Journal of Applied Analysis and Computation
Volume 7, Number 1, February 2017, 134–146
Website:http://jaac-online.com/
DOI:10.11948/2017010

EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS FOR A SYSTEM OF DIFFERENCE EQUATIONS WITH COUPLED BOUNDARY CONDITIONS*

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Abstract We study the existence and multiplicity of positive solutions for a system of nonlinear second-order difference equations subject to coupled multi-point boundary conditions.

Keywords Difference equations, coupled multi-point boundary conditions, positive solutions, existence, multiplicity.

MSC(2010) 39A10.

1. Introduction

The mathematical modeling of many nonlinear problems from computer science, economics, mechanical engineering, control systems, biological neural networks and others, leads to the consideration of nonlinear difference equations (see Kelley and Peterson [16], Lakshmikantham and Trigiante [17]). In the last decades, many authors have investigated such problems by using various methods, such as fixed point theorems, the critical point theory, upper and lower solutions, the fixed point index theory and the topological degree theory (see for example [1, 3–9, 15, 18–22]).

In this paper, we consider the system of nonlinear second-order difference equations

$$(S) \qquad \begin{cases} \Delta^2 u_{n-1} + f(n, v_n) = 0, & n = \overline{1, N-1}, \\ \Delta^2 v_{n-1} + g(n, u_n) = 0, & n = \overline{1, N-1}, \end{cases}$$

with the coupled multi-point boundary conditions

$$(BC) \qquad u_0 = 0, \;\; u_N = \sum_{i=1}^{p} a_i v_{\xi_i}, \;\; v_0 = 0, \;\; v_N = \sum_{i=1}^{q} b_i u_{\eta_i},$$

where $N \in \mathbb{N}$, $N \geq 2$, p, $q \in \mathbb{N}$, Δ is the forward difference operator with stepsize 1, $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_{n-1} = u_{n+1} - 2u_n + u_{n-1}$, $n = \overline{k, m}$ means that $n = k, k+1, \ldots, m$ for k, $m \in \mathbb{N}$, $a_i \in \mathbb{R}$, $\xi_i \in \mathbb{N}$ for all $i = \overline{1, p}$, $b_i \in \mathbb{R}$, $\eta_i \in \mathbb{N}$ for all $i = \overline{1, q}$, $1 \leq \xi_1 < \cdots < \xi_p \leq N-1$ and $1 \leq \eta_1 < \cdots < \eta_q \leq N-1$.

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*The work of R. Luca was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0557.

Under sufficient conditions on the functions f and g, we study the existence and multiplicity of positive solutions of problem $(S)-(BC)$ by using some theorems from the fixed point index theory. By a positive solution of problem $(S)-(BC)$ we mean a pair of sequences $(u,v) = ((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}})$ satisfying (S) and (BC), with $u_n \geq 0$, $v_n \geq 0$ for all $n = \overline{0,N}$ and $(u,v) \neq (0,0)$.

The existence of positive solutions for the system of nonlinear second-order difference equations with two parameters λ and μ, namely the system

$$(S_1) \qquad \begin{cases} \Delta^2 u_{n-1} + \lambda f(n, u_n, v_n) = 0, & n = \overline{1, N-1}, \\ \Delta^2 v_{n-1} + \mu g(n, u_n, v_n) = 0, & n = \overline{1, N-1}, \end{cases}$$

with the coupled boundary conditions (BC) was investigated in Henderson and Luca [13]. The system (S_1) with the uncoupled boundary conditions

$$(BC_1) \qquad u_0 = \sum_{i=1}^{p} a_i u_{\xi_i}, \ \ u_N = \sum_{i=1}^{q} b_i u_{\eta_i}, \ \ v_0 = \sum_{i=1}^{r} c_i v_{\zeta_i}, \ \ v_N = \sum_{i=1}^{l} d_i v_{\rho_i},$$

has been investigated in Henderson and Luca [11], and in Henderson and Luca [12] by using the Guo-Krasnosel'skii fixed point theorem. We also mention the paper Henderson etc [14], where the authors studied the existence and multiplicity of positive solutions for the system (S) with the multi-point boundary conditions (BC_1).

In Section 2, we present some auxiliary results from Henderson and Luca [13] which investigate a system of second-order difference equations subject to the coupled boundary conditions (BC). In Section 3, we prove the main theorems for the existence and multiplicity of the positive solutions with respect to a cone for our problem $(S)-(BC)$ which are based on three theorems from the fixed point index theory. An example is presented in Section 4 to illustrate our main results.

2. Auxiliary results

In this section, we present some auxiliary results from Henderson and Luca [13] related to the following system of second-order difference equations

$$\begin{cases} \Delta^2 u_{n-1} + x_n = 0, & n = \overline{1, N-1}, \\ \Delta^2 v_{n-1} + y_n = 0, & n = \overline{1, N-1}, \end{cases} \tag{2.1}$$

with the coupled multi-point boundary conditions

$$u_0 = 0, \ \ u_N = \sum_{i=1}^{p} a_i v_{\xi_i}, \ \ v_0 = 0, \ \ v_N = \sum_{i=1}^{q} b_i u_{\eta_i}, \tag{2.2}$$

where $N \in \mathbb{N}$, $N \geq 2$, p, $q \in \mathbb{N}$, $a_i \in \mathbb{R}$, $\xi_i \in \mathbb{N}$ for all $i = \overline{1,p}$, $b_i \in \mathbb{R}$, $\eta_i \in \mathbb{N}$ for all $i = \overline{1,q}$, $1 \leq \xi_1 < \cdots < \xi_p \leq N-1$, $1 \leq \eta_1 < \cdots < \eta_q \leq N-1$, x_i, $y_i \in \mathbb{R}$ for all $i = \overline{1, N-1}$.

Lemma 2.1 (Henderson and Luca [13]). *If $a_i \in \mathbb{R}$, $\xi_i \in \mathbb{N}$ for all $i = \overline{1,p}$, $b_i \in \mathbb{R}$, $\eta_i \in \mathbb{N}$ for all $i = \overline{1,q}$, $1 \leq \xi_1 < \ldots < \xi_p \leq N-1$, $1 \leq \eta_1 < \cdots < \eta_q \leq N-1$,*

$\Delta_0 = N^2 - (\sum_{i=1}^{p} a_i\xi_i)(\sum_{i=1}^{q} b_i\eta_i) \neq 0$, *and* $x_i,\ y_i \in \mathbb{R}$ *for all* $i = \overline{1, N-1}$, *then the unique solution of (2.1)–(2.2) is given by*

$$\begin{aligned} u_n &= \sum_{j=1}^{N-1} G_1(n,j)x_j + \sum_{j=1}^{N-1} G_2(n,j)y_j, \quad n = \overline{0,N}, \\ v_n &= \sum_{j=1}^{N-1} G_3(n,j)y_j + \sum_{j=1}^{N-1} G_4(n,j)x_j, \quad n = \overline{0,N}, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} G_1(n,j) &= g_0(n,j) + \frac{n}{\Delta_0}\left(\sum_{i=1}^{p} a_i\xi_i\right)\left(\sum_{i=1}^{q} b_i g_0(\eta_i,j)\right), \\ G_2(n,j) &= \frac{nN}{\Delta_0}\sum_{i=1}^{p} a_i g_0(\xi_i,j), \\ G_3(n,j) &= g_0(n,j) + \frac{n}{\Delta_0}\left(\sum_{i=1}^{q} b_i\eta_i\right)\left(\sum_{i=1}^{p} a_i g_0(\xi_i,j)\right), \\ G_4(n,j) &= \frac{nN}{\Delta_0}\sum_{i=1}^{q} b_i g_0(\eta_i,j), \end{aligned} \tag{2.4}$$

and

$$g_0(n,j) = \frac{1}{N}\begin{cases} j(N-n), & 1 \le j \le n \le N, \\ n(N-j), & 0 \le n \le j \le N-1, \end{cases}$$

for all $n = \overline{0,N}$ *and* $j = \overline{1,N-1}$.

Lemma 2.2 (Henderson and Luca [13]). *If* $a_i \ge 0$, $\xi_i \in \mathbb{N}$ *for all* $i = \overline{1,p}$, $b_i \ge 0$, $\eta_i \in \mathbb{N}$ *for all* $i = \overline{1,q}$, $1 \le \xi_1 < \cdots < \xi_p \le N-1$, $1 \le \eta_1 < \cdots < \eta_q \le N-1$, *and* $\Delta_0 = N^2 - (\sum_{i=1}^{p} a_i\xi_i)(\sum_{i=1}^{q} b_i\eta_i) > 0$, *then the functions* G_i, $i = \overline{1,4}$, *given by (2.4), satisfy* $G_i(n,j) \ge 0$ *for all* $n = \overline{0,N}$, $j = \overline{1,N-1}$, $i = \overline{1,4}$. *Moreover, if* $x_n \ge 0$, $y_n \ge 0$ *for all* $n = \overline{1,N-1}$, *then the solution* $((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}})$ *of problem (2.1)–(2.2) (given by (2.3)) satisfies* $u_n \ge 0$, $v_n \ge 0$ *for all* $n = \overline{0,N}$.

Lemma 2.3 (Henderson and Luca [13]). *Assume that* $a_i \ge 0$, $\xi_i \in \mathbb{N}$ *for all* $i = \overline{1,p}$, $b_i \ge 0$, $\eta_i \in \mathbb{N}$ *for all* $i = \overline{1,q}$, $1 \le \xi_1 < \cdots < \xi_p \le N-1$, $1 \le \eta_1 < \cdots < \eta_q \le N-1$, *and* $\Delta_0 > 0$. *Then the functions* G_i, $i = \overline{1,4}$ *satisfy the inequalities*

a_1) $G_1(n,j) \le I_1(j)$, $\forall\, n = \overline{0,N}$, $j = \overline{1,N-1}$, *where*

$$I_1(j) = g_0(j,j) + \frac{N}{\Delta_0}\left(\sum_{i=1}^{p} a_i\xi_i\right)\left(\sum_{i=1}^{q} b_i g_0(\eta_i,j)\right).$$

a_2) *For every* $c \in \{1,\ldots,[\![N/2]\!]\}$, *we have* $\min\limits_{n=\overline{c,N-c}} G_1(n,j) \ge \frac{c}{N} I_1(j)$, $\forall\, j = \overline{1,N-1}$.

b_1) $G_2(n,j) \le I_2(j)$, $\forall\, n = \overline{0,N}$, $j = \overline{1,N-1}$, *where* $I_2(j) = \frac{N^2}{\Delta_0}\sum_{i=1}^{p} a_i g_0(\xi_i,j)$.

b_2) *For every* $c \in \{1,\ldots,[\![N/2]\!]\}$, *we have* $\min\limits_{n=\overline{c,N-c}} G_2(n,j) \ge \frac{c}{N} I_2(j)$, $\forall\, j = \overline{1,N-1}$.

$c_1)$ $G_3(n,j) \le I_3(j),\ \ \forall\, n=\overline{0,N},\ \ j=\overline{1,N-1}$, *where*

$$I_3(j) = g_0(j,j) + \frac{N}{\Delta_0}\left(\sum_{i=1}^{q} b_i\eta_i\right)\left(\sum_{i=1}^{p} a_i g_0(\xi_i,j)\right).$$

$c_2)$ *For every* $c \in \{1,\dots,[\![N/2]\!]\}$, *we have* $\min\limits_{n=\overline{c,N-c}} G_3(n,j) \ge \dfrac{c}{N} I_3(j),\ \ \forall\, j = \overline{1,N-1}$.

$d_1)$ $G_4(n,j) \le I_4(j),\ \ \forall\, n=\overline{0,N},\ \ j=\overline{1,N-1}$, *where* $I_4(j) = \dfrac{N^2}{\Delta_0}\sum\limits_{i=1}^{q} b_i g_0(\eta_i,j)$.

$d_2)$ *For every* $c \in \{1,\dots,[\![N/2]\!]\}$, *we have* $\min\limits_{n=\overline{c,N-c}} G_4(n,j) \ge \dfrac{c}{N} I_4(j),\ \ \forall\, j = \overline{1,N-1}$, *where* $[\![N/2]\!]$ *is the largest integer not greater than* $N/2$.

Lemma 2.4 (Henderson and Luca [13])**.** *Assume that* $a_i \ge 0$, $\xi_i \in \mathbb{N}$ *for all* $i=\overline{1,p}$, $b_i \ge 0$, $\eta_i \in \mathbb{N}$ *for all* $i=\overline{1,q}$, $1 \le \xi_1 < \cdots < \xi_p \le N-1$, $1 \le \eta_1 < \cdots < \eta_q \le N-1$, $\Delta_0 > 0$, $c \in \{1,\dots,[\![N/2]\!]\}$, *and* $x_n,\ y_n \ge 0$ *for all* $n=\overline{1,N-1}$. *Then the solution of problem (2.1)–(2.2) satisfies the inequalities*

$$\min_{n=\overline{c,N-c}} u_n \ge \frac{c}{N}\max_{m=\overline{0,N}} u_m, \quad \min_{n=\overline{c,N-c}} v_n \ge \frac{c}{N}\max_{m=\overline{0,N}} v_m.$$

3. Main results

In this section, we investigate the existence and multiplicity of positive solutions for our problem $(S)-(BC)$ under various assumptions on the functions f and g.

We present the basic assumptions that we shall use in the sequel.

$(A1)$ $a_i \ge 0$, $\xi_i \in \mathbb{N}$ for all $i=\overline{1,p}$, $b_i \ge 0$, $\eta_i \in \mathbb{N}$ for all $i=\overline{1,q}$, $1 \le \xi_1 < \cdots < \xi_p \le N-1$, $1 \le \eta_1 < \cdots < \eta_q \le N-1$ and $\Delta_0 = N^2 - \left(\sum_{i=1}^{p} a_i\xi_i\right)\left(\sum_{i=1}^{q} b_i\eta_i\right) > 0$.

$(A2)$ The functions $f,\ g : \{1,\dots,N-1\}\times[0,\infty) \to [0,\infty)$ are continuous.

By using the functions G_i, $i=\overline{1,4}$ from Section 2 (Lemma 2.1), our problem $(S)-(BC)$ can be written equivalently as the following system

$$\begin{cases} u_n = \displaystyle\sum_{i=1}^{N-1} G_1(n,i) f(i,v_i) + \sum_{i=1}^{N-1} G_2(n,i) g(i,u_i), & n=\overline{0,N}, \\ v_n = \displaystyle\sum_{i=1}^{N-1} G_3(n,i) g(i,u_i) + \sum_{i=1}^{N-1} G_4(n,i) f(i,v_i), & n=\overline{0,N}. \end{cases}$$

We consider the Banach space $X=\mathbb{R}^{N+1} = \{u=(u_0,u_1,\dots,u_N),\ \ u_i\in\mathbb{R},\ \ i=\overline{0,N}\}$ with the maximum norm $\|\cdot\|$, $\|u\| = \max\limits_{n=\overline{0,N}}|u_n|$, and the Banach space $Y = X\times X$ with the norm $\|(u,v)\|_Y = \|u\|+\|v\|$. We define the cone $P\subset Y$ by

$$P = \left\{(u,v)\in Y;\ \ u=(u_n)_{n=\overline{0,N}},\ \ v=(v_n)_{n=\overline{0,N}},\ \ u_n\ge 0,\ v_n\ge 0,\ \forall\, n=\overline{0,N}\right\}.$$

We introduce the operators Q_1, $Q_2 : Y \to X$ and $\mathcal{Q} : Y \to Y$ defined by $Q_1(u,v) = (Q_1(u,v))_{n=\overline{0,N}}$, $Q_2(u,v) = (Q_2(u,v))_{n=\overline{0,N}}$,

$$(Q_1(u,v))_n = \sum_{i=1}^{N-1} G_1(n,i)f(i,v_i) + \sum_{i=1}^{N-1} G_2(n,i)g(i,u_i), \quad n = \overline{0,N},$$
$$(Q_2(u,v))_n = \sum_{i=1}^{N-1} G_3(n,i)g(i,u_i) + \sum_{i=1}^{N-1} G_4(n,i)f(i,v_i), \quad n = \overline{0,N},$$

and $\mathcal{Q}(u,v) = (Q_1(u,v), Q_2(u,v))$, $(u,v) = ((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}}) \in Y$.

Under the assumptions $(A1)$ and $(A2)$, it is easy to see that the operator $\mathcal{Q} : P \to P$ is completely continuous (see also Lemma 3.1 from Henderson and Luca [13]). Thus the existence and multiplicity of positive solutions of problem $(S)-(BC)$ are equivalent to the existence and multiplicity of fixed points of operator $\mathcal{Q}$.

Theorem 3.1. *Assume that $(A1)$ and $(A2)$ hold. If the functions f and g also satisfy the conditions*
$(A3)$ *There exists $c \in \{1, \ldots, [\![N/2]\!]\}$ such that*

$$i)\ f_\infty^i = \lim_{u\to\infty} \min_{n=\overline{c,N-c}} \frac{f(n,u)}{u} = \infty. \quad ii)\ g_\infty^i = \lim_{u\to\infty} \min_{n=\overline{c,N-c}} \frac{g(n,u)}{u} = \infty.$$

$(A4)$ *There exist $p_1 \geq 1$ and $q_1 \geq 1$ such that*

$$i)\ f_0^s = \lim_{u\to 0^+} \max_{n=\overline{1,N-1}} \frac{f(n,u)}{u^{p_1}} = 0. \quad ii)\ g_0^s = \lim_{u\to 0^+} \max_{n=\overline{1,N-1}} \frac{g(n,u)}{u^{q_1}} = 0,$$

then problem $(S)-(BC)$ has at least one positive solution $\Big((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}}\Big)$.

Proof. For c given in $(A3)$, we define the cone

$$P_0 = \Big\{(u,v) \in P,\ u = (u_n)_{n=\overline{0,N}},\ v = (v_n)_{n=\overline{0,N}},$$
$$\min_{n=\overline{c,N-c}} u_n \geq \frac{c}{N}\|u\|,\ \min_{n=\overline{c,N-c}} v_n \geq \frac{c}{N}\|v\|\Big\}.$$

From our assumptions and Lemma 2.4, for any $(u,v) \in P$, we deduce that $\mathcal{Q}(u,v) = (Q_1(u,v), Q_2(u,v)) \in P_0$, that is $\mathcal{Q}(P) \subset P_0$.

We consider the sequences $u^0 = (u_n^0)_{n=\overline{0,N}}$, $v^0 = (v_n^0)_{n=\overline{0,N}}$, defined by

$$\begin{cases} u_n^0 = \displaystyle\sum_{i=1}^{N-1} G_1(n,i) + \sum_{i=1}^{N-1} G_2(n,i), & n = \overline{0,N}, \\ v_n^0 = \displaystyle\sum_{i=1}^{N-1} G_3(n,i) + \sum_{i=1}^{N-1} G_4(n,i), & n = \overline{0,N}, \end{cases}$$

that is (u^0, v^0) is the solution of problem (2.1)–(2.2) with $x^0 = (x_n^0)_{n=\overline{1,N-1}}$, $y^0 = (y_n^0)_{n=\overline{1,N-1}}$, $x_n^0 = 1$, $y_n^0 = 1$ for all $n = \overline{1,N-1}$. Hence $(u^0,v^0) = \mathcal{Q}(x^0,y^0) \in P_0$.

We define the set

$$M = \{(u,v) \in P,\ \text{there exists } \lambda \geq 0 \text{ such that } (u,v) = \mathcal{Q}(u,v) + \lambda(u^0,v^0)\}.$$

We will show that $M \subset P_0$ and M is a bounded set of Y. If $(u,v) \in M$, then there exists $\lambda \geq 0$ such that $(u,v) = \mathcal{Q}(u,v) + \lambda(u^0, v^0)$ or equivalently

$$\begin{cases} u_n = \sum_{i=1}^{N-1} G_1(n,i)(f(i,v_i)+\lambda) + \sum_{i=1}^{N-1} G_2(n,i)(g(i,u_i)+\lambda), \;\; n = \overline{0,N}, \\ v_n = \sum_{i=1}^{N-1} G_3(n,i)(g(i,u_i)+\lambda) + \sum_{i=1}^{N-1} G_4(n,i)(f(i,v_i)+\lambda), \;\; n = \overline{0,N}. \end{cases}$$

By Lemma 2.4, we obtain $(u,v) = ((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}}) \in P_0$, hence $M \subset P_0$, and

$$\|u\| \leq \frac{N}{c} \min_{n=\overline{c,N-c}} u_n, \;\; \|v\| \leq \frac{N}{c} \min_{n=\overline{c,N-c}} v_n, \;\; \forall\,(u,v) \in M. \tag{3.1}$$

From $(A3)$, we conclude that for $\varepsilon_1 = \frac{2N}{cm_4} > 0$ and $\varepsilon_2 = \frac{2N}{cm_2} > 0$, there exist C_1, $C_2 > 0$ such that

$$f(n,u) \geq \varepsilon_1 u - C_1, \;\; g(n,u) \geq \varepsilon_2 u - C_2, \;\; \forall\, n = \overline{c,N-c}, \;\; u \in [0,\infty), \tag{3.2}$$

where $m_i = \sum_{j=c}^{N-c} I_i(j)$, $i = 2,4$, and I_i, $i = 2,4$ are defined in Lemma 2.3.

For $(u,v) = ((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}}) \in M$ and $n = \overline{c,N-c}$, by using Lemma 2.3 and relations (3.2), it follows that

$$\begin{aligned} u_n &= (Q_1(u,v))_n + \lambda u_n^0 \geq (Q_1(u,v))_n = \sum_{i=1}^{N-1} G_1(n,i)f(i,v_i) + \sum_{i=1}^{N-1} G_2(n,i)g(i,u_i) \\ &\geq \sum_{i=c}^{N-c} G_2(n,i)g(i,u_i) \geq \frac{c}{N}\sum_{i=c}^{N-c} I_2(i)(\varepsilon_2 u_i - C_2) \\ &\geq \frac{c\varepsilon_2 m_2}{N} \min_{i=\overline{c,N-c}} u_i - \frac{cm_2C_2}{N} = 2 \min_{i=\overline{c,N-c}} u_i - C_3, \;\; C_3 = \frac{cm_2C_2}{N}, \\ v_n &= (Q_2(u,v))_n + \lambda v_n^0 \geq (Q_2(u,v))_n = \sum_{i=1}^{N-1} G_3(n,i)g(i,u_i) + \sum_{i=1}^{N-1} G_4(n,i)f(i,v_i) \\ &\geq \sum_{i=c}^{N-c} G_4(n,i)f(i,v_i) \geq \frac{c}{N}\sum_{i=c}^{N-c} I_4(i)(\varepsilon_1 v_i - C_1) \\ &\geq \frac{c\varepsilon_1 m_4}{N} \min_{i=\overline{c,N-c}} v_i - \frac{cm_4C_1}{N} = 2 \min_{i=\overline{c,N-c}} v_i - C_4, \;\; C_4 = \frac{cm_4C_1}{N}. \end{aligned}$$

Therefore, we deduce

$$\min_{i=\overline{c,N-c}} u_i \leq C_3, \;\; \min_{i=\overline{c,N-c}} v_i \leq C_4, \;\; \forall\,(u,v) = ((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}}) \in M. \tag{3.3}$$

From relations (3.1) and (3.3), we obtain $\|u\| \leq \frac{NC_3}{c}$, $\|v\| \leq \frac{NC_4}{c}$, and then $\|(u,v)\|_Y = \|u\| + \|v\| \leq \frac{NC_3}{c} + \frac{NC_4}{c} =: C_5$, for all $(u,v) \in M$, that is M is a bounded set of Y.

Besides, there exists a sufficiently large $R_1 > 1$ such that

$$(u,v) \neq \mathcal{Q}(u,v) + \lambda(u^0,v^0), \;\; \forall\,(u,v) \in \partial B_{R_1} \cap P, \;\; \forall\, \lambda \geq 0.$$

From Amann [2], we deduce that the fixed point index of operator $\mathcal{Q}$ over $B_{R_1} \cap P$ with respect to P is

$$i(\mathcal{Q}, B_{R_1} \cap P, P) = 0. \tag{3.4}$$

Next, from assumption $(A4)$, we conclude that for $\varepsilon_3 = \min\{\frac{1}{4M_1}, \frac{1}{4M_4}\}$ and $\varepsilon_4 = \min\{\frac{1}{4M_2}, \frac{1}{4M_3}\}$, there exists $r_1 \in (0,1]$ such that

$$f(n,u) \le \varepsilon_3 u^{p_1}, \ \ g(n,u) \le \varepsilon_4 u^{q_1}, \ \ \forall\, n = \overline{1, N-1}, \ u \in [0, r_1], \tag{3.5}$$

where $M_i = \sum_{j=1}^{N-1} I_1(j)$, $i = \overline{1,4}$.

By using (3.5), we deduce that for all $(u,v) \in \bar{B}_{r_1} \cap P$ and $n = \overline{0,N}$

$$\begin{aligned}
(Q_1(u,v))_n &\le \sum_{i=1}^{N-1} I_1(i)\varepsilon_3 v_i^{p_1} + \sum_{i=1}^{N-1} I_2(i)\varepsilon_4 u_i^{q_1} \\
&\le \varepsilon_3 M_1 \|v\|^{p_1} + \varepsilon_4 M_2 \|u\|^{q_1} \le \tfrac{1}{4}\|v\| + \tfrac{1}{4}\|u\| = \tfrac{1}{4}\|(u,v)\|_Y, \\
(Q_2(u,v))_n &\le \sum_{i=1}^{N-1} I_3(i)\varepsilon_4 u_i^{q_1} + \sum_{i=1}^{N-1} I_4(i)\varepsilon_3 v_i^{p_1} \\
&\le \varepsilon_4 M_3 \|u\|^{q_1} + \varepsilon_3 M_4 \|v\|^{p_1} \le \tfrac{1}{4}\|u\| + \tfrac{1}{4}\|v\| = \tfrac{1}{4}\|(u,v)\|_Y.
\end{aligned}$$

These imply that $\|Q_1(u,v)\| \le \frac{1}{4}\|(u,v)\|_Y$, $\|Q_2(u,v)\| \le \frac{1}{4}\|(u,v)\|_Y$, and so

$$\|\mathcal{Q}(u,v)\|_Y = \|Q_1(u,v)\| + \|Q_2(u,v)\| \le \frac{1}{2}\|(u,v)\|_Y, \ \ \forall\,(u,v) \in \partial B_{r_1} \cap P.$$

From Amann [2], we conclude that the fixed point index of operator $\mathcal{Q}$ over $B_{r_1} \cap P$ with respect to P is

$$i(\mathcal{Q}, B_{r_1} \cap P, P) = 1. \tag{3.6}$$

Combining (3.5) and (3.6), we obtain

$$i(\mathcal{Q}, (B_{R_1} \setminus \bar{B}_{r_1}) \cap P, P) = i(\mathcal{Q}, B_{R_1} \cap P, P) - i(\mathcal{Q}, B_{r_1} \cap P, P) = -1.$$

We deduce that $\mathcal{Q}$ has at least one fixed point $(u,v) \in (B_{R_1} \setminus \bar{B}_{r_1}) \cap P$, that is $r_1 < \|(u,v)\|_Y < R_1$. The proof of Theorem 3.1 is completed. □

Theorem 3.2. *Assume that $(A1)$ and $(A2)$ hold. If the functions f and g also satisfy the conditions*

$(A5)$ *i)* $f_\infty^s = \lim\limits_{u\to\infty} \max\limits_{n=\overline{1,N-1}} \dfrac{f(n,u)}{u} = 0.$ *ii)* $g_\infty^s = \lim\limits_{u\to\infty} \max\limits_{n=\overline{1,N-1}} \dfrac{g(n,u)}{u} = 0.$

$(A6)$ *There exist $c \in \{1, \ldots, [\![N/2]\!]\}$, $p_2 \in (0,1]$ and $q_2 \in (0,1]$ such that*

$$i)\ \ f_0^i = \lim_{u\to 0^+} \min_{n=\overline{c,N-c}} \frac{f(n,u)}{u^{p_2}} = \infty. \ \ ii)\ \ g_0^i = \lim_{u\to 0^+} \min_{n=\overline{c,N-c}} \frac{g(n,u)}{u^{q_2}} = \infty,$$

then problem $(S)-(BC)$ has at least one positive solution $\Big((u_n)_{n=\overline{0,N}}, (v_n)_{n=\overline{0,N}}\Big)$.

Proof. From the assumption $(A5)$, we deduce that for $\varepsilon_5 = \min\{\frac{1}{4M_1}, \frac{1}{4M_4}\}$ and $\varepsilon_6 = \min\{\frac{1}{4M_2}, \frac{1}{4M_3}\}$ there exist C_6, $C_7 > 0$ such that

$$f(n,u) \le \varepsilon_5 u + C_6, \ \ g(n,u) \le \varepsilon_6 u + C_7, \ \ \forall\, n = \overline{1, N-1}, \ u \in [0, \infty). \tag{3.7}$$

Hence for $(u,v) \in P$, by using (3.7), we obtain

$$\begin{aligned}
(Q_1(u,v))_n &\le \sum_{i=1}^{N-1} I_1(i)(\varepsilon_5 v_i + C_6) + \sum_{i=1}^{N-1} I_2(i)(\varepsilon_6 u_i + C_7)\\
&\le \varepsilon_5 \|v\| \sum_{i=1}^{N-1} I_1(i) + C_6 \sum_{i=1}^{N-1} I_1(i) + \varepsilon_6 \|u\| \sum_{i=1}^{N-1} I_2(i) + C_7 \sum_{i=1}^{N-1} I_2(i)\\
&= \varepsilon_5 \|v\| M_1 + C_6 M_1 + \varepsilon_6 \|u\| M_2 + C_7 M_2\\
&\le \tfrac{1}{4}\|v\| + \tfrac{1}{4}\|u\| + C_8\\
&= \tfrac{1}{4}\|(u,v)\|_Y + C_8, \ \ \forall\, n = \overline{0,N}, \ \ C_8 := C_6 M_1 + C_7 M_2,\\
(Q_2(u,v))_n &\le \sum_{i=1}^{N-1} I_3(i)(\varepsilon_6 u_i + C_7) + \sum_{i=1}^{N-1} I_4(i)(\varepsilon_5 v_i + C_6)\\
&\le \varepsilon_6 \|u\| \sum_{i=1}^{N-1} I_3(i) + C_7 \sum_{i=1}^{N-1} I_3(i) + \varepsilon_5 \|v\| \sum_{i=1}^{N-1} I_4(i) + C_6 \sum_{i=1}^{N-1} I_4(i)\\
&= \varepsilon_6 \|u\| M_3 + C_7 M_3 + \varepsilon_5 \|v\| M_4 + C_6 M_4\\
&\le \tfrac{1}{4}\|u\| + \tfrac{1}{4}\|v\| + C_9\\
&= \tfrac{1}{4}\|(u,v)\|_Y + C_9, \ \ \forall\, n = \overline{0,N}, \ \ C_9 := C_7 M_3 + C_6 M_4,
\end{aligned}$$

and so

$$\|\mathcal{Q}(u,v)\|_Y = \|Q_1(u,v)\| + \|Q_2(u,v)\| \le \frac{1}{2}\|(u,v)\|_Y + C_{10}, \ \ C_{10} := C_8 + C_9.$$

Then there exists a sufficiently large $R_2 \ge \max\{4C_{10}, 1\}$ such that

$$\|\mathcal{Q}(u,v)\|_Y \le \frac{3}{4}\|(u,v)\|_Y, \ \ \forall\,(u,v) \in P, \ \ \|(u,v)\|_Y \ge R_2.$$

Hence $\|\mathcal{Q}(u,v)\|_Y < \|(u,v)\|_Y$ for all $(u,v) \in \partial B_{R_2} \cap P$ and from Amann [2], we have

$$i(\mathcal{Q}, B_{R_2} \cap P, P) = 1. \tag{3.8}$$

On the other hand, from $(A6)$ we conclude that for $\varepsilon_7 = \frac{N}{c(m_3+m_4)}$ and $\varepsilon_8 = \frac{N}{c(m_1+m_2)}$ there exists $r_2 \in (0,1)$ such that

$$f(n,u) \ge \varepsilon_7 u^{p_2}, \ \ g(n,u) \ge \varepsilon_8 u^{q_2}, \ \ \forall\, n = \overline{c, N-c}, \ \ u \in [0, r_2], \tag{3.9}$$

where $m_i = \sum_{j=c}^{N-c} I_i(j)$, $i = \overline{1,4}$.

From (3.9) and Lemma 2.3, we deduce for any $(u,v) \in \bar{B}_{r_2} \cap P$

$$\begin{aligned}
(Q_1(u,v))_n &\ge \sum_{i=c}^{N-c} G_1(n,i) f(i,v_i) + \sum_{i=c}^{N-c} G_2(n,i) g(i,u_i)\\
&\ge \varepsilon_7 \sum_{i=c}^{N-c} G_1(n,i) v_i^{p_2} + \varepsilon_8 \sum_{i=c}^{N-c} G_2(n,i) u_i^{q_2}\\
&\ge \varepsilon_7 \sum_{i=c}^{N-c} G_1(n,i) v_i + \varepsilon_8 \sum_{i=c}^{N-c} G_2(n,i) u_i =: (L_1(u,v))_n, \ \ \forall\, n = \overline{0,N},
\end{aligned}$$

$$
\begin{aligned}
(Q_2(u,v))_n &\geq \sum_{i=c}^{N-c} G_3(n,i)g(i,u_i) + \sum_{i=c}^{N-c} G_4(n,i)f(i,v_i)\\
&\geq \varepsilon_8 \sum_{i=c}^{N-c} G_3(n,i)u_i^{q_2} + \varepsilon_7 \sum_{i=c}^{N-c} G_4(n,i)v_i^{p_2}\\
&\geq \varepsilon_8 \sum_{i=c}^{N-c} G_3(n,i)u_i + \varepsilon_7 \sum_{i=c}^{N-c} G_4(n,i)v_i =: (L_2(u,v))_n, \ \ \forall\, n=\overline{0,N}.
\end{aligned}
$$

Hence

$$
\mathcal{Q}(u,v) \geq L(u,v), \ \ \forall\,(u,v) \in \partial B_{r_2} \cap P, \tag{3.10}
$$

where the linear operator $L : P \to P$ is defined by $L(u,v) = (L_1(u,v), L_2(u,v))$.

We consider now $(\widetilde{u}^0, \widetilde{v}^0) \in P \setminus \{(0,0)\}$ with $\widetilde{u}^0 = (\widetilde{u}^0_n)_{n=\overline{0,N}}$ and $\widetilde{v}^0 = (\widetilde{v}^0_n)_{n=\overline{0,N}}$ defined by

$$
\begin{aligned}
\widetilde{u}^0_n &= \sum_{i=c}^{N-c} G_1(n,i) + \sum_{i=c}^{N-c} G_2(n,i), \ \ n=\overline{0,N},\\
\widetilde{v}^0_n &= \sum_{i=c}^{N-c} G_3(n,i) + \sum_{i=c}^{N-c} G_4(n,i), \ \ n=\overline{0,N}.
\end{aligned}
$$

Then $L(\widetilde{u}^0, \widetilde{v}^0) = (L_1(\widetilde{u}^0, \widetilde{v}^0), L_2(\widetilde{u}^0, \widetilde{v}^0))$, and

$$
\begin{aligned}
(L_1(\widetilde{u}^0, \widetilde{v}^0))_n =& \varepsilon_7 \sum_{i=c}^{N-c} G_1(n,i) \left(\sum_{j=c}^{N-c} G_3(i,j) + \sum_{j=c}^{N-c} G_4(i,j) \right)\\
&+ \varepsilon_8 \sum_{i=c}^{N-c} G_2(n,i) \left(\sum_{j=c}^{N-c} G_1(i,j) + \sum_{j=c}^{N-c} G_2(i,j) \right)\\
\geq& \varepsilon_7 \sum_{i=c}^{N-c} G_1(n,i) \left(\sum_{j=c}^{N-c} \frac{c}{N} I_3(j) + \sum_{j=c}^{N-c} \frac{c}{N} I_4(j) \right)\\
&+ \varepsilon_8 \sum_{i=c}^{N-c} G_2(n,i) \left(\sum_{j=c}^{N-c} \frac{c}{N} I_1(j) + \sum_{j=c}^{N-c} \frac{c}{N} I_2(j) \right)\\
=& \frac{\varepsilon_7 c}{N}(m_3+m_4) \sum_{i=c}^{N-c} G_1(n,i) + \frac{\varepsilon_8 c}{N}(m_1+m_2) \sum_{i=c}^{N-c} G_2(n,i)\\
=& \sum_{i=c}^{N-c} G_1(n,i) + \sum_{i=c}^{N-c} G_2(n,i) = \widetilde{u}^0_n, \ \ \forall\, n=\overline{0,N},\\
(L_2(\widetilde{u}^0, \widetilde{v}^0))_n =& \varepsilon_8 \sum_{i=c}^{N-c} G_3(n,i) \left(\sum_{j=c}^{N-c} G_1(i,j) + \sum_{j=c}^{N-c} G_2(i,j) \right)\\
&+ \varepsilon_7 \sum_{i=c}^{N-c} G_4(n,i) \left(\sum_{j=c}^{N-c} G_3(i,j) + \sum_{j=c}^{N-c} G_4(i,j) \right)\\
\geq& \varepsilon_8 \sum_{i=c}^{N-c} G_3(n,i) \left(\sum_{j=c}^{N-c} \frac{c}{N} I_1(j) + \sum_{j=c}^{N-c} \frac{c}{N} I_2(j) \right)
\end{aligned}
$$

$$\begin{aligned}
&+ \varepsilon_7 \sum_{i=c}^{N-c} G_4(n,i) \left(\sum_{j=c}^{N-c} \frac{c}{N} I_3(j) + \sum_{j=c}^{N-c} \frac{c}{N} I_4(j) \right) \\
=&\frac{\varepsilon_8 c}{N}(m_1+m_2) \sum_{i=c}^{N-c} G_3(n,i) + \frac{\varepsilon_7 c}{N}(m_3+m_4) \sum_{i=c}^{N-c} G_4(n,i) \\
=&\sum_{i=c}^{N-c} G_3(n,i) + \sum_{i=c}^{N-c} G_4(n,i) = \widetilde{v}_n^0, \ \ \forall\, n = \overline{0,N}.
\end{aligned}$$

So

$$L(\widetilde{u}^0, \widetilde{v}^0) \geq (\widetilde{u}^0, \widetilde{v}^0). \tag{3.11}$$

We may suppose that $\mathcal{Q}$ has no fixed point on $\partial B_{r_2} \cap P$ (otherwise the proof is finished). From (3.10) and (3.11), and Lemma 2.3 from Zhou and Xu [23], we conclude that

$$i(\mathcal{Q}, B_{r_2} \cap P, P) = 0. \tag{3.12}$$

Therefore, from (3.8) and (3.12), we have

$$i(\mathcal{Q}, (B_{R_2} \setminus \bar{B}_{r_2}) \cap P, P) = i(\mathcal{Q}, B_{R_2} \cap P, P) - i(\mathcal{Q}, B_{r_2} \cap P, P) = 1.$$

Then $\mathcal{Q}$ has at least one fixed point in $(B_{R_2} \setminus \bar{B}_{r_2}) \cap P$, that is $r_2 < \|(u,v)\|_Y < R_2$. Thus problem $(S)-(BC)$ has at least one positive solution $(u,v) \in P$. This completes the proof of Theorem 3.2. □

Theorem 3.3. *Assume that $(A1)-(A3)$ and $(A6)$ hold. If the functions f and g also satisfy the condition*
$(A7)$ For each $n = \overline{1,N-1}$, $f(n,u)$ and $g(n,u)$ are nondecreasing with respect to u, and there exists a constant $R_0 > 0$ such that

$$f(n,N) < \frac{R_0}{4m_0}, \ \ g(n,N) < \frac{R_0}{4m_0}, \ \ \forall\, n = \overline{1,N-1},$$

where $m_0 = \max\{M_i, \ i = \overline{1,4}\}$, $(M_i = \sum_{j=1}^{N-1} I_i(j), \ i = \overline{1,4})$, then problem $(S)-(BC)$ has at least two positive solutions $\left((u_n^1)_{n=\overline{0,N}}, (v_n^1)_{n=\overline{0,N}}\right)$ and $\left((u_n^2)_{n=\overline{0,N}}\right.$, $\left.(v_n^2)_{n=\overline{0,N}}\right)$.

Proof. By using $(A7)$, for any $(u,v) \in \partial B_{R_0} \cap P$, we obtain

$$\begin{aligned}
(Q_1(u,v))_n &\leq \sum_{i=1}^{N-1} G_1(n,i) f(i,N) + \sum_{i=1}^{N-1} G_2(n,i) g(i,N) \\
&\leq \sum_{i=1}^{N-1} I_1(i) f(i,N) + \sum_{i=1}^{N-1} I_2(i) g(i,N) \\
&< \frac{R_0}{4m_0} \sum_{i=1}^{N-1} I_1(i) + \frac{R_0}{4m_0} \sum_{i=1}^{N-1} I_2(i) \\
&= \frac{R_0 M_1}{4m_0} + \frac{R_0 M_2}{4m_0} \leq \frac{R_0}{2}, \ \ \forall\, n = \overline{0,N},
\end{aligned}$$

$$\begin{aligned}(Q_2(u,v))_n &\leq \sum_{i=1}^{N-1} G_3(n,i)g(i,N) + \sum_{i=1}^{N-1} G_4(n,i)f(i,N)\\ &\leq \sum_{i=1}^{N-1} I_3(i)g(i,N) + \sum_{i=1}^{N-1} I_4(i)f(i,N)\\ &< \frac{R_0}{4m_0}\sum_{i=1}^{N-1} I_3(i) + \frac{R_0}{4m_0}\sum_{i=1}^{N-1} I_4(i)\\ &= \frac{R_0 M_3}{4m_0} + \frac{R_0 M_4}{4m_0} \leq \frac{R_0}{2}, \quad \forall\, n = \overline{0,N}.\end{aligned}$$

Then we deduce

$$\|\mathcal{Q}(u,v)\|_Y = \|Q_1(u,v)\| + \|Q_2(u,v)\| < R_0 = \|(u,v)\|_Y, \quad \forall\,(u,v) \in \partial B_{R_0} \cap P.$$

By Amann [2], we conclude that

$$i(\mathcal{Q}, B_{R_0} \cap P, P) = 1. \tag{3.13}$$

On the other hand, from $(A3)$, $(A6)$ and the proofs of Theorems 3.1 and 3.2 we know that there exists a sufficiently large $R_1 > R_0$ and a sufficiently small $r_2 \in (0, R_0)$ such that

$$i(\mathcal{Q}, B_{R_1} \cap P, P) = 0, \quad i(\mathcal{Q}, B_{r_2} \cap P, P) = 0. \tag{3.14}$$

From the relations (3.13) and (3.14), we obtain

$$\begin{aligned}&i(\mathcal{Q}, (B_{R_1} \setminus \bar{B}_{R_0}) \cap P, P) = i(\mathcal{Q}, B_{R_1} \cap P, P) - i(\mathcal{Q}, B_{R_0} \cap P, P) = -1,\\ &i(\mathcal{Q}, (B_{R_0} \setminus \bar{B}_{r_2}) \cap P, P) = i(\mathcal{Q}, B_{R_0} \cap P, P) - i(\mathcal{Q}, B_{r_2} \cap P, P) = 1.\end{aligned}$$

Then $\mathcal{Q}$ has at least one fixed point $(u^1, v^1) \in (B_{R_1} \setminus \bar{B}_{R_0}) \cap P$ and has at least one fixed point $(u^2, v^2) \in (B_{R_0} \setminus \bar{B}_{r_2}) \cap P$. Therefore, problem $(S) - (BC)$ has two distinct positive solutions (u^1, v^1) and (u^2, v^2). The proof of Theorem 3.3 is completed. □

4. An example

Let $N = 30$, $p = 3$, $q = 2$, $a_1 = 3$, $a_2 = 1$, $a_3 = 1/2$, $\xi_1 = 5$, $\xi_2 = 15$, $\xi_3 = 25$, $b_1 = 1$, $b_2 = 1/2$, $\eta_1 = 10$, $\eta_2 = 20$.

We consider the system of second-order difference equations

$$(S_0) \qquad \begin{cases} \Delta^2 u_{n-1} + f(n, v_n) = 0, & n = \overline{1,29},\\ \Delta^2 v_{n-1} + g(n, u_n) = 0, & n = \overline{1,29},\end{cases}$$

with the multi-point boundary conditions

$$(BC_0) \qquad u_0 = 0, \ \ u_{30} = 3v_5 + v_{15} + v_{25}/2, \ \ v_0 = 0, \ \ v_{30} = u_{10} + u_{20}/2,$$

where the functions f and g are given by $f(n,u) = a_0(u^{\alpha_0} + u^{\beta_0})$, $g(n,u) = b_0(u^{\gamma_0} + u^{\delta_0})$ for $n = \overline{1,29}$ and $u \in [0,\infty)$, with $\alpha_0 > 1$, $0 < \beta_0 < 1$, $\gamma_0 > 1$, $0 < \delta_0 < 1$, a_0, $b_0 > 0$. We have $\Delta_0 = N^2 - \left(\sum_{i=1}^{p} a_i \xi_i\right)\left(\sum_{i=1}^{q} b_i \eta_i\right) = 50 > 0$.

The functions $I_i,\ i=\overline{1,4}$ from Lemma 2.3 are given by

$$I_1(j)=\begin{cases}\frac{1}{60}(1335j-2j^2),\ 1\le j\le 10,\\ \frac{1}{60}(15300-195j-2j^2),\ 11\le j\le 20,\\ \frac{1}{30}(15300-480j-j^2),\ 21\le j\le 29,\end{cases}\quad I_2(j)=\begin{cases}\frac{111}{2}j,\ 1\le j\le 5,\\ \frac{3}{2}(180+j),\ 6\le j\le 15,\\ \frac{3}{2}(360-11j),\ 16\le j\le 25,\\ \frac{3}{2}(510-17j),\ 26\le j\le 29,\end{cases}$$

$$I_3(j)=\begin{cases}\frac{1}{30}(1140j-j^2),\ 1\le j\le 5,\\ \frac{1}{30}(5400+60j-j^2),\ 6\le j\le 15,\\ \frac{1}{30}(10800-300j-j^2),\ 16\le j\le 25,\\ \frac{1}{30}(15300-480j-j^2),\ 26\le j\le 29,\end{cases}\quad I_4(j)=\begin{cases}15j,\ 1\le j\le 10,\\ 180-3j,\ 11\le j\le 20,\\ 360-12j,\ 21\le j\le 29.\end{cases}$$

We also deduce $M_1=\sum_{j=1}^{29} I_1(j)\approx 3974.83333333$, $M_2=\sum_{j=1}^{29} I_2(j)=5962.5$, $M_3=\sum_{j=1}^{29} I_3(j)\approx 4124.83333333$ and $M_4=\sum_{j=1}^{29} I_4(j)=2700$. Then $m_0=\max_{i=\overline{1,4}} M_i=M_2$. The functions $f(n,u)$ and $g(n,u)$ are nondecreasing with respect to u, for any $n=\overline{1,29}$, and for $p_2=q_2=1$ and $c\in\{1,\dots,15\}$, the assumptions $(A3)$ and $(A6)$ are satisfied; indeed we obtain $f_\infty^i=\infty$, $g_\infty^i=\infty$, $f_0^i=\infty$ and $g_0^i=\infty$. We take $R_0=1$ and then $f(n,R_0)=2a_0$, $g(n,R_0)=2b_0$ for all $n=\overline{1,29}$. If $a_0<\frac{1}{8m_0}$ and $b_0<\frac{1}{8m_0}$, then the assumption $(A7)$ is satisfied. For example, if $a_0\le 2.096\cdot 10^{-5}$ and $b_0\le 2.096\cdot 10^{-5}$, then by Theorem 3.3, we deduce that problem $(S_0)-(BC_0)$ has at least two positive solutions.

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