Journal of Applied Analysis and Computation
Volume 7, Number 1, February 2017, 119–133
Website:http://jaac-online.com/
DOI:10.11948/2017009

INTERACTIONS OF DELTA SHOCK WAVES FOR THE AW-RASCLE TRAFFIC MODEL WITH SPLIT DELTA FUNCTIONS*

Zhiqiang Shao[1,†] and Meixiang Huang[1]

Abstract This paper is concerned with the interactions of δ-shock waves for the Aw-Rascle traffic model with split delta functions. The solutions are obtained constructively when the initial data are three piecewise constant states. The global structure and large time-asymptotic behaviors of the solutions are analyzed case by case. Moreover, it can be found that the Riemann solutions are stable for such small perturbations with initial data by studying the limits of the solutions when the perturbed parameter $\varepsilon \to 0$.

Keywords Aw-Rascle model, Riemann problem, Delta shock wave, wave interaction.

MSC(2010) 35L65, 35L67.

1. Introduction

Consider the Aw-Rascle model of traffic flow in the conservative form [1]

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho(v + p(\rho))) + \partial_x(\rho v(v + p(\rho))) = 0, \end{cases} \tag{1.1}$$

where ρ and v represent the traffic density and velocity of the cars located at position x at time t, respectively; the "pressure" function takes the form

$$p(\rho) = -\rho^{-1}. \tag{1.2}$$

The model (1.1) is now widely used to study the formation and dynamics of traffic jams. It was proposed by Aw and Rascle [1] to remedy the deficiencies of second order models of car traffic pointed out by Daganzo [7] and had also been independently derived by Zhang [28]. Since its introduction, it had received extensive attention (see [9,14,20], etc.). Recently, Pan and Han [19] studied the system (1.1) for the Chaplygin gas pressure. While eq. (1.2) was introduced by Chaplygin [4], Tsien [27] and von Karman [12] as a suitable mathematical approximation for calculating the lifting force on a wing of an airplane in aerodynamics. The sound speed $c = \rho^{-1}$ tends to zero as the density ρ tends to infinity. This unusual property allows mass concentrations in finite time. The Chaplygin gas has been advertised as a possible model for dark energy [2,3,10,21].

†the corresponding author. Email address: zqshao@fzu.edu.cn(Z. Shao)
1College of Mathematics and Computer Science, Fuzhou University, Fuzhou 350116, China
*The authors were supported by Natural Science Foundation of Fujian Province (No. 2015J01014).

Delta-shock is a very interesting topic in the theory of systems of conservations laws. It is a generalization of an ordinary shock. Speaking informally, it is a kind of discontinuity, on which at least one of the state variables may develop an extreme concentration in the form of a weighted Dirac delta function with the discontinuity as its support. From the physical point of view, it represents the process of concentration of the mass. For related researches of delta-shocks, we refer the readers to [5,6,8,13,15–18,22–26] and the references cited therein for more details.

Recently, Guo, Zhang and Yin [11] studied the Interaction of delta shock waves for the Chaplygin gas equations with spilt delta functions. However, it has noticed that few literatures contribute to system (1.1) for interaction of delta shock waves with spilt delta functions so far. Motivated by [11], the main purpose of the present article is to use the same method to investigate various possible interactions of delta shock waves and contact discontinuities for (1.1)–(1.2). It is important to investigate the interactions of elementary waves not only because of their significance in practical applications but also because of their basic role as building blocks for the general mathematical theory of quasi-linear hyperbolic equations. And the results on interactions can be also used in a procedure similar to the wave-front tracking approximation for general initial data. Thus, we take three pieces constant initial data instead of the Riemann data and then the solutions beyond the interactions are constructed. Furthermore, we prove that the solutions of the perturbed initial value problem converge to the corresponding Riemann solutions as $\varepsilon \to 0$, which shows the stability of the Riemann solutions for the small perturbation.

It is difficult to deal with the interactions of delta shock waves with the other elementary waves, for it will give rise to the product of $\delta(x)$ and $H(x)$. To overcome this problem, we adopt the method of splitting of delta function along a regular curve in $\overline{R_+^2}$ proposed by Nedeljkov and Oberguggenberger [16–18]. By using the method of splitting of delta functions, the product of the piecewise smooth function and discontinuity along such curve makes sense and the differentiation is defined by mapping into the usual Radon measure space. By employing the method of splitting of delta functions, the interaction including the delta shock waves and other elementary waves were widely investigated in [23,24].

The paper is organized as follows. In Section 2, we restate the Riemann problem to the Aw-Rascle traffic model (1.1)–(1.2) and the solution concept based on splitting of delta measures along a regular curve in $\overline{R_+^2}$ for readers convenience. In Section 3, the interactions of the delta shock waves and contact discontinuities are discussed for all kinds when the initial data are three piece constant states. And the solutions are constructed globally and the stability of the Riemann solutions is analyzed by letting $\varepsilon \to 0$.

2. Preliminaries

In this section, we briefly review the Riemann solutions of (1.1) and (1.2) with initial data

$$(\rho, v)(0, x) = (\rho_\pm, v_\pm), \qquad \pm x > 0, \tag{2.1}$$

where $\rho_\pm > 0$, the detailed study of which can be found in [19].

The eigenvalues of system (1.1)–(1.2) are

$$\lambda_1(\rho,v)=v-\frac{1}{\rho},\quad \lambda_2(\rho,v)=v,$$

with corresponding right eigenvectors

$$r_1(\rho,v)=(1,-\frac{1}{\rho^2})^T,\quad r_2(\rho,v)=(1,0)^T.$$

By a direct calculation, we obtain $\nabla\lambda_i\cdot r_i=0,\ i=1,2$, which means that the system (1.1) with (1.2) is full linear degenerate and the associated elementary waves are contact discontinuities.

Since system (1.1), (1.2) and the Riemann data (2.1) are invariant under stretching of coordinates: $(t,x)\to(\alpha t,\alpha x)$ (α is constant), we seek the self-similar solution

$$(\rho,v)(t,x)=(\rho,v)(\xi),\quad \xi=\frac{x}{t}.$$

Then the Riemann problem (1.1) and (2.1) is reduced to the following boundary value problem of the ordinary differential equations:

$$\begin{cases}-\xi\rho_\xi+(\rho v)_\xi=0,\\ -\xi(\rho v-1)_\xi+(\rho v^2-v)_\xi=0,\end{cases}\tag{2.2}$$

with $(\rho,v)(\pm\infty)=(\rho_\pm,v_\pm)$.

For any smooth solution, system (2.2) can be written as

$$\begin{pmatrix} v-\xi & \rho\\ -\xi v+v^2 & -\xi\rho+2\rho v-1\end{pmatrix}\begin{pmatrix}\rho_\xi\\ v_\xi\end{pmatrix}=0.\tag{2.3}$$

It provides either the general solutions (constant states)

$$(\rho,v)(\xi)=constant\quad(\rho>0),\tag{2.4}$$

or singular solutions

$$\xi=v-\frac{1}{\rho}=v_--\frac{1}{\rho_-},\quad \xi=v=v_-.\tag{2.5}$$

For a bounded discontinuity at $\xi=\sigma$, the Rankine-Hugoniot conditions holds:

$$\begin{cases}-\sigma[\rho]+[\rho v]=0,\\ -\sigma[\rho v-1]+[\rho v^2-v]=0,\end{cases}\tag{2.6}$$

where $[\rho]=\rho-\rho_-$, and σ is the velocity of the discontinuity. By solving (2.6), we obtain

$$\sigma=v-\frac{1}{\rho}=v_--\frac{1}{\rho_-},\quad \sigma=v=v_-.\tag{2.7}$$

From (2.5) and (2.7), we find that the rarefaction waves coincide with the shock waves in the phase plane, which correspond to contact discontinuities:

$$J_1:\xi=v-\frac{1}{\rho}=v_--\frac{1}{\rho_-},\tag{2.8}$$

$$J_2:\xi=v=v_-.\tag{2.9}$$

In the phase plane, through the point (ρ_-, v_-), we draw a branch of curve (2.8) for $\rho > 0$, which have two asymptotic line $v = v_- - \frac{1}{\rho_-}$ and $\rho = 0$, denote by J_1. Through the point (ρ_-, v_-), we also draw a branch of curve (2.9) for $\rho > 0$, denote by J_2. Through the point $\left(\rho_-, v_- - \frac{1}{\rho_-}\right)$, we draw the curve (2.9), denote by S. It easy to know the phase plane can be divided into five regions, as show in Figure 1.

For any given right state (ρ_+, v_+), according to Figure 1, we can construct the unique global Riemann solution connecting two constant states $(\rho_\pm, u_\pm)$. When $(\rho_+, v_+) \in I \cup II \cup III \cup IV$, the Riemann solution contains a 1-contact discontinuity, a 2-contact discontinuity, a nonvacuum intermediate constant state (ρ_*, v_*), where

$$v_* = v_+, \qquad \frac{1}{\rho_*} = v_+ - v_- + \frac{1}{\rho_-}. \tag{2.10}$$

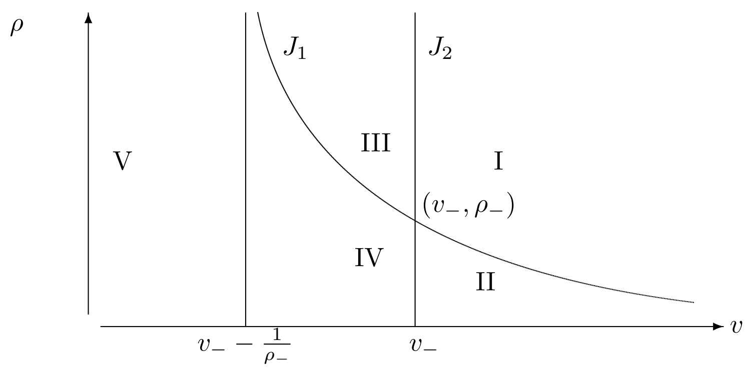

Figure 1. Contact discontinuity curves in phase plane.

When $(\rho_+, v_+) \in V$, the characteristics originating from the origin will overlap in a domain Ω as shown in Figure 2. So, singularity must happen in Ω. It is easy to know that the singularity is impossible to be a jump with finite amplitude because the Rankine-Hugoniot condition is not satisfied on the bounded jump. In other words, there is no solution which is piecewise smooth and bounded. Motivated by [16], we seek solutions with delta distribution at the jump. In fact, the appearance of delta shock wave is due to the overlap of linear degenerate characteristic lines.

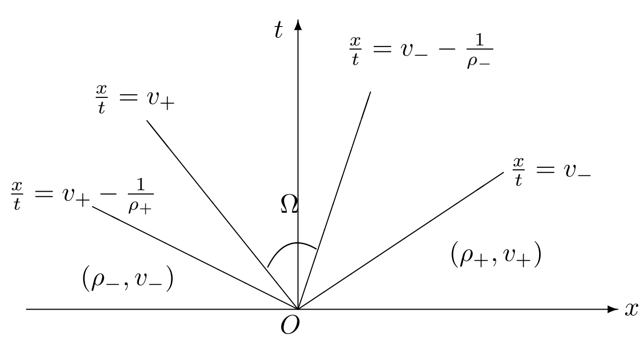

Figure 2. Analysis of characteristics for the delta shock wave.

For system (1.1)–(1.2), the definition of solution in the sense of distributions can be given as follows.

Definition 2.1. A pair (ρ, v) constitutes a solution of (1.1) and (1.2) in the sense of distributions if it satisfies

$$\begin{cases} \int_0^{+\infty}\int_{-\infty}^{+\infty}(\rho\varphi_t + \rho v\varphi_x)dxdt = 0, \\ \int_0^{+\infty}\int_{-\infty}^{+\infty}((\rho(v+p))\varphi_t + (\rho v(v+p))\varphi_x)dxdt = 0, \end{cases} \tag{2.11}$$

for all test functions $\varphi \in C_0^\infty(R^+ \times R^1)$.

Moreover, we define a two-dimensional weighted delta function in the following way.

Definition 2.2. A two-dimensional weighted delta function $w(s)\delta_L$ supported on a smooth curve $L = \{(t(s), x(s)) : a < s < b\}$ is defined by

$$\langle w(s)\delta_L, \varphi\rangle = \int_a^b w(s)\varphi(t(s), x(s))ds \tag{2.12}$$

for all test functions $\varphi \in C_0^\infty(R^2)$.

Let us consider a piecewise smooth solution of (1.1) and (1.2) in the form

$$(\rho, v)(t, x) = \begin{cases} (\rho_-, v_-), & x < \sigma t, \\ (w(t)\delta(x-\sigma t), \sigma), & x = \sigma t, \\ (\rho_+, v_+), & x > \sigma t, \end{cases} \tag{2.13}$$

where $w(t)$ and σ are weight and velocity of Dirac delta wave respectively, satisfying the generalized Rankine-Hugoniot conditions

$$\begin{cases} \frac{dx(t)}{dt} = \sigma, \\ \frac{dw(t)}{dt} = \sigma[\rho] - [\rho v], \\ \frac{d(w(t)\sigma)}{dt} = \sigma[\rho v - 1] - [\rho v^2 - v], \end{cases} \tag{2.14}$$

where $[\rho] = \rho_+ - \rho_-$, with initial data

$$(x, w)(0) = (0, 0). \tag{2.15}$$

By solving (2.14), we can get

$$w(t) = \sqrt{\rho_+\rho_-[v]^2 + [\rho][v]}\ t, \tag{2.16}$$

$$\sigma = \frac{[\rho v] + \sqrt{\rho_+\rho_-[v]^2 + [\rho][v]}}{[\rho]}, \tag{2.17}$$

as $\rho_+ \neq \rho_-$, and

$$w(t) = (\rho_- v_- - \rho_+ v_+)t, \quad \sigma = \frac{[\rho v^2 - v]}{2[\rho v]}, \tag{2.18}$$

as $\rho_+ = \rho_-$, We also can justify the delta shock wave satisfies the entropy condition:

$$v_+ < \sigma < v_- - \frac{1}{\rho_-}, \tag{2.19}$$

which means that all the characteristics on both sides of the delta shock are incoming.

Thus, we have obtained the global solution of the Aw-Rascle traffic model for Chaplygin pessure.

Next, we briefly introduce the concept of left- and right-hand side delta functions which will be extensively used later and the detailed study can be found in [17,18,23,24].

Divide $\overline{R_+^2}$ into two open sets Ω_1 and Ω_2 by a piecewise smooth curve Γ, with $\Omega_1 \cap \Omega_2 = \emptyset$ and $\Omega_1 \cup \Omega_2 = \overline{R_+^2}$. Let $C(\Omega_i)$ and $M(\Omega_i)$ be the space of bounded and continuous real-valued functions equipped with the L^∞-norm and the space of measures on $\Omega_i (i = 1, 2)$, respectively. Denote $C_\Gamma = (C(\Omega_1), C(\Omega_2))$ and $M_\Gamma = (M(\Omega_1), M(\Omega_2))$, the product of $G = (G_1, G_2) \in C_\Gamma$ and $D = (D_1, D_2) \in M_\Gamma$, which is defined as an element $GD = (G_1D_1, G_2D_2) \in M_\Gamma$, where $G_iD_i (i = 1, 2)$ is defined as the usual product of a continuous function and a measure. So, the product defined as above makes sense.

Every measure on $\overline{\Omega_i}$ as a measure on $\overline{R_+^2}$ with support in $\overline{\Omega_i}(i = 1, 2)$. From this viewpoint, the mapping $m : M_\Gamma \to M(\overline{R_+^2})$ can be obtained by taking $m(D) = D_1 + D_2$. Similarly, we have $m(GD) = G_1D_1 + G_2D_2$.

The solution concept used in this paper can be described as follows: carry out the multiplication and composition in the space M_Γ and then take the mapping $m : M_\Gamma \to M(\overline{R_+^2})$ before differentiation in the space of distributions.

3. Interactions of delta shock waves and contact discontinuity

In this section, we consider the initial value problem (1.1)–(1.2) with three pieces constant initial data as follows:

$$(\rho, v)(0, x) = \begin{cases} (\rho_-, v_-), & -\infty < x < -\varepsilon, \\ (\rho_m, v_m), & -\varepsilon < x < \varepsilon, \\ (\rho_+, v_+), & \varepsilon < x < +\infty, \end{cases} \tag{3.1}$$

where $\varepsilon > 0$ is arbitrarily small. The data (3.1) is a perturbation of the corresponding Riemann initial data (2.1). We face the question of determining whether the Riemann solutions of (1.1)–(1.2) and (2.1) are the limits $(\rho_\varepsilon, v_\varepsilon)(t, x)$ as $\varepsilon \to 0$, where $(\rho_\varepsilon, v_\varepsilon)(t, x)$ are the solutions of (1.1)–(1.2) and (3.1). We will deal with this problem case by case along with constructing the solutions.

In order to cover all the cases completely, we divide our discussion into the following four cases according to the different combinations of the delta shock waves and contact discontinuities starting from $(-\varepsilon, 0)$ and $(\varepsilon, 0)$. In the following cases, we only consider $\rho_+ \neq \rho_-$, the situation $\rho_+ = \rho_-$ is the same as that.

Case 3.1. $v_m < v_- - \frac{1}{\rho_-}$, $v_+ < v_m - \frac{1}{\rho_m}$.

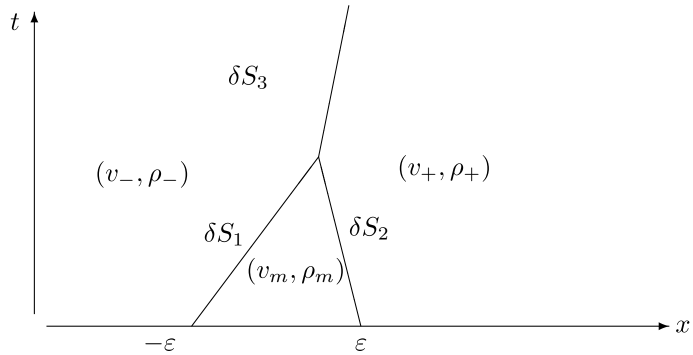

Figure 3. $v_m < v_- - \frac{1}{\rho_-}$ and $v_+ < v_m - \frac{1}{\rho_m}$.

In this case, when t is small enough, the solution of the initial value problem (1.1)–(1.2) and (3.1) can be expressed briefly as follows (see Figure 3):

$$(\rho_-, v_-) + \delta S_1 + (\rho_m, v_m) + \delta S_2 + (\rho_+, v_+),$$

where "+" means "followed by".

The propagation speed of the two delta shock waves (δS_1 and δS_2) are σ_1 and σ_2, where σ_1 and σ_2 satisfy

$$v_m < \sigma_1 < v_- - \frac{1}{\rho_-}, \quad v_+ < \sigma_2 < v_- - \frac{1}{\rho_-}.$$

Thus, it is easy to see that the δS_1 will overtake δS_2 at a finite time. The intersection (x_1, t_1) is determined by

$$\begin{cases} x_1 + \varepsilon = \sigma_1 t_1, \\ x_1 - \varepsilon = \sigma_2 t_1, \end{cases} \tag{3.2}$$

where

$$\sigma_1 = \frac{\rho_m v_m - \rho_- v_- + \frac{dw_1(t)}{dt}}{\rho_m - \rho_-}, \ w_1(t) = \sqrt{\rho_- \rho_m (v_m - v_-)^2 + (\rho_m - \rho_-)(v_m - v_-)}\ t;$$

$$\sigma_2 = \frac{\rho_+ v_+ - \rho_m v_m + \frac{dw_2(t)}{dt}}{\rho_+ - \rho_m}, \ w_2(t) = \sqrt{\rho_+ \rho_m (v_+ - v_m)^2 + (\rho_+ - \rho_m)(v_+ - v_m)}\ t.$$

By solving (3.2), we obtain

$$(x_1, t_1) = (\frac{2\varepsilon\sigma_1}{\sigma_1 - \sigma_2} - \varepsilon, \frac{2\varepsilon}{\sigma_1 - \sigma_2}). \tag{3.3}$$

At the intersection (x_1, t_1), the new initial data are formed as follows:

$$v|_{t=t_1} = \begin{cases} v_-, & x < x_1, \\ v_+, & x > x_1, \end{cases} \quad \rho|_{t=t_1} = \begin{cases} \rho_-, & x < x_1, \\ \rho_+, & x > x_1, \end{cases} + \beta(t_1)\delta(x_1, t_1),$$

where $\beta(t_1)$ denotes the sum of the strengths of incoming delta shock waves δS_1 and δS_2 at the time t_1, which can be calculated by

$$\begin{aligned}\beta(t_1) =& \sqrt{\rho_-\rho_m(v_m - v_-)^2 + (\rho_m - \rho_-)(v_m - v_-)}\ t_1 \\ &+ \sqrt{\rho_+\rho_m(v_+ - v_m)^2 + (\rho_+ - \rho_m)(v_+ - v_m)}\ t_1. \end{aligned} \tag{3.4}$$

A new delta shock wave will generate after interaction and we denote it with δS_3, which can be expressed as

$$\begin{cases} v(x,t) = v_- + (v_+ - v_-)H, \\ \rho(x,t) = \rho_- + (\rho_+ - \rho_-)H + \beta_-(t)D^- + \beta_+(t)D^+, \end{cases} \tag{3.5}$$

where H is the Heaviside function and $\beta(t)D = \beta_-(t)D^- + \beta_+(t)D^+$ is a split delta function. All of them are supported on the line $x = x_1 + (t - t_1)\sigma_3$, i.e., they are the functions of $x = x_1 + (t - t_1)\sigma_3$, here σ_3 is the propagating speed of δS_3. Although they are supported on the same line, D^- is the delta measure on the set $\overline{R_+^2} \cap \{(x,t)|x \le x_1 + (t - t_1)\sigma_3\}$ and D^+ is the delta measure on the set $\overline{R_+^2} \cap \{(x,t)|x \ge x_1 + (t - t_1)\sigma_3\}$. From (3.5), we can compute

$$\rho_t(x,t) = (-\sigma(\rho_+ - \rho_-) + \beta_-'(t) + \beta_+'(t))\delta - \sigma(\beta_-(t) + \beta_+(t))\delta', \tag{3.6}$$

$$(\rho v)_x(x,t) = (\rho_+ v_+ - \rho_- v_-)\delta + (v_-\beta_-(t) + v_+\beta_+(t))\delta', \tag{3.7}$$

$$(\rho v)_t(x,t) = (-\sigma(\rho_+ v_+ - \rho_- v_-) + v_-\beta_-'(t) + v_+\beta_+'(t))\delta - \sigma(v_-\beta_-(t) + v_+\beta_+(t))\delta', \tag{3.8}$$

$$(\rho v^2 - v)_x(x,t) = (\rho_+ v_+^2 - \rho_- v_-^2 - (v_+ - v_-))\delta + (v_-^2\beta_-(t) + v_+^2\beta_+(t))\delta'. \tag{3.9}$$

Substituting (3.6)–(3.7) into the first equation of (1.1), we have the relations

$$\begin{cases} -\sigma(\rho_+ - \rho_-) + \beta_-'(t) + \beta_+'(t) + \rho_+ v_+ - \rho_- v_- = 0, \\ -\sigma(\beta_-(t) + \beta_+(t)) + v_-\beta_-(t) + v_+\beta_+(t) = 0. \end{cases} \tag{3.10}$$

Substituting (3.8)–(3.9) into $(1.1)_2$, we obtain

$$\begin{cases} -\sigma(\rho_+ v_+ - \rho_- v_-) + v_-\beta_-'(t) + v_+\beta_+'(t) + \rho_+ v_+^2 - \rho_- v_-^2 - v_+ + v_- = 0, \\ -\sigma(v_-\beta_-(t) + v_+\beta_+(t)) + v_-^2\beta_-(t) + v_+^2\beta_+(t) = 0. \end{cases} \tag{3.11}$$

From (3.10) and (3.11), we know that the equations are overdetermined. Noting the initial condition (3.4), from (3.10), we can calculate

$$\beta(t) = \beta_-(t) + \beta_+(t) = \beta(t_1) + \sqrt{\rho_+\rho_-(v_+ - v_-)^2 + (\rho_+ - \rho_-)(v_+ - v_-)}(t - t_1). \tag{3.12}$$

From $(3.11)_1$ and $(3.10)_2$, we have

$$\beta(t) = \beta_-(t) + \beta_+(t) = \beta(t_1) + \sqrt{\rho_+\rho_-(v_+ - v_-)^2 + (\rho_+ - \rho_-)(v_+ - v_-)}(t - t_1). \tag{3.13}$$

So Eqs. (3.10)–(3.11) are compatible.

It is easy to see that (x_1, t_1) tend to $(0,0)$ as $\varepsilon \to 0$ from (3.3), moreover we have $\beta(t_1) \to 0$ as $\varepsilon \to 0$ from (3.4), Thus, the limit of the solution of (1.1)–(1.2) and

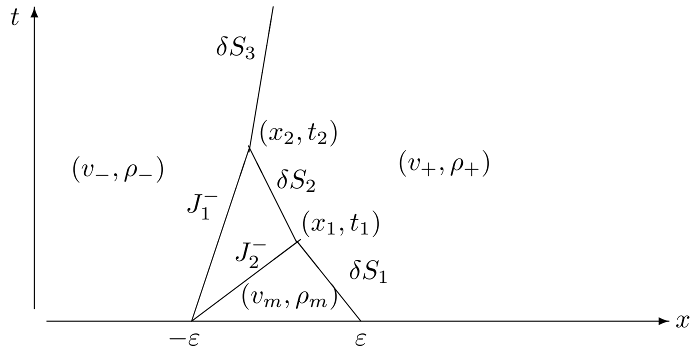

Figure 4. $v_+ < v_m - \frac{1}{\rho_m}$ and $v_+ < v_- - \frac{1}{\rho_-} < v_m$.

(3.1) is still a single delta shock wave, which is exactly the corresponding Riemann solution of (1.1)–(1.2) and (2.1) in this case.

Case 2. $v_- - \frac{1}{\rho_-} < v_m, v_+ < v_m - \frac{1}{\rho_m}$.

In this case, when t is small enough, the solution of the initial value problem (1.1)–(1.2) and (3.1) can be expressed briefly as follows (see Figure 4 or Figure 5):

$$(\rho_-, v_-) + J_1^- + (\rho_1, v_1) + J_2^- + (\rho_m, v_m) + \delta S_1 + (\rho_+, v_+).$$

Moreover, from (2.10), we have

$$v_1 = v_m, \quad \frac{1}{\rho_1} = v_m - v_- + \frac{1}{\rho_-}. \tag{3.14}$$

The propagating speed of the 2-contact discontinuity J_2^- is v_m and that of the delta shock wave δS_1 is σ_1 satisfies $v_+ < \sigma_1 < v_m - \frac{1}{\rho_m}$. Thus, it is easy to see that J_2^- and δS_1 will meet at a finite time. The interaction (x_1, t_1) is determined by

$$\begin{cases} x_1 + \varepsilon = v_m t_1, \\ x_1 - \varepsilon = \sigma_1 t_1, \end{cases} \tag{3.15}$$

which means that

$$(x_1, t_1) = (\frac{2\varepsilon v_m}{v_m - \sigma_1} - \varepsilon, \frac{2\varepsilon}{v_m - \sigma_1}), \tag{3.16}$$

where

$$\sigma_1 = \frac{\rho_+ v_+ - \rho_m v_m + \frac{dw_1(t)}{dt}}{\rho_+ - \rho_m}, \; w_1(t) = \sqrt{\rho_+ \rho_m (v_+ - v_m)^2 + (\rho_+ - \rho_m)(v_+ - v_m)}\; t.$$

The strength of δS_1 can be calculated by

$$\beta(t_1) = w_1(t_1) = \sqrt{\rho_+ \rho_m (v_+ - v_m)^2 + (\rho_+ - \rho_m)(v_+ - v_m)}\; t_1. \tag{3.17}$$

Now at the time $t = t_1$, we have a new Riemann problem with initial data

$$(\rho, v)(t_1, x) = \begin{cases} (\rho_1, v_1), & x < x_1, \\ (\rho_+, v_+), & x > x_1. \end{cases} \tag{3.18}$$

We further divide our discussion into the following two subcases.

Subcase 2.1. $v_+ < v_- - \frac{1}{\rho_-}$.

In this subcase, a new delta shock wave will generate after interaction and we denote it with δS_2 (see Figure 4). Similar to case 1, we express δS_2 as

$$\begin{cases} v(x,t) = v_1 + (v_+ - v_1)H, \\ \rho(x,t) = \rho_1 + (\rho_+ - \rho_1)H + \beta_1(t)D^- + \beta_+(t)D^+. \end{cases} \tag{3.19}$$

From (3.19), we can compute

$$\rho_t(x,t) = (-\sigma(\rho_+ - \rho_1) + \beta_1'(t) + \beta_+'(t))\delta - \sigma(\beta_1(t) + \beta_+(t))\delta', \tag{3.20}$$

$$(\rho v)_x(x,t) = (\rho_+ v_+ - \rho_1 v_1)\delta + (v_1\beta_1(t) + v_+\beta_+(t))\delta', \tag{3.21}$$

$$(\rho v)_t(x,t) = (-\sigma(\rho_+ v_+ - \rho_1 v_1) + v_1\beta_-'(t) + v_+\beta_+'(t))\delta - \sigma(v_1\beta_1(t) + v_+\beta_+(t))\delta', \tag{3.22}$$

$$(\rho v^2 - v)_x(x,t) = (\rho_+ v_+^2 - \rho_1 v_1^2 - (v_+ - v_1))\delta + (v_1^2\beta_1(t) + v_+^2\beta_+(t))\delta'. \tag{3.23}$$

Substituting (3.20)–(3.21) into the first equation of (1.1), we have the relations

$$\begin{cases} -\sigma(\rho_+ - \rho_1) + \beta_1'(t) + \beta_+'(t) + \rho_+ v_+ - \rho_1 v_1 = 0, \\ -\sigma(\beta_1(t) + \beta_+(t)) + v_1\beta_1(t) + v_+\beta_+(t) = 0. \end{cases} \tag{3.24}$$

Substituting (3.22)–(3.23) into $(1.1)_2$, we obtain

$$\begin{cases} -\sigma(\rho_+ v_+ - \rho_1 v_1) + v_1\beta_1'(t) + v_+\beta_+'(t) + \rho_+ v_+^2 - \rho_1 v_1^2 - v_+ + v_1 = 0, \\ -\sigma(v_1\beta_1(t) + v_+\beta_+(t)) + v_1^2\beta_1(t) + v_+^2\beta_+(t) = 0. \end{cases} \tag{3.25}$$

From (3.24) and (3.25), we know that the equations are overdetermined. Noting the initial condition (3.17), it can be easily derived from (3.24) that

$$\beta(t) = \beta_1(t) + \beta_+(t) = \beta(t_1) + \sqrt{\rho_+\rho_1(v_+ - v_1)^2 + (\rho_+ - \rho_1)(v_+ - v_1)}(t - t_1).$$

Combining $(3.25)_1$ with $(3.24)_2$, we can get

$$\beta(t) = \beta_1(t) + \beta_+(t) = \beta(t_1) + \sqrt{\rho_+\rho_1(v_+ - v_1)^2 + (\rho_+ - \rho_1)(v_+ - v_1)}(t - t_1),$$

which means (3.24)–(3.25) are compatible.

The propagating speed of the 1-contact discontinuity J_1^- is $v_- - \frac{1}{\rho_-}$ and that of the delta shock wave δS_2 is σ_2, where σ_2 satisfies $v_+ < \sigma_2 < v_- - \frac{1}{\rho_-}$. Thus, the contact discontinuity J_1^- will overtake the delta shock δS_2, and they begin to interact with each other at (x_2, t_2), which satisfies

$$\begin{cases} x_2 + \varepsilon = \big(v_- - \frac{1}{\rho_-}\big)t_2, \\ x_2 - x_1 = \sigma_2(t_2 - t_1). \end{cases} \tag{3.26}$$

This gives

$$(x_2, t_2) = \left(\frac{(\varepsilon + x_1 - \sigma_2 t_1)\big(v_- - \frac{1}{\rho_-}\big)}{v_- - \frac{1}{\rho_-} - \sigma_2} - \varepsilon, \frac{\varepsilon + x_1 - \sigma_2 t_1}{v_- - \frac{1}{\rho_-} - \sigma_2}\right). \tag{3.27}$$

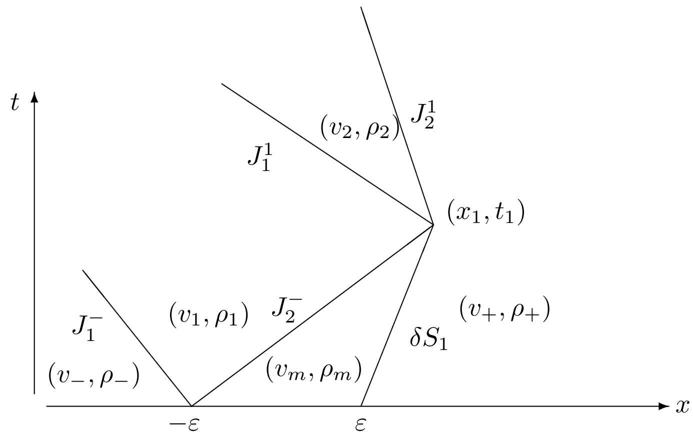

Figure 5. $v_- - \frac{1}{\rho_-} < v_+ < v_m - \frac{1}{\rho_m}$.

The strength of δS_2 at (x_2, t_2) can be calculated by

$$\beta(t_2) = \beta(t_1) + \sqrt{\rho_+\rho_1(v_+ - v_1)^2 + (\rho_+ - \rho_1)(v_+ - v_1)}(t_2 - t_1). \tag{3.28}$$

Now at the time $t = t_2$, we again have a Riemann problem with initial data

$$(\rho, v)(t_2, x) = \begin{cases} (\rho_-, v_-), & x < x_2, \\ (\rho_+, v_+), & x > x_2. \end{cases} \tag{3.29}$$

Since $v_+ < v_- - \frac{1}{\rho_-}$., a new delta shock wave δS_3 will generate after interaction. From (3.5), we can calculate

$$\rho_t(x,t) = (-\sigma(\rho_+ - \rho_-) + \beta_-^{'}(t) + \beta_+^{'}(t))\delta - \sigma(\beta_-(t) + \beta_+(t))\delta^{'}, \tag{3.30}$$

$$(\rho v)_x(x,t) = (\rho_+ v_+ - \rho_- v_-)\delta + (v_-\beta_-(t) + v_+\beta_+(t))\delta^{'}, \tag{3.31}$$

$$\begin{aligned}(\rho v)_t(x,t) = {} & (-\sigma(\rho_+ v_+ - \rho_- v_-) + v_-\beta_-^{'}(t) + v_+\beta_+^{'}(t))\delta \\ & - \sigma(v_-\beta_-(t) + v_+\beta_+(t))\delta^{'},\end{aligned} \tag{3.32}$$

$$(\rho v^2 - v)_x(x,t) = (\rho_+ v_+^2 - \rho_- v_-^2 - (v_+ - v_-))\delta + (v_-^2\beta_-(t) + v_+^2\beta_+(t))\delta^{'}. \tag{3.33}$$

Substituting (3.30)–(3.31) into $(1.1)_1$ and comparing the coefficients of δ and $\delta^{'}$, we have

$$\begin{cases} -\sigma(\rho_+ - \rho_-) + \beta_-^{'}(t) + \beta_+^{'}(t) + \rho_+ v_+ - \rho_- v_- = 0, \\ -\sigma(\beta_-(t) + \beta_+(t)) + v_-\beta_-(t) + v_+\beta_+(t) = 0. \end{cases} \tag{3.34}$$

From (3.32)–(3.33) and $(1.1)_2$, we obtain

$$\begin{cases} -\sigma(\rho_+ v_+ - \rho_- v_-) + v_-\beta_-^{'}(t) + v_+\beta_+^{'}(t) + \rho_+ v_+^2 - \rho_- v_-^2 - v_+ + v_- = 0, \\ -\sigma(v_-\beta_-(t) + v_+\beta_+(t)) + v_-^2\beta_-(t) + v_+^2\beta_+(t) = 0. \end{cases} \tag{3.35}$$

Noting the initial condition (3.28), from (3.34), we can calculate

$$\beta(t)=\beta_-(t)+\beta_+(t)=\beta(t_2)+\sqrt{\rho_+\rho_-(v_+-v_-)^2+(\rho_+-\rho_-)(v_+-v_-)}(t-t_2).$$

From $(3.35)_1$ and $(3.34)_2$, we have

$$\beta(t)=\beta_-(t)+\beta_+(t)=\beta(t_2)+\sqrt{\rho_+\rho_-(v_+-v_-)^2+(\rho_+-\rho_-)(v_+-v_-)}(t-t_2),$$

which means (3.34)–(3.35) are compatible.

It is easy to see that (x_1,t_1) and (x_2,t_2) tend to $(0,0)$ as $\varepsilon\to 0$ from (3.16) and (3.17), moreover we have $\beta(t_1)\to 0$ and $\beta(t_2)\to 0$ as $\varepsilon\to 0$ from (3.17) and (3.28). Thus, the limit of the solution of (1.1)–(1.2) and (3.1) is still a single delta shock wave, which is exactly the corresponding Riemann solution of (1.1)–(1.2) and (2.1) in this subcase.

Subcase 2.2: $\quad v_- - \frac{1}{\rho_-} < v_+$.

In this subcase, the interaction of the 2-contact discontinuity and the delta shock wave will produce two new contact discontinuities J_1^1 and J_2^1 (see Figure 5). The intermediate state (ρ_2,v_2) satisfy

$$v_2=v_+,\quad \frac{1}{\rho_2}=v_+-v_1+\frac{1}{\rho_1}, \tag{3.36}$$

where v_1 and ρ_1 satisfy (3.14).

So, when $t>t_1$, the solution of (1.1)–(1.2) and (3.1) can be expressed as

$$(\rho_-,v_-)+J_1^-+(\rho_1,v_1)+J_1^1+(\rho_2,v_2)+J_2^1+(\rho_+,v_+).$$

Case 3. $v_- - \frac{1}{\rho_-} < v_m,\ \ v_m-\frac{1}{\rho_m}<v_+$.

In this case, when t is small enough, the solution of the initial value problem (1.1)–(1.2) and (3.1) can be expressed as follows:

$$(\rho_-,v_-)+J_1^-+(\rho_1,v_1)+J_2^-+(\rho_m,v_m)+J_1^++(\rho_2,v_2)+J_2^++(\rho_+,v_+).$$

Moreover, from (2.10), we have

$$v_1=v_m,\ \frac{1}{\rho_1}=v_m-v_-+\frac{1}{\rho_-}, \tag{3.37}$$

$$v_2=v_+,\ \frac{1}{\rho_2}=v_+-v_m+\frac{1}{\rho_m}. \tag{3.38}$$

The propagating speed of the 2-contact discontinuity J_2^- is v_m and that of the 1-contact discontinuity J_1^+ is $v_m-\frac{1}{\rho_m}$. Thus, it is easy to see that J_2^- and J_1^+ will meet at a finite time and the interaction (x_1,t_1) satisfies

$$\begin{cases} x_1+\varepsilon=v_m t_1,\\ x_1-\varepsilon=\big(v_m-\frac{1}{\rho_m}\big)t_1, \end{cases} \tag{3.39}$$

which gives

$$(x_1,t_1)=(2\varepsilon\rho_m v_m-\varepsilon,2\varepsilon\rho_m). \tag{3.40}$$

Now at the time $t = t_1$, we again have a Riemann problem with initial data

$$(\rho, v)(t_1, x) = \begin{cases} (\rho_1, v_1), & x < x_1, \\ (\rho_2, v_2), & x > x_1. \end{cases} \tag{3.41}$$

Consider the states (ρ_1, v_1) and (ρ_2, v_2), we should divide our discussion into the following two subcases.

Subcase 3.1. $v_+ < v_- - \frac{1}{\rho_-}$

A new delta shock wave will generate after interaction, we denote it as δS, which can be express as

$$\begin{cases} v(x,t) = v_1 + (v_2 - v_1)H, \\ \rho(x,t) = \rho_1 + (\rho_2 - \rho_1)H + \beta_1(t)D^- + \beta_2(t)D^+. \end{cases} \tag{3.42}$$

From (3.42), we can compute

$$\rho_t(x,t) = (-\sigma(\rho_2 - \rho_1) + \beta_1^{'}(t) + \beta_2^{'}(t))\delta - \sigma(\beta_1(t) + \beta_2(t))\delta^{'}, \tag{3.43}$$

$$(\rho v)_x(x,t) = (\rho_2 v_2 - \rho_1 v_1)\delta + (v_1\beta_1(t) + v_2\beta_2(t))\delta^{'}, \tag{3.44}$$

$$(\rho v)_t(x,t) = (-\sigma(\rho_2 v_2 - \rho_1 v_1) + v_1\beta_1^{'}(t) + v_2\beta_2^{'}(t))\delta - \sigma(v_1\beta_1(t) + v_2\beta_2(t))\delta^{'}, \tag{3.45}$$

$$(\rho v^2 - v)_x(x,t) = (\rho_2 v_2^2 - \rho_1 v_1^2 - (v_2 - v_1))\delta + (v_1^2\beta_1(t) + v_2^2\beta_2(t))\delta^{'}. \tag{3.46}$$

Substituting (3.43)–(3.44) into the first equation of (1.1), we have

$$\begin{cases} -\sigma(\rho_2 - \rho_1) + \beta_1^{'}(t) + \beta_2^{'}(t) + \rho_2 v_2 - \rho_1 v_1 = 0, \\ -\sigma(\beta_1(t) + \beta_2(t)) + v_1\beta_1(t) + v_2\beta_2(t) = 0. \end{cases} \tag{3.47}$$

From (3.45)–(3.46) and $(1.1)_2$, we obtain

$$\begin{cases} -\sigma(\rho_2 v_2 - \rho_1 v_1) + v_1\beta_1^{'}(t) + v_2\beta_2^{'}(t) + \rho_2 v_2^2 - \rho_1 v_1^2 - v_2 + v_1 = 0, \\ -\sigma(v_1\beta_1(t) + v_2\beta_2(t)) + v_1^2\beta_1(t) + v_2^2\beta_2(t) = 0. \end{cases} \tag{3.48}$$

From (3.47), we can get

$$\beta(t) = \beta_1(t) + \beta_2(t) = \sqrt{\rho_2\rho_1(v_2 - v_1)^2 + (\rho_2 - \rho_1)(v_2 - v_1)}(t - t_1),$$

where v_1, v_2, ρ_1, ρ_2 satisfy (3.37) and (3.38). From (3.48), we can calculate

$$\beta(t) = \beta_1(t) + \beta_2(t) = \sqrt{\rho_2\rho_1(v_2 - v_1)^2 + (\rho_2 - \rho_1)(v_2 - v_1)}(t - t_1).$$

So Eqs. (3.47)–(3.48) are compatible.

The propagating speed σ satisfies $v_+ < \sigma < v_- - \frac{1}{\rho_-}$. So the contact discontinuities J_1^- and J_2^+ will interact with delta shock in finite time, then the new delta shock generate. We can similarly analyze it as case 2, so we omit it.

Letting $\varepsilon \to 0$, we can easily see that the solution of (1.1)–(1.2) and (3.1) is the corresponding Riemann solution of (1.1)–(1.2) and (2.1).

Subcase 3.2. $v_- - \frac{1}{\rho_-} < v_+$.

In this subcase, we know that, the interaction of the 2-contact discontinuity J_2^- and the 1-contact discontinuity J_1^+ will produce two new contact discontinuities J_1 and J_2. The intermediate state (ρ_3, v_3) satisfy

$$v_3 = v_2, \quad \frac{1}{\rho_3} = v_2 - v_1 + \frac{1}{\rho_1}, \tag{3.49}$$

where v_1, v_2, ρ_1, ρ_2 satisfy (3.37) and (3.38). So, when $t > t_1$, the solution of (1.1)–(1.2) and (3.1) can be expressed as

$$(\rho_-, v_-) + J_1^- + (\rho_1, v_1) + J_1 + (\rho_3, v_3) + J_2 + (\rho_2, v_2) + J_2^+ + (\rho_+, v_+).$$

Letting $\varepsilon \to 0$, we can easily see that the solution of (1.1)–(1.2) and (3.1) is the corresponding Riemann solution of (1.1)–(1.2) and (2.1).

Case 4. $v_m < v_- - \frac{1}{\rho_-}$, $v_m - \frac{1}{\rho_m} < v_+$.

This case can be discussed similarly to case 2, here we omit.

So far, we have finished the discussion for all kinds of interactions and the global solutions for the perturbed initial value problem (1.1)–(1.2) and (3.1) have been constructed completely. We summarize our results in the following.

Theorem 3.1. *The limits of solutions of the perturbed initial value problem (1.1)–(1.2) and (3.1) are exactly the corresponding Riemann solutions of (1.1)–(1.2) and (2.1) as $\varepsilon \to 0$. Thus, we can draw the conclusion that the Riemann solutions of (1.1)–(1.2) and (2.1) are stable with respect to such small perturbations.*

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