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EXACT TRAVELLING WAVE SOLUTIONS OF REACTION-DIFFUSION MODELS OF FRACTIONAL ORDER*

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Abstract Reaction-diffusion models are used in different areas of chemistry problems. Also, coupled reaction-diffusion systems describing the spatio- temporal dynamics of competition models have been widely applied in many real world problems. In this paper, we consider a coupled fractional system with diffusion and competition terms in ecology, and reaction-diffusion growth model of fractional order with Allee effect describing and analyzing the spread dynamic of a single population under different dispersal and growth rates. Finding the exact solutions of such models are very helpful in the theories and numerical studies. Exact traveling wave solutions of the above reaction-diffusion models are found by means of the Q-function method. Moreover, graphic illustrations in two and three dimensional plots of some of the obtained solutions are also given to predict their behaviours.

Keywords Reaction-diffusion models, fractional calculus, exact solution, Q-function method.

MSC(2010) 35Q51, 35Q53.

1. Introduction

Diffusion-reaction systems are well-established and play an important role in different life-science disciplines [2, 7]. As results of various uses, for the past few decades, much attention has been paid to the problem of finding exact solutions of reaction-diffusion models [14, 16]. With the use of these solutions, many researchers may give better insight into the various aspects, such as physical, biological and ecological effect of the models considered. Kraenkel et al [9] obtained exact solutions for a system of two coupled nonlinear partial differential equations describing the spatio-temporal dynamics of a predator-prey system where the prey per capita growth rate is subject to the Allee effect. A system of two nonlinear differential equations which arises in biology is considered and the variational iteration method is implemented for finding the solution [19]. Numerical solutions of the population dynamics model with density-dependent migrations and the Allee effects using the homotopy perturbation method is obtained in [13].

In this article, first we consider a mathematical coupled model of fractional order with diffusion and competition terms in ecology. In particular, the considered specific model attributed to Lotka-Volterra system which is an extension of the basic logistic model of single species. Consider the spread of two competing species

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using the fractional coupled PDEs

$$\begin{cases} \dfrac{\partial^\alpha U}{\partial T^\alpha} = D_U \dfrac{\partial^2 U}{\partial X^2} + AU\left(1 - \dfrac{U}{K_U} - \dfrac{C_{UV}}{K_U}V\right), \\ \dfrac{\partial^\alpha V}{\partial T^\alpha} = D_V \dfrac{\partial^2 V}{\partial X^2} - BV\left(1 - \dfrac{1}{K_V}V - \dfrac{C_{VU}}{K_V}U\right), \end{cases} \tag{1.1}$$

where $U(X,T)$ and $V(X,T)$ represents the frequencies of competition populations at position X and time T; D_U is a diffusion coefficient of U and D_V is a diffusion coefficient of V which represent species-specific diffusion rates; K_U denotes carrying capacity of U and K_V represents carrying capacity of V, A is a growth rate of U and B is a growth rate of V; C_{UV} and C_{VU} are the competition coefficients which expressing the per capita competitive effect of populations V on U, and U on V, on the growth rate and realized carrying capacity of the rival populations. Moreover, in the absence of competition ($C_{UV} = C_{VU} = 0$), each population grows to its respective carrying capacity and also in the presence of competition, one or the other rival may survive while its competitor dies out, or else the rivals may coexist. The above coupled fractional system is the fractional counter part of the model considered in [12].

Spatial dispersal is modeled by the diffusion processes $D_U \frac{\partial^2 U}{\partial X^2}$ and $D_V \frac{\partial^2 V}{\partial X^2}$, where D_U and D_V describe the diffusivity of both populations. Two competitive species occur a competitive coexistence (stability) if it yields

$$C_{UV} < \frac{K_U}{K_V}, \quad C_{VU} < \frac{K_V}{K_U}, \tag{1.2}$$

which means that $C_{UV}C_{VU} < 1$ [15]. Contrary, they are competitive instability if it holds

$$C_{UV} > \frac{K_U}{K_V}, \quad C_{VU} > \frac{K_V}{K_U}, \tag{1.3}$$

which means that $C_{UV}C_{VU} > 1$, then eventually one population will go extinct, but which one depends on the carrying capacities [15].

In the last two decades, there has been a renewed interest in a biological phenomenon called the Allee effect and dynamics of the population models with the Allee effect can be found in [2,4,6,22]. Also, the Allee effect can be described by the positive relationship between any component of individual fitness and either numbers or densities of conspecifics [5]. Abbas et al [1] studied a fractional differential equation model of the single species multiplicative Allee effect. Shu and Weng [20] investigated a diffusive SI epidemic model with strong Allee effect. Stephens et al [21] described several scenarios that cause the Allee effect in both animals and plants. Now, we consider a fractional reaction-diffusion model with Allee effect as follows;

$$\frac{\partial^\alpha P(X,T)}{\partial T^\alpha} = D\frac{\partial^2 P(X,T)}{\partial X^2} + RP(X,T)\left(\frac{P(X,T)}{A} - 1\right)\left(1 - \frac{P(X,T)}{K}\right), \tag{1.4}$$

where $P(X,T)$ stands for the frequencies of population at position X and time T; D is a diffusion coefficient of P; K is a carrying capacity of P; A is an Allee threshold and R is the inherent per unit growth rate. For the analysis of the exact solutions of the fractional reaction-diffusion model with Allee effect, we consider that the per capita growth rate $\left(\frac{\partial^\alpha P}{\partial T^\alpha}\right)$ is negative above the carrying capacity K and positive below. However, in the presence of an Allee effect, it also decreases below a given population size, and can even become negative below a critical population threshold A [3].

Recently, it is proved that many phenomena in reaction-diffusion models can be described by fractional model using mathematical tools from fractional calculus. However, it is not easy to find exact solutions for fractional partial differential equations and hence an effective technique for solving such equations is necessary [17,18]. The method of Q-functions is a very powerful technique for deriving exact traveling wave solutions of nonlinear ordinary and partial differential equations arising in real-world problems and also its advantages are discussed in the papers [11]. The Adomian's decomposition method is implemented for finding the exact solutions of a more general biological population models in [8]. In this work, we implement Q-function method for solving reaction-diffusion models of fractional order. More precisely, in this paper, we obtain exact traveling wave solutions of reaction-diffusion models such as the fractional competition system and fractional reaction-diffusion system with Allee effect by using fractional sub-equation method in the sense of the modified Riemann-Liouville derivative defined by Jumarie [10] together with Q-function technique.

2. Convert to ODE with some relationships of coefficients in (1.1) and (1.4)

In this section, we provide main steps of the fractional sub-equation method with modified Riemann-Liouville derivative for converting reaction-diffusion models (1.1) and (1.4) to ordinary differential equations. The Jumarie's modified Riemann-Liouville derivative of order α is defined by the following expression [10]

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dt} \displaystyle\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, \ 0<\alpha<1, \\ (f^{(n)}(t))^{(\alpha-n)}, \ n \le \alpha < n+1, \ n \ge 1. \end{cases} \tag{2.1}$$

Also, we present some important properties for the modified Riemann-Liouville derivative as follows:

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, \tag{2.2}$$

$$D_t^\alpha (f(t)g(t)) = g(t) D_t^\alpha f(t) + f(t) D_t^\alpha g(t), \tag{2.3}$$

$$D_t^\alpha f[g(t)] = f'_g[g(t)] D_t^\alpha g(t) = D_t^\alpha f[g(t)](g'(t))^\alpha. \tag{2.4}$$

Moreover, substituting the expressions $u = \frac{U}{K_U}, v = \frac{C_{UV}}{K_U}V, t = \frac{AT^\alpha}{\Gamma(1+\alpha)}, x = \sqrt{\frac{A}{D_V}}X, d = \frac{D_U}{D_V}, a = \frac{BK_U C_{VU}}{K_V A}, b = \frac{K_V}{K_U C_{VU}}, c = C_{UV}C_{VU}$ into coupled system (1.1), we obtain the following system of PDEs

$$\begin{cases} \dfrac{\partial u}{\partial t} = d \dfrac{\partial^2 u}{\partial x^2} + u(1-u-v), \\ \dfrac{\partial v}{\partial t} = d \dfrac{\partial^2 v}{\partial x^2 - av(b - \frac{1}{c}v - u)}. \end{cases} \tag{2.5}$$

Now, using the wave transformation $u(x,t) = \mathsf{y}(\mathsf{z}), v(x,t) = \mathsf{w}(\mathsf{z}), \mathsf{z} = kx - \omega t (l \neq 0)$, system (2.5) converted into an ordinary differential equation(ODE) in the following form

$$\begin{cases} \omega \mathsf{y}' + \mathsf{y}(1 - \mathsf{y} - \mathsf{w}) + dk^2 \mathsf{y}'' = 0, \\ \omega \mathsf{w}' - a\mathsf{w}(b - \dfrac{1}{c}\mathsf{w} - \mathsf{y}) + k^2 \mathsf{w}'' = 0, \end{cases} \tag{2.6}$$

where $\mathsf{y}' = \frac{d\mathsf{y}}{d\mathsf{z}}, \mathsf{y}'' = \frac{d^2\mathsf{y}}{d\mathsf{z}^2}, \mathsf{w}' = \frac{d\mathsf{w}}{d\mathsf{z}}$ and $\mathsf{w}'' = \frac{d^2\mathsf{w}}{d\mathsf{z}^2}$.

Next, by substituting the expressions $u = \frac{P}{KA}, t = \frac{RT^\alpha}{\Gamma(1+\alpha)}, x = \sqrt{R}X$ into equation (1.4), we obtain the following PDE

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + u(Ku - 1)(1 - Au). \tag{2.7}$$

Next, for the reaction-diffusion model with Allee effect, consider the transformation $u(x,t) = \mathsf{y}(\mathsf{z}), \mathsf{z} = lx - \omega t (l \neq 0)$. By using the transformation, equation (2.7) converted to the ODE

$$\omega \mathsf{y}' + Dl^2 \mathsf{y}'' + \mathsf{y}(K\mathsf{y} - 1)(1 - A\mathsf{y}) = 0, \tag{2.8}$$

where $\mathsf{y}' = \frac{d\mathsf{y}}{d\mathsf{z}}$ and $\mathsf{y}'' = \frac{d^2\mathsf{y}}{d\mathsf{z}^2}$.

3. Exact solutions of fractional competition models

Let us find the traveling exact wave solutions of the system (1.1). To determine the parameter m, we balance the order between the highest order derivative term and nonlinear term in Eq. (2.6). This in turn gives $m = 2$. Thus, we can look solutions of the system (2.6) in the form

$$\begin{cases} \mathsf{y} = A_0 + A_1 Q(\mathsf{z}) + A_2 Q^2(\mathsf{z}), \\ \mathsf{w} = B_0 + B_1 Q(\mathsf{z}) + B_2 Q^2(\mathsf{z}), \end{cases} \tag{3.1}$$

where $A_i, B_i, i = 0, 1, 2$ are constants to be determined later and $Q(\mathsf{z})$ is the logistic function in the form $Q(\mathsf{z}) = \frac{1}{1+e^{-\mathsf{z}-\mathsf{z}_0}}$ in which z_0 is an arbitrary constant.

One can see that $Q(\mathsf{z})$ satisfies the Riccati equation in the form

$$Q_{\mathsf{z}} = Q - Q^2. \tag{3.2}$$

Taking into account expression (3.1) and Equation (3.2),we can obtain the derivatives $\mathsf{y}', \mathsf{w}', \mathsf{y}''$ and w'' expressed via the function $Q(\mathsf{z})$. Substituting $\mathsf{y}(\mathsf{z}), \mathsf{w}(\mathsf{z})$, $\mathsf{y}', \mathsf{w}', \mathsf{y}''$ and w'' expressed via $Q(\mathsf{z})$ into the system (2.6) and equating to zero the expressions with the same degree of $Q(\mathsf{z})$, we can obtain eight solutions at different values of parameters of system (2.5) with substitutions provided there and hence we can obtain exact solution of system (1.1).

When $a = -\frac{5(d-1)}{2d}, b = \frac{5d-2}{5(d-1)}$ and $c = \frac{25(d-1)}{2(5d-2)}$ the exact traveling wave solution of coupled competition model (1.1) can be obtained as

$$\begin{cases} U(X,T) = K_U \left(Q(\mathsf{z}) - Q^2(\mathsf{z})\right), \\ V(X,T) = \dfrac{5K_U}{2C_{UV}} Q^2(\mathsf{z}), \\ \mathsf{z} = \pm \dfrac{1}{2\sqrt{d}} \sqrt{\dfrac{A}{D_V}} X - \dfrac{5AT^\alpha}{4\Gamma(1+\alpha)}. \end{cases} \tag{3.3}$$

Figs. 1 and 2 represent the three-dimensional and two dimensional plot of the solution (3.3).

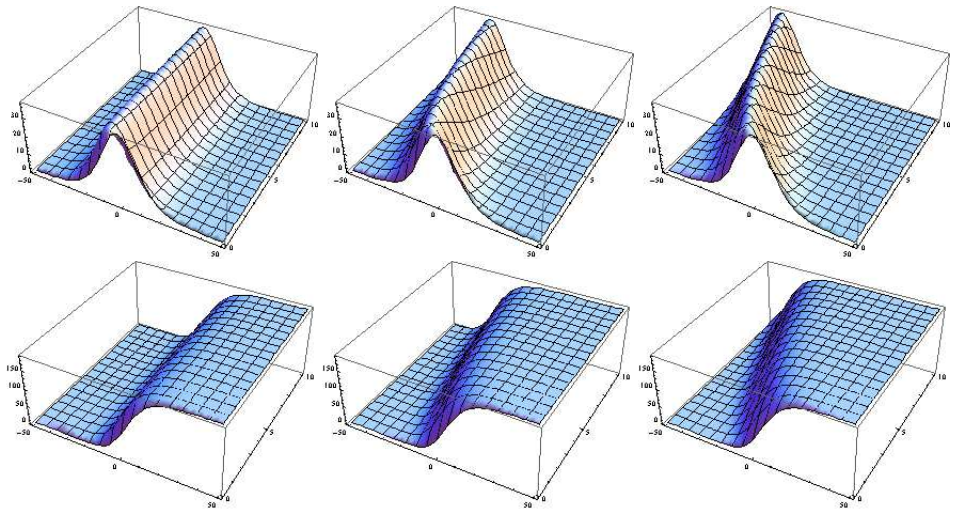

Figure 1. Solutions U and V of (3.3) when $D_U = 10, D_V = 5, A = 1, B = 3, C_{UV} = 2, K_U = 150, K_V = 100, \mathsf{z}_0 = 0$ and $\alpha = 0.1, 0.5, 0.7$.

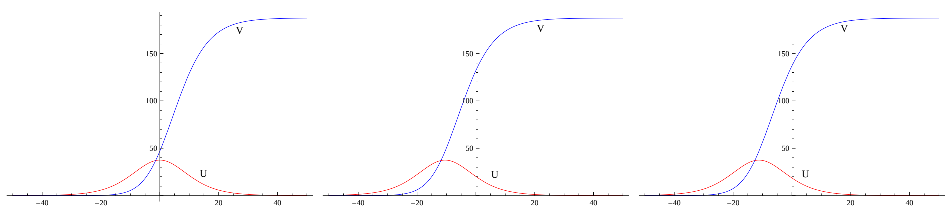

Figure 2. Graph of (3.3) which represents the extinct and survival of populations U and V, respectively when $\alpha = 0.1$ and $T = 0, 10, 20$ with $D_U = 10, D_V = 5, A = 1, B = 3, C_{UV} = 2, K_U = 150, K_V = 100, \mathsf{z}_0 = 0$.

For the case $a = \frac{k^2d+1-5k^2}{2k^2d-1}, b = -\frac{2(2k^2d-1)(k^2d+1-2k^2)}{k^2d+1-5k^2}$ and $c = -\frac{(k^2d+1-5k^2)(k^2d+1)}{(k^2d+1-2k^2)(2k^2d-1)}$, the solution of system (1.1) can be obtained as

$$\begin{cases} U(X,T) = K_U\left((2-4k^2d)Q(\mathsf{z}) + (4k^2d-2)Q^2(\mathsf{z})\right), \\ V(X,T) = \dfrac{(2k^2d+2)K_U}{C_{UV}}Q^2(\mathsf{z}), \\ \mathsf{z} = k\sqrt{\dfrac{A}{D_V}}X + (k^2d+1)\dfrac{AT^\alpha}{\Gamma(1+\alpha)}. \end{cases} \tag{3.4}$$

Moreover, two and three dimensional plots of the solutions (3.4) are provided in Figs. 3 and 4. The exact solution of coupled competition model (1.1) which corresponds to $a = \frac{2(k^2d+1-5k^2)}{6k^2d-1}, b = -\frac{(6k^2d-1)(k^2d+1-2k^2)}{k^2d+1-5k^2}$ and

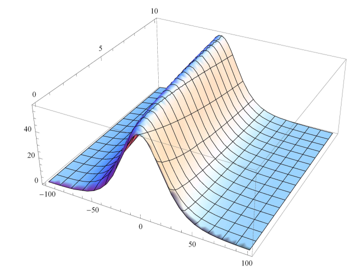

Figure 3. Solutions U and V of (3.4) under $D_U = 1, D_V = 2, A = 0.15, B = 3, C_{UV} = 0.5, K_U = 100, K_V = 150, k = -0.5, \mathsf{z}_0 = 0, \alpha = 0.5$.

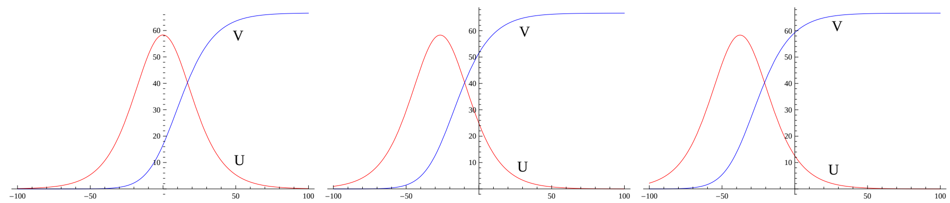

Figure 4. Graph of (3.4) which represents that populations U and V are extinct when $T = 0, 10, 20$.

$c = \frac{2d(k^2d+1-5k^2)(6k^2d+1)}{(4k^2d^2-14k^2d+4d-3)(6k^2d-1)}$ is given by

$$\begin{cases} U(X,T) = K_U\left((1-6k^2d)Q(\mathsf{z}) + \dfrac{4k^2d(6k^2d-1)}{2k^2d-1}Q^2(\mathsf{z})\right), \\ V(X,T) = -\dfrac{2k^2d(6k^2d+1)K_U}{(2k^2d-1)C_{UV}}Q^2(\mathsf{z}), \\ \mathsf{z} = k\sqrt{\dfrac{A}{D_V}}X + (k^2d+1)\dfrac{AT^\alpha}{\Gamma(1+\alpha)}. \end{cases} \tag{3.5}$$

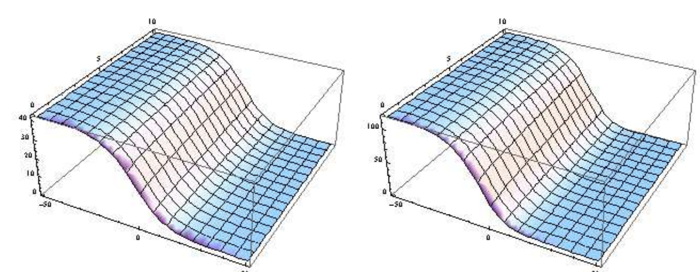

Figure 5. Solution (3.5) represents traveling wave solution U and V under $D_U = 1, D_V = 2, A = 0.15, B = 3, C_{UV} = 0.5, K_U = 100, K_V = 150, k = -0.5, \mathsf{z}_0 = 0, \alpha = 0.5$.

Figs. 5 and 6 show the profile of the exact solution U and V of (3.5) for some fixed parameters values. Further, the exact solution associated with the values

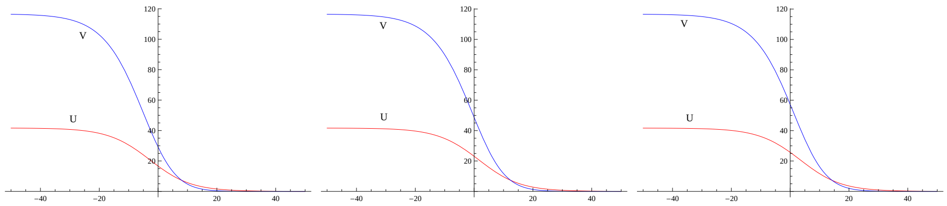

Figure 6. Figure of (3.5) represents that populations U and V are extinct when $T = 0, 10, 20$.

$a = \frac{2(k^2d-1-5k^2)}{4k^2d-1}, b = -\frac{k^2(4k^2d^2-8k^2d-4d+3)}{k^2d-1-5k^2}$ and $c = -\frac{2d(k^2d-1-5k^2)}{4k^2d^2-8k^2d-4d+3}$ is given by

$$\begin{cases} U(X,T) = K_U\left(1-(4k^2d+1)Q(\mathsf{z})+4k^2dQ^2(\mathsf{z})\right), \\ V(X,T) = \dfrac{2k^2dK_U}{C_{UV}}Q^2(\mathsf{z}), \\ \mathsf{z} = k\sqrt{\dfrac{A}{D_V}}X-(1-k^2d)\dfrac{AT^\alpha}{\Gamma(1+\alpha)}. \end{cases} \tag{3.6}$$

For populations U and V in solutions (3.6), one population U is surviving and other population V is extinct that satisfy $C_{UV} > \frac{K_U}{K_V}$ which is given in Figs. 7 and 8.

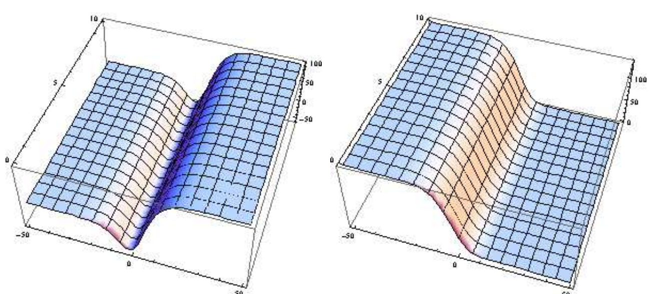

Figure 7. Solution (3.6) represents traveling wave solutions U and V under $D_U = 10, D_V = 3.5, A = 0.5, B = 3, C_{UV} = 1.6, K_U = 150, K_V = 100, k = -0.6, \mathsf{z}_0 = 0, \alpha = 0.5$.

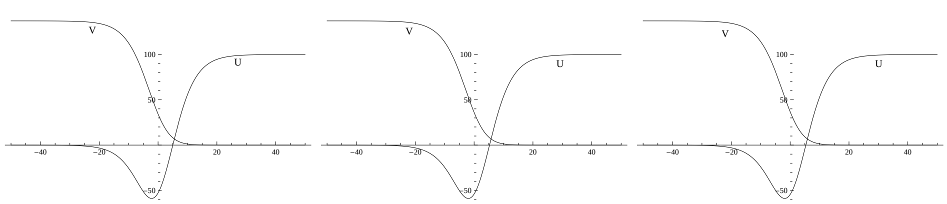

Figure 8. Graph of (3.6) represents the survival and extinct of populations U and V when $T = 0, 10, 20$.

Further, when $a = k^2d-1-5k^2, b = -\frac{k^2+k^2d-1}{k^2d-1-5k^2}$ and $c = -\frac{(k^2d-1-5k^2)(6k^2d-1)}{k^2+k^2d-1}$, the obtained exact solutions can be written as

$$\begin{cases} U(X,T) = K_U\left(1 - 2Q(\mathsf{z}) + Q^2(\mathsf{z})\right), \\ V(X,T) = \dfrac{(6k^2-1)dK_U}{C_{UV}}Q^2(\mathsf{z}), \\ \mathsf{z} = k\sqrt{\dfrac{A}{D_V}}X - (1-k^2d)\dfrac{AT^\alpha}{\Gamma(1+\alpha)}. \end{cases} \tag{3.7}$$

For species U and V in solutions (3.7), one population U is extinct and other population V is surviving that satisfy $C_{UV} > \frac{K_U}{K_V}$ which is provided in Figs. 9 and 10. It is concluded from the graphs that two species are rapidly changing as position X varies.

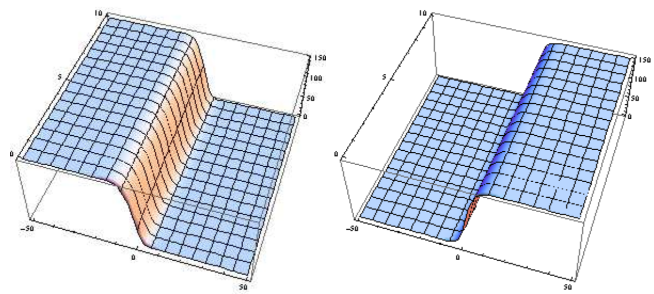

Figure 9. Graph of (3.7) represents traveling wave solutions U and V under $D_U = 1, D_V = 1.5, A = 0.5, B = 0.3, C_{UV} = 1.51, K_U = 150, K_V = 100, k = 0.8, \mathsf{z}_0 = 0, \alpha = 0.5$.

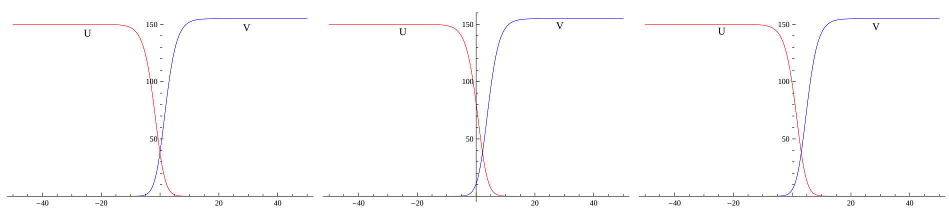

Figure 10. Graph of (3.7) represents the survival and extinct of populations V and U when $T = 0, 10, 20$.

In the case of $a = \frac{k^2d-1-5k^2}{3k^2d}, b = -\frac{-7k^2d+1+5k^2-12k^4d+6k^4d^2}{k^2d-1-5k^2}$ and $c = \frac{(k^2d-1-5k^2)(k^2d-1)}{k^2(2k^2d^2-7k^2d-2d+3)}$, the solutions can be written as

$$\begin{cases} U(X,T) = K_U\left(1 - 6k^2dQ(\mathsf{z}) + \dfrac{12k^4d^2}{k^2d+1}Q^2(\mathsf{z})\right), \\ V(X,T) = -\dfrac{6k^2d(k^2d-1)dK_U}{(k^2d+1)C_{UV}}Q^2(\mathsf{z}), \\ \mathsf{z} = k\sqrt{\dfrac{A}{D_V}}X - (1-k^2d)\dfrac{AT^\alpha}{\Gamma(1+\alpha)}. \end{cases} \tag{3.8}$$

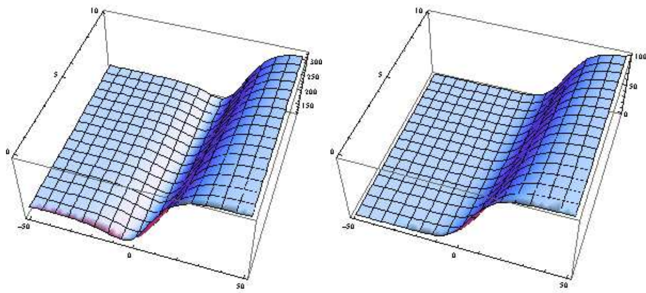

Figure 11. Solution (3.8) presents traveling wave solutions U and V under $D_U = 10, D_V = 5, A = 1, B = 3, C_{UV} = 1.3, K_U = 150, K_V = 100, k = 0.35, \mathsf{z}_0 = 0, \alpha = 0.5$.

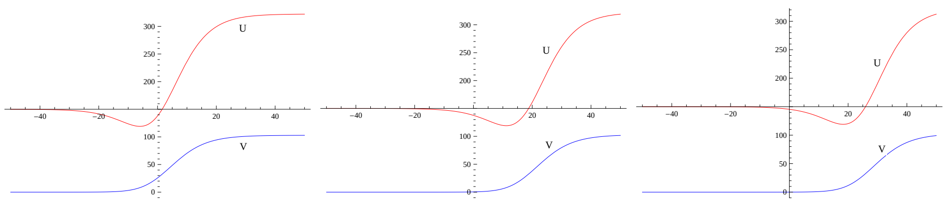

Figure 12. Graph of (3.8) represents two populations V and U when $T = 0, 10, 20$.

Moreover, if $a = \frac{5(d-1)}{3d}, b = \frac{25d-16}{10(d-1)}$ and $c = \frac{25(d-1)}{10d-19}$, then the exact solution of system is given by

$$\begin{cases} U(X,T) = K_U \left(1 + \dfrac{3}{2}Q(\mathsf{z}) + Q^2(\mathsf{z})\right), \\ V(X,T) = -\dfrac{5K_U}{2C_{UV}}Q^2(\mathsf{z}), \\ \mathsf{z} = \pm\dfrac{1}{2\sqrt{d}}\mathsf{i}\sqrt{\dfrac{A}{D_V}}X + \dfrac{5AT^\alpha}{2\Gamma(1+\alpha)}, \end{cases} \tag{3.9}$$

where i is an imaginary number.

When $a = -\frac{5(d-1)}{4d}, b = \frac{5d+1}{5(d-1)}$ and $c = \frac{25(d-1)}{2(5d+1)}$, then the corresponding exact solution to the system is given by

$$\begin{cases} U(X,T) = K_U \left(1 - 2Q(\mathsf{z}) + Q^2(\mathsf{z})\right), \\ V(X,T) = -\dfrac{5K_U}{2C_{UV}}Q^2(\mathsf{z}), \\ \mathsf{z} = \pm\dfrac{1}{2\sqrt{d}}\mathsf{i}\sqrt{\dfrac{A}{D_V}}X + \dfrac{5AT^\alpha}{2\Gamma(1+\alpha)}, \end{cases} \tag{3.10}$$

where i is an imaginary number. The Figs. 11 and 12 represent the two and three dimensional plot of the solution curve (3.8) under different parameter values.

4. Exact solutions of fractional reaction-diffusion model with Allee effect

In this section, let us find the traveling wave solutions of system (1.4). It can be easily seen that the equation in system (2.8) has the second-order pole solution. Therefore, we can look for solutions of Equation (2.8) in the form

$$\mathsf{y} = A_0 + A_1 Q(\mathsf{z}), \tag{4.1}$$

where $Q(\mathsf{z}) = \frac{1}{1+e^{-\mathsf{z}-\mathsf{z}_0}}$ is the logistic function satisfies the Riccati equation (3.2); z_0 is an arbitrary constant; A_0 and A_1 are constants to be determined later.

By following the similar steps and procedures as in Section 3, we can obtain three traveling wave solutions for the system (1.4) First, we obtain the traveling wave solutions of Eq.(1.4) in the form

$$P(X,T) = A - \frac{A}{1+e^{-\mathsf{z}-\mathsf{z}_0}}, \tag{4.2}$$

where $\mathsf{z} = \pm\sqrt{\frac{AR}{2DK}}X - \frac{A-2K}{2K}\frac{RT^\alpha}{\Gamma(1+\alpha)}$.

Fig. 13 represents the behaviour of the solution (4.2). As T is varying, P is moving left which represents that there is no Allee effect between the population size and per capita growth rate;

$$\frac{\partial^\alpha P(X,T)}{\partial T^\alpha} = -\frac{(A-2K)R}{2KA}P(X,T)(A-P(X,T)), \tag{4.3}$$

where $A < 2K$.

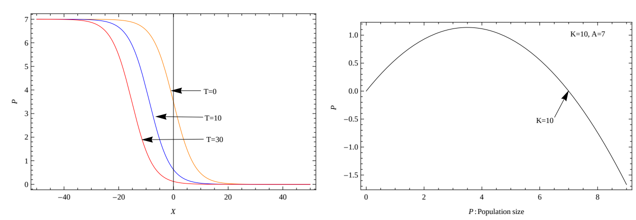

Figure 13. First figure represents the solution (4.2) when $K = 10, R = 0.3, D = 20, A = 7, \mathsf{z}_0 = 0, \alpha = 0.5$ and second figure represents there is no Allee effect between the population size and per capita growth rate.

Next, we obtain the following traveling wave solution of Eq. (1.4):

$$P(X,T) = A - \frac{A-K}{1+e^{-\mathsf{z}-\mathsf{z}_0}}, \tag{4.4}$$

where $\mathsf{z} = \pm(A-K)\sqrt{\frac{R}{2DKA}}X - \frac{A^2-K^2}{2KA}\frac{RT^\alpha}{\Gamma(1+\alpha)}$.

The behaviour of the obtained solution (4.4) is provided in Fig. 14. Fig. 15 represents a strong Allee effect with a strong Allee threshold A throughout between the population size and the per capita growth rate with the following relation:

$$\frac{\partial^\alpha P(X,T)}{\partial T^\alpha} = -\frac{(A+K)R}{2KA}(P(X,T)-A)(P(X,T)-K). \tag{4.5}$$

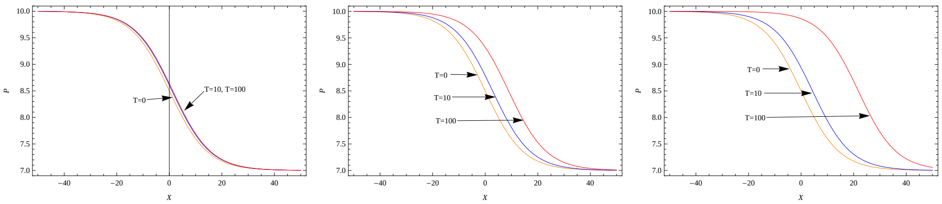

Figure 14. Graph of (4.4) when $K = 10, R = 0.3, D = 20, A = 7, \mathsf{z}_0 = 0$ as $\alpha = 0.1, 0.5, 0.7$. with different time $T = 0, 10, 30$.

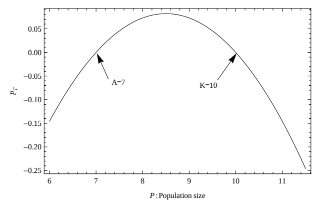

Figure 15. Graph of (4.4) represents a strong Allee effect between P and $P_T = \frac{\partial^\alpha P}{\partial T^\alpha}$ when $K = 10, R = 0.3, D = 20, A = 7$.

Finally, we obtain the following traveling wave solution of Eq. (1.4) as follows

$$P(X,T) = K + \frac{A-K}{1+e^{-\mathsf{z}-\mathsf{z}_0}}, \tag{4.6}$$

where $\mathsf{z} = \pm(A-K)\sqrt{\frac{R}{2DKA}}X + \frac{A^2-K^2}{2KA}\frac{RT^\alpha}{\Gamma(1+\alpha)}$.

Fig. 16 denotes the behaviors of the solution (4.6). Fig. 17 presents a strong Allee effect with a strong Allee threshold A between the population size and the per capita growth rate with the following relation:

$$\frac{\partial^\alpha P(X,T)}{\partial T^\alpha} = -\frac{(A+K)R}{2KA}(P(X,T)-A)(P(X,T)-K). \tag{4.7}$$

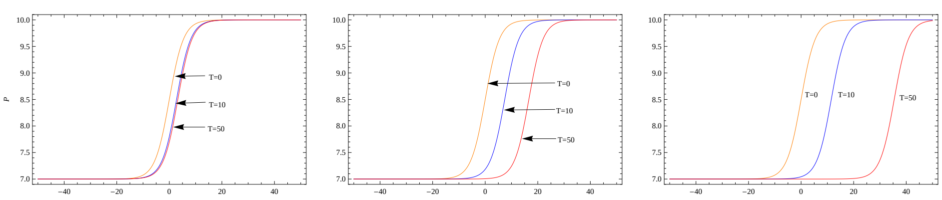

Figure 16. Curve of (4.6) when $K = 10, R = 0.3, D = 20, A = 7, \mathsf{z}_0 = 0$ and $\alpha = 0.1, 0.5, 0.7$. with different times $T = 0, 10, 30$.

5. Conclusion

This paper presents the exact solutions of reaction-diffusion models of fractional order. In particular, the exact solutions are obtained for the coupled reaction-diffusion systems describing the spatio-temporal dynamics of competition models and reaction-diffusion growth model of fractional order with Allee effect. In this

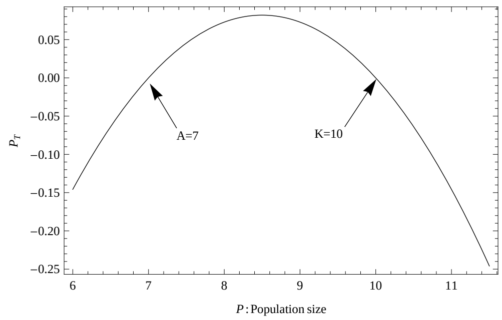

Figure 17. Graph of (4.6) represents a strong Allee effect between P and $P_T = \frac{\partial^\alpha P}{\partial T^\alpha}$.

paper, we implemented Q-function method for obtaining exact solutions of such models. The results reveal that the Q-function technique is an effective tool for obtaining exact solutions of reaction-diffusion models of fractional order.

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