(t)) - y_0 \sin(\lambda t + w(t)), \\ y(t) &= x_0 \sin(\lambda t + w(t)) + y_0 \cos(\lambda t + w(t)). \end{aligned}$$

It is clear now that for any $t \ge 0$, $x(t)^2 + y(t)^2 = x_0^2 + y_0^2$ a.s. Therefore the linear system (1.3) has a stochastic center at the origin.

We shall study a quadratic perturbation of the linear system (1.3), that is systems of the form

$$\begin{aligned} dx(t) &= (-\tfrac{1}{2}x + \lambda y + F_2)dt + (-y + P_2)dw(t), \\ dy(t) &= (-\lambda x - \tfrac{1}{2}y + G_2)dt + (x + Q_2)dw(t), \end{aligned} \tag{1.4}$$

where F_2, G_2, P_2, Q_2 are all homogeneous quadratic polynomials in x, y. We shall give necessary and sufficient conditions for the above system to have a stochastic center at the origin.

The present paper is organized as follows. We first give a simple criterion in Section 2 to determine if a general system possesses a stochastic center at the origin. In Section 3 we study quadratic systems (1.4) and give necessary and sufficient conditions for it to have a stochastic center at the origin.

2. A criterion for a stochastic differential system to have a stochastic center at the origin

Local invariant curves for stochastic differential system of the form (1.1) have been studied in [9], where necessay and sufficient conditions are given. For completeness of the paper we give the conditions and the proof here.

Theorem 2.1. *Let notations be as in Definition 1.1. Then (1.2) is verified for H for any (x_0, y_0) in a neighborhood U of the origin, if and only if for all $(x, y) \in U$,*

$$g_1(x,y)\frac{\partial H}{\partial x}(x,y) + g_2(x,y)\frac{\partial H}{\partial y}(x,y) = 0, \tag{2.1}$$

$$L_2(H) = 0, \tag{2.2}$$

where L_2 is the operator

$$L_2 = f_1(x,y)\frac{\partial}{\partial x} + f_2(x,y)\frac{\partial}{\partial y} + \frac{1}{2}\Big(g_1^2\frac{\partial^2}{\partial x^2} + 2g_1g_2\frac{\partial^2}{\partial x\partial y} + g_2^2\frac{\partial^2}{\partial y^2}\Big).$$

Proof. For any $(x_0, y_0) \in U$, according to Itô's formula, one has for all $0 \le t < \sigma_{(x_0,y_0)}$,

$$\begin{aligned} &H(x(t), y(t)) \\ =&H(x_0, y_0) + \int_0^t L_2(H(x(s), y(s)))ds \\ &+ \int_0^t \Big(g_1(x(s), y(s))\frac{\partial H}{\partial x}(x(s), y(s)) + g_2(x(s), y(s))\frac{\partial H}{\partial y}(x(s), y(s))\Big)dw(s). \end{aligned}$$

It is clear that if (2.1) and (2.2) hold, (1.2) follows.

We now prove the necessity. Since $H(x(t), y(t)) = H(x_0, y_0)$ a.s., one has that

$$\int_0^t L_2(H(x(s), y(s)))ds = -\int_0^t \Big(g_1(x(s), y(s))\frac{\partial H}{\partial x} + g_2(x(s), y(s))\frac{\partial H}{\partial y}\Big)dw(s), \ \textit{a.s.}$$

The right-hand side of the above equation is a local martingale, contiuous a.s., with the variance $\int_0^t \Big(g_1(x(s),y(s))\frac{\partial H}{\partial x}+g_2(x(s),y(s))\frac{\partial H}{\partial y}\Big)^2 ds$. And the left-hand side is absolutely continous a.s. It follows that (see [16]), for all $0 \le t < \sigma_{(x_0,y_0)}$,

$$\int_0^t L_2(H(x(s),y(s)))ds = 0,$$

and

$$\int_0^t \Big(g_1(x(s),y(s))\frac{\partial H}{\partial x} + g_2(x(s),y(s))\frac{\partial H}{\partial y}\Big)^2 ds = 0.$$

If there exists $(x_1,y_1) \in U$ such that $L_2(H)(x_1,y_1) \neq 0$, say $L_2(H)(x_1,y_1) > 0$, then there exist an neighborhood U_0 of (x_1,y_1) such that $L_2(H)(x,y) > 0$ for all $(x,y) \in U_0$.

Let $(x(t),y(t))$ be the solution of (1.1) with the initial condition $x(0) = x_1, y(0) = y_1$. Then there exists $t_1 > 0$ such that $L_2(H(x(s),y(s))) > 0$ for all $0 \le s \le t_1$. Therefore

$$\int_0^{t_1} L_2(H(x(s),y(s)))ds \neq 0,$$

which is a contradiction. Hence we obtain condition (2.2).

Again, if there exists $(x_0,y_0) \in U$ such that

$$g_1(x_0,y_0)\frac{\partial H}{\partial x}(x_0,y_0) + g_2(x_0,y_0)\frac{\partial H}{\partial y}(x_0,y_0) \neq 0,$$

then there exists a neighborhoof U_0 of (x_0,y_0) such that

$$g_1(x,y)\frac{\partial H}{\partial x}(x,y) + g_2(x,y)\frac{\partial H}{\partial y}(x,y) \neq 0,$$

for all $(x,y) \in U_0$. Let $(x(t),y(t))$ be the maximal solution of system (1.1) with the initial condition $x(0) = x_0, y(0) = y_0$. Then according to the continuity, there exists a $t_1 > 0$ such that for $0 < s \le t_1$, one has

$$g_1(x(s),y(s))\frac{\partial H}{\partial x}(x(s),y(s)) + g_2(x(s),y(s))\frac{\partial H}{\partial y}(x(s),y(s)) \neq 0.$$

Hence

$$\int_0^{t_1} \Big(g_1(x(s),y(s))\frac{\partial H}{\partial x} + g_2(x(s),y(s))\frac{\partial H}{\partial y}\Big)^2 ds \neq 0,$$

which is a contradiction. Condition (2.1) follows. □

We now give the following criterion for a stochastic differential system to have a stochastic center at the origin.

Theorem 2.2. *Consider stochastic differential system* (1.1)*. Then it has a stochastic center at the origin if and only if the differential system*

$$\frac{dx}{dt} = g_1(x,y), \quad \frac{dy}{dt} = g_2(x,y), \tag{2.3}$$

is integrable at the origin and there exists a neighborhood of the origin in which

$$g_1 f_2 - g_2 f_1 - \frac{g_1}{2}\Big(\frac{\partial g_1 g_2}{\partial x} + \frac{\partial g_2^2}{\partial y}\Big) + \frac{g_2}{2}\Big(\frac{\partial g_1^2}{\partial x} + \frac{\partial g_1 g_2}{\partial y}\Big) = 0. \tag{2.4}$$

Proof. System (1.1) has a stochastic center at the origin if and only if conditions (2.1) and (2.2) are satisfied in a neighborhood of the origin. It is clear that condition (2.1) signifies that differential system (2.3) is integrable at the origin. We now suppose that (2.1) is satisfied and prove that condition (2.2) is equivalent to condition (2.4).

Multiplying (2.1) by g_1 and differentiating with respect to x lead to

$$g_1^2\frac{\partial^2 H}{\partial x^2}+\frac{\partial g_1^2}{\partial x}\frac{\partial H}{\partial x}+g_1g_2\frac{\partial^2 H}{\partial x\partial y}+\frac{\partial(g_1g_2)}{\partial x}\frac{\partial H}{\partial y}=0.$$

Similarly multiplying (2.1) by g_2 and differentiating with respect to y lead to

$$g_2^2\frac{\partial^2 H}{\partial y^2}+\frac{\partial g_2^2}{\partial y}\frac{\partial H}{\partial y}+g_1g_2\frac{\partial^2 H}{\partial x\partial y}+\frac{\partial(g_1g_2)}{\partial y}\frac{\partial H}{\partial x}=0.$$

Then

$$g_1^2\frac{\partial^2 H}{\partial x^2}+2g_1g_2\frac{\partial^2 H}{\partial x\partial y}+g_2^2\frac{\partial^2 H}{\partial y^2}=-\Big(\frac{\partial g_1^2}{\partial x}+\frac{\partial(g_1g_2)}{\partial y}\Big)\frac{\partial H}{\partial x}-\Big(\frac{\partial(g_1g_2)}{\partial x}+\frac{\partial g_2^2}{\partial y}\Big)\frac{\partial H}{\partial y}.$$

Hence

$$L_2(H)=\tilde f_1(x,y)\frac{\partial H}{\partial x}+\tilde f_2(x,y)\frac{\partial H}{\partial y},$$

where

$$\tilde f_1=f_1-\frac{1}{2}\Big(\frac{\partial g_1^2}{\partial x}+\frac{\partial g_1g_2}{\partial y}\Big),\quad \tilde f_2=f_2-\frac{1}{2}\Big(\frac{\partial g_1g_2}{\partial x}+\frac{\partial g_2^2}{\partial y}\Big).$$

Using (2.1), one has $L_2(H)=0$ in a neighborhood of the origin if and only if $g_1\tilde f_2-g_2\tilde f_1=0$ in a neighborhood of the origin. □

3. Necessary and sufficient conditions for the existence of a stochastic center for quadratic systems

We now study quadratic systems with stochastic centers. We assume that the linear part of the (non-random) differential system $\dot x=g_1(x,y)$, $\dot y=g_2(x,y)$ has a center at the origin, *i.e.*

$$g_1=-y+P_2,\quad g_2=x+Q_2,$$

where P_2,Q_2 are quadratic polynomials. Since Dulac, many authors have studied the necessary and sufficient conditions for such a system to be integrable at the origin (see [1, 6, 10]. Complete conditions are given in [1]. In [17] complexes variables are used to state the center conditions in a very simple form. We state it here for later use. Let $z=x+iy$ and consider

$$\dot z=iz+Az^2+Bz\bar z+C\bar z^2,$$

where $A=a_1+ia_2, B=b_1+ib_2$ and $C=c_1+ic_2$ are complex constants.

If $B\neq 0$ then one can change it to 1 by a complex scaling. The above equation in the real form is the following systems according to $B=0$ or $B=1$:

$$\begin{aligned}\dot x&=g_1(x,y)=-y+(a_1+c_1)x^2+(2c_2-2a_2)xy+(-a_1-c_1)y^2,\\ \dot y&=g_2(x,y)=x+(a_2+c_2)x^2+(2a_1-2c_1)xy+(-a_2-c_2)y^2,\end{aligned}\tag{3.1}$$

or

$$
\begin{aligned}
\dot{x} &= g_1(x,y) = -y + (1+a_1+c_1)x^2 + (2c_2-2a_2)xy + (1-a_1-c_1)y^2, \\
\dot{y} &= g_2(x,y) = x + (a_2+c_2)x^2 + (2a_1-2c_1)xy + (-a_2-c_2)y^2.
\end{aligned} \tag{3.2}
$$

The integrability conditions of [7] in the real form state as follows.

Theorem 3.1. *Let notations be as above.*

(1) *System* (3.1) *is integrable at the origin.*

System (3.2) *is integrable at the origin if and only if one of the following conditions is satisfied,*

(2) $2a_1 + 1 = a_2 = 0$,

(3) $a_2 = c_2 = 0$,

(4) $a_1 - 1 = a_2 = c_1^2 + c_2^2 - 1 = 0$.

In this section we consider stochastic differential systems of the form (1.4) with g_1, g_2 as in (3.1) or (3.2). Let F_2, G_2 be in the following form

$$
\begin{aligned}
F_2 &= A_1x^2 + A_2xy + A_3y^2, \\
G_2 &= A_4x^2 + A_5xy + A_6y^2,
\end{aligned} \tag{3.3}
$$

where the A_i are real constants.

We first remark that if system (3.1) or (3.2) is integrable at the origin, then its first integral is in the form $H(x,y) = x^2 + y^2 + \sum_{i+j\geq 3} h_{ij}x^iy^j$. Hence $H(x,y) = C$ is a closed curve in a neighborhood of the origin. Therefore if system (1.4) has a stochastic center at the origin, then there exists a neighborhood of the origin such that any solution with initial conditions (x_0, y_0) near the origin is defined in $[0, +\infty)$.

Our aim is to look for conditions on the coefficients of F_2, G_2, P_2, Q_2 such that a stochastic differential system of the form (1.4) has a stochastic center at the origin. We have the following results.

Theorem 3.2. *Consider the following stochastic differential equations*

$$
\begin{aligned}
dx(t) &= (-\tfrac{1}{2}x + \lambda y + F_2)dt + (-y + P_2)dw(t), \\
dy(t) &= (-\lambda x - \tfrac{1}{2}y + G_2)dt + (x + Q_2)dw(t),
\end{aligned} \tag{3.4}
$$

where $g_1 = -y + P_2, g_2 = x + Q_2$ *are as in* (3.1) *or* (3.2) *and* F_2, G_2 *are as in* (3.3). *Then it has a stochastic center at the origin if and only if one of the following conditions is fulfilled,*

(1.1) g_1, g_2 *are as in* (3.1) *with* $a_1^2 + a_2^2 - c_1^2 - c_2^2 = 0$, $(a_1, a_2) \neq (0,0)$, *and*

$$
\begin{aligned}
&A_1 = -(a_1+c_1)\lambda - \tfrac{3}{2}a_2 + \tfrac{1}{2}c_2, && A_2 = 2(a_2-c_2)\lambda - a_1 - c_1, \\
&A_3 = (a_1+c_1)\lambda - \tfrac{1}{2}a_2 - \tfrac{1}{2}c_2, && A_4 = -(a_2+c_2)\lambda - \tfrac{1}{2}a_1 - \tfrac{1}{2}c_1, \\
&A_5 = 2(-a_1+c_1)\lambda - a_2 - c_2, && A_6 = (a_2+c_2)\lambda - \tfrac{3}{2}a_1 + \tfrac{1}{2}c_1.
\end{aligned}
$$

(1.2) *g_1, g_2 are as in (3.1) with $a_1 = c_1 = a_2 = c_2 = 0$, and $A_1 = A_6 = 0, A_2 = -A_4, A_5 = -A_3$.*

(2.1) *g_1, g_2 are as in (3.2) with $a_1 = c_1 = -\frac{1}{2}, a_2 = c_2 = 0$, and $A_1 = A_4 = A_5 = 0, A_2 = 2, A_3 = -2\lambda, A_6 = 1$.*

(3.1) *g_1, g_2 are as in (3.2) with $a_2 = c_2 = 0$, $c_1 = a_1$, and*

$$\begin{aligned} &A_1 = -(2a_1+1)\lambda, \ A_2 = -2a_1 + 1, \ A_3 = (2a_1 - 1)\lambda, \\ &A_4 = -a_1 - \tfrac{1}{2}, \qquad A_5 = 0, \qquad\qquad A_6 = -a_1 + \tfrac{1}{2}. \end{aligned}$$

(3.2) *g_1, g_2 are as in (3.2) with $a_2 = c_2 = 0$, $c_1 = 0, a_1 = 1$, and $A_1 = -2\lambda, A_2 = A_6 = -1, A_3 = A_4 = 0, A_5 = -2\lambda$.*

Proof. Let

$$P = g_1 f_2 - g_2 f_1 - \frac{g_1}{2}\Big(\frac{\partial g_1 g_2}{\partial x} + \frac{\partial g_2^2}{\partial y}\Big) + \frac{g_2}{2}\Big(\frac{\partial g_1^2}{\partial x} + \frac{\partial g_1 g_2}{\partial y}\Big).$$

Then according to Theorem 2.2, system (3.4) has a stochastic center at the origin if and only if the differential system $\dot{x} = g_1(x,y), \dot{y} = g_2(x,y)$ is integrable and $P = 0$. Then it is necessary that g_1, g_2 be in one of the 4 cases in Theorem 3.1.

We study the 4 cases with g_1, g_2 verifying the conditions in Theorem 3.1. Since $g_1 = -y + \cdots, g_2 = x + \cdots$ where the dots represent higher order terms, one has that P is a polynomial of degree 5 and in all cases the homogeneous quadratic part in P is zero. Then we can write P in the form

$$P = \sum_{i=0}^{3} \alpha_i x^{3-i} y^i + \sum_{i=0}^{4} \beta_i x^{4-i} y^i + \sum_{i=0}^{5} \mu_i x^{5-i} y^i.$$

3.1. Case (1).

In the actual case $b_1 = b_2 = 0$, one can compute the coefficients of P to obtain

$$\begin{aligned} &\alpha_0 = -A_1 - \tfrac{3}{2}a_2 + \tfrac{1}{2}c_2 - (a_1 + c_1)\lambda, \\ &\alpha_1 = -\tfrac{3}{2}a_1 - \tfrac{3}{2}c_1 - A_4 - A_2 + (a_2 - 3c_2)\lambda, \\ &\alpha_2 = -A_3 - A_5 - \tfrac{3}{2}a_2 - \tfrac{3}{2}c_2 + (3c_1 - a_1)\lambda, \\ &\alpha_3 = \tfrac{1}{2}c_1 - \tfrac{3}{2}a_1 + (a_2 + c_2)\lambda - A_6. \end{aligned}$$

Hence from $\alpha_i = 0$ for all i, one obtains

$$\begin{aligned} &A_1 = -\tfrac{3}{2}a_2 + \tfrac{1}{2}c_2 - (a_1 + c_1)\lambda, \qquad A_2 = -A_4 - \tfrac{3}{2}a_1 - \tfrac{3}{2}c_1 + (a_2 - 3c_2)\lambda, \\ &A_5 = -A_3 - \tfrac{3}{2}a_2 - \tfrac{3}{2}c_2 + (3c_1 - a_1)\lambda, \ A_6 = \tfrac{1}{2}c_1 - \tfrac{3}{2}a_1 + (a_2 + c_2)\lambda. \end{aligned}$$

Substituting them in P, we obtain, by denoting $K = a_1^2 + a_2^2 - c_1^2 - c_2^2$,

$$\begin{aligned} &\mu_0 = -(a_2 + c_2)K, \quad \mu_1 = (-a_1 + 3c_1)K, \ \ \mu_2 = 2(c_2 - a_2)K, \\ &\mu_3 = 2(-a_1 + c_1)K, \ \ \mu_4 = (-a_2 + 3c_2)K, \ \ \mu_5 = -(c_1 + a_1)K, \end{aligned}$$

and

$$\begin{aligned}
\beta_0 =& (c_1+a_1)A_4+(a_2+c_2)(c_1+a_1)\lambda+\frac{3}{2}c_1^2-\frac{1}{2}a_1^2+c_2^2+a_1c_1-a_2^2,\\
\beta_1 =& \left(3c_2-a_2\right)A_4-\left(a_1+c_1\right)A_3+\left(3c_2^2+a_1^2+2a_2c_2-a_2^2+c_1^2+2a_1c_1\right)\lambda\\
&-\left(a_2-c_2\right)\left(c_1+a_1\right),\\
\beta_2 =& \left(a_2-3\,c_2\right)A_3+(a_1-3c_1)A_4+(-4c_1a_2+4a_1c_2)\lambda\\
&-\frac{3}{2}a_2^2-\frac{3}{2}a_1^2+\frac{1}{2}c_1^2-a_1c_1-a_2c_2+\frac{1}{2}c_2^2,\\
\beta_3 =& \left(3c_1-a_1\right)A_3+\left(-a_2-c_2\right)A_4+\left(a_1^2-a_2^2-3c_1^2-c_2^2-2a_1c_1-2a_2c_2\right)\lambda\\
&-\left(a_2+c_2\right)\left(-c_1+a_1\right),\\
\beta_4 =& \left(a_2+c_2\right)A_3-\left(a_2+c_2\right)\left(c_1+a_1\right)\lambda+a_2c_2-\frac{1}{2}a_2^2+\frac{3}{2}c_2^2-a_1^2+c_1^2.
\end{aligned}$$

Since it is necessary that $\mu_j=0$ for all j, we have necessarily $K=0$ and in this case $\mu_j=0$ for all j.

If $c_1+a_1\neq 0$, then one can find A_3, A_4 from $\beta_0=\beta_1=0$. By using $K=0$, we have

$$A_4=\left(-a_2-c_2\right)\lambda-\frac{1}{2}a_1-\frac{1}{2}c_1,\quad A_3=(a_1+c_1)\lambda-\frac{1}{2}a_2-\frac{1}{2}c_2.$$

Finally we have also, by using $K=0$,

$$\begin{aligned}
&A_1=-(a_1+c_1)\lambda-\frac{3}{2}a_2+\frac{1}{2}c_2,\ \ A_2=2(a_2-c_2)\lambda-a_1-c_1,\\
&A_5=2(-a_1+c_1)\lambda-a_2-c_2,\qquad A_6=(a_2+c_2)\lambda-\tfrac{3}{2}a_1+\tfrac{1}{2}c_1.
\end{aligned}$$

Using these results, we obtain $\beta_2=\beta_3=\beta_4=0$ since $K=0$. Hence $P=0$ and we get condition (1.1).

Now we consider the case with $c_1=-a_1$. In this case we have

$$\beta_0=(c_2-a_2)(c_2+a_2)=0.$$

- If $c_2=a_2\neq 0$, then

$$\begin{aligned}
&\beta_0=0,\ \ \beta_1=2a_2(2a_2\lambda+A_4),\ \ \beta_2=-2a_2A_3+4a_1A_4+2a_2(4a_1\lambda-a_2),\\
&\beta_3=-4a_1A_3-2a_2A_4-4a_2(a_2\lambda+a_1),\ \ \beta_4=2a_2(a_2+A_3).
\end{aligned}$$

  Then $A_3=-a_2, A_4=-2a_2\lambda$ and hence $\beta_i=0$ for all i. This case is included in condition (1.1).

- If $c_2=-a_2$, then

$$\beta_0=0,\ \ \beta_1=-4a_2A_4,\ \ \beta_2=4(a_2A_3+a_1A_4),\ \ \beta_3=-a_1A_3,\ \ \beta_4=0.$$

  Hence we get $a_1A_3=a_2A_4=a_2A_3+a_1A_4=0$ which is included in case (1.1) if $(a_1,a_2)\neq(0,0)$.
  If $a_1=a_2=0$, then $c_1=c_2=0$, $\beta_i=0$ for all i with any A_3, A_4. We obtain condition (1.2).

3.2. Case (2)

Now g_1, g_2 are as in (3.2). Substituing $a_1 = -\frac{1}{2}, a_2 = 0$, in g_1, g_2, we obtain

$$\alpha_0 = (-\tfrac{1}{2} - c_1)\lambda + \tfrac{1}{2}c_2 - A_1, \qquad \alpha_1 = \tfrac{5}{4} - \tfrac{3}{2}c_1 - 3c_2\lambda - A_4 - A_2,$$
$$\alpha_2 = (3c_1 - \tfrac{1}{2})\lambda - \tfrac{3}{2}c_2 - A_5 - A_3, \quad \alpha_3 = \tfrac{1}{2}c_1 + \tfrac{5}{4} + c_2\lambda - A_6.$$

Hence from $\alpha_i = 0$ for all i, we obtain

$$A_1 = (-\tfrac{1}{2} - c_1)\lambda + \tfrac{1}{2}c_2, \qquad A_2 = \tfrac{5}{4} - \tfrac{3}{2}c_1 - 3c_2\lambda - A_4,$$
$$A_5 = (3c_1 - \tfrac{1}{2})\lambda - \tfrac{3}{2}c_2 - A_3, \; A_6 = \tfrac{1}{2}c_1 + \tfrac{5}{4} + c_2\lambda.$$

Again by putting them in P, we have

$$\begin{aligned}
\beta_0 &= (\tfrac{1}{2} + c_1)A_4 + \tfrac{1}{2}c_2(1 + 2c_1)\lambda + \tfrac{3}{2}c_1 + \tfrac{3}{2}c_1^2 + \tfrac{3}{8} + c_2^2,\\
\beta_1 &= \left(-\tfrac{1}{2} - c_1\right) A_3 + 3c_2A_4 + \left(3c_2^2 - \tfrac{3}{4} - c_1 + c_1^2\right) \lambda + \tfrac{1}{2}c_2 \left(2c_1 + 5\right),\\
\beta_2 &= -3c_2A_3 + \left(\tfrac{1}{2} - 3c_1\right) A_4 + \tfrac{1}{2}{c_2}^2 + \tfrac{1}{2}{c_1}^2 - \tfrac{1}{2}c_1 - \tfrac{3}{8} - 4c_2\lambda,\\
\beta_3 &= \left(3c_1 - \tfrac{1}{2}\right) A_3 - c_2A_4 + \left(-3{c_1}^2 - \tfrac{3}{4} - c_2^2 + 5c_1\right) \lambda + \tfrac{1}{2}c_2 \left(2c_1 + 3\right),\\
\beta_4 &= c_2A_3 - \tfrac{1}{2}c_2(2c_1 - 3)\lambda - \tfrac{3}{4} + \tfrac{3}{2}c_2^2 - c_1 + c_1^2,
\end{aligned}$$

and

$$\mu_0 = \tfrac{1}{4}c_2 \left(1 + 4c_2^2 + 4c_1^2 + 4c_1\right), \qquad \mu_1 = \left(\tfrac{1}{2} - 3c_1\right) c_2^2 - \tfrac{3}{8} \left(1 + 2c_1\right)^3,$$
$$\mu_2 = -\tfrac{1}{2}c_2 \left(20c_1 + 9 + 4c_1^2 + 4c_2^2\right), \quad \mu_3 = \tfrac{3}{4} - 9c_2^2 + \tfrac{5}{2}c_1 + c_1^2 - 2c_1^3 - 2c_1c_2^2,$$
$$\mu_4 = -\tfrac{1}{4}c_2 \left(3 + 12c_2^2 - 20c_1 + 12c_1^2\right), \; \mu_5 = \tfrac{1}{8}\left(2c_1 - 3\right)\left(4c_1^2 - 4c_1 + 4c_2^2 - 3\right).$$

By considering μ_4 and μ_5, we have the following:

- If $c_1 = \frac{3}{2}$, then we get $\mu_4 = -3c_2^3$. Hence $c_2 = 0$. Substituing it in μ_1 leads to $\mu_1 = -24 \neq 0$. Therefore it is impossible to have $P = 0$. One then has $c_1 \neq \frac{3}{2}$.
- If $c_2 \neq 0$, then from $\mu_4 = \mu_5 = 0$ we have

  $$K_1 = 3 + 12c_2^2 - 20c_1 + 12c_1^2 = 0, \quad K_2 = 4c_1^2 - 4c_1 + 4c_2^2 - 3 = 0.$$

  Since $K_1 - 3K_2 = 4(3 - 2c_1) \neq 0$, this is a contradiction.

Therefore $c_2 = 0$, and $\mu_5 = \frac{1}{8}(2c_1 + 1)(2c_1 - 3)^2$, which yields to $c_1 = -\frac{1}{2}$. Then we get

$$\beta_2 = 2A_4, \quad \beta_3 = -2\,A_3 - 4\,\lambda.$$

One then has $A_4 = 0, A_3 = -2\lambda$, in which case we have $P = 0$. We obtain also $A_1 = A_5 = 0, A_2 = 2, A_6 = 1$, which are condition (2.1).

3.3. Case (3)

We obtain by using $a_2 = c_2 = 0$,

$$\alpha_0 = (-a_1 - 1 - c_1)\,\lambda - A_1, \qquad \alpha_1 = -\tfrac{3}{2}a_1 - \tfrac{3}{2}c_1 + \tfrac{1}{2} - A_4 - A_2,$$
$$\alpha_2 = (-a_1 + 3c_1 - 1)\lambda - A_3 - A_5, \quad \alpha_3 = -\tfrac{3}{2}a_1 + \tfrac{1}{2}c_1 + \tfrac{1}{2} - A_6.$$

Hence from $\alpha_i = 0$ for all i, one obtains

$$A_1 = (-a_1 - 1 - c_1)\,\lambda, \qquad A_2 = -\tfrac{3}{2}a_1 - \tfrac{3}{2}c_1 + \tfrac{1}{2} - A_4,$$
$$A_5 = (-a_1 + 3c_1 - 1)\lambda - A_3,\ A_6 = -\tfrac{3}{2}a_1 + \tfrac{1}{2}c_1 + \tfrac{1}{2}.$$

Substituing them in P, we get

$$\begin{aligned}
\beta_0 &= (a_1 + c_1 + 1)\,A_4 - \tfrac{1}{2}(a_1 + c_1 + 1)(a_1 - 3c_1 - 1),\\
\beta_1 &= -(a_1 + c_1 + 1)A_3 + \lambda(a_1 + c_1 + 1)(a_1 + c_1 - 1),\\
\beta_2 &= (-3c_1 + a_1 + 1)\,A_4 - \tfrac{1}{2}\,(a_1 + c_1 - 1)\,(3a_1 + 1 - c_1)\,,\\
\beta_3 &= (-a_1 + 3c_1 - 1)\,A_3 + \lambda\,(a_1 + c_1 - 1)\,(-3c_1 + a_1 + 1)\,,\\
\beta_4 &= -\,(a_1 + c_1 - 1)\,(a_1 - c_1)\,,
\end{aligned}$$

and

$$\begin{aligned}
&\mu_0 = \mu_2 = \mu_4 = 0,\\
&\mu_1 = -\,(a_1 - 3c_1 - 1)\,(a_1 + c_1 + 1)\,(a_1 - c_1)\,,\\
&\mu_3 = -2\,(a_1 + c_1 - 1)\,(a_1 - c_1)^2\,,\\
&\mu_5 = -\,(a_1 - c_1)\,(a_1 + c_1 - 1)^2\,.
\end{aligned}$$

From $\mu_5 = 0$, one has either $c_1 = a_1$ or $c_1 = 1 - a_1$.

- If $c_1 = a_1$, then

  $$\beta_0 = (2a_1 + 1)(A_4 + a_1 + \tfrac{1}{2}),\ \beta_1 = (2a_1 + 1)[-A_3 + (2a_1 - 1)\lambda],$$
  $$\beta_2 = (1 - 2a_1)(A_4 + a_1 + \tfrac{1}{2}),\ \beta_3 = (2a_1 - 1)[A_3 - (2a_1 - 1)\lambda].$$

  One gets $A_4 = -a_1 - \frac{1}{2}$, $A_3 = (2a_1 - 1)\lambda$. Therefore

  $$A_1 = -(2a_1 + 1)\lambda, A_2 = -2a_1 + 1, A_5 = 0, A_6 = -a_1 + \frac{1}{2},$$

  which is condition (3.1).

- If $c_1 = 1 - a_1$, then $\mu_1 = -8(a_1 - 1)(2a_1 - 1)$. One obtains a new case for $a_1 = 1, c_1 = 0$. Hence $A_3 = A_4 = 0$ and $A_1 = -2\lambda, A_2 = A_6 = -1, A_5 = -2\lambda$. This is condition (3.2).

3.4. Case (4)

We now consider case (4) in Theorem 3.1. In this case $a_1 = 2$, $a_2 = 0$, $c_1^2 + c_2^2 - 1 = 0$. One obtains for example

$$\mu_1 = -(5c_1 - 1)(c_1 - 3), \quad \mu_5 = -(c_1 + 1)^2.$$

Therefore it is impossible to have $\mu_1 = \mu_5 = 0$. Hence no system in this case can have a stochastic center.

This completes the proof of Theorem 3.2. $\square$

4. Numerical simulations

Numerical simulations are done in Matlab by using the Euler-Maruyama method (see for example [8]). We present here simulation results of some of the cases of Theorem 3.2. Figure 1 corresponds to the Case 1.1, and Figure 2 to the case 2.1.

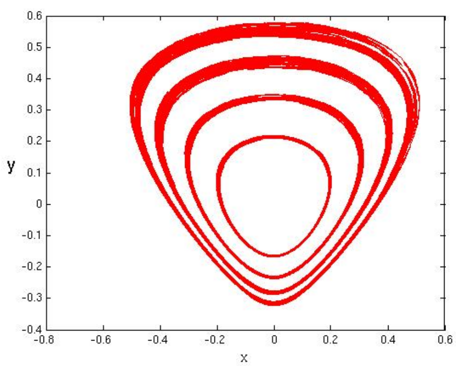

Figure 1. Some results in case 1.1. with $\lambda = 1$, $a_1 = c_1 = 1$, $a_2 = c_2 = 0$ and different initial conditions.

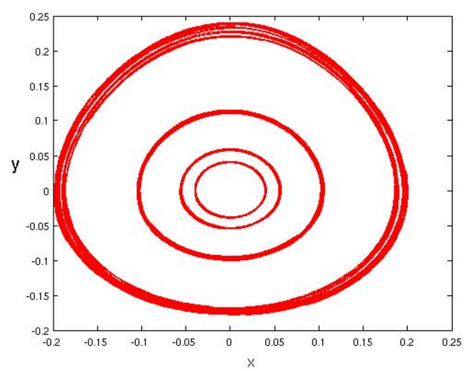

Figure 2. Some results in case 2.1. with $\lambda = 2$ and different initial conditions.

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