Journal of Applied Analysis and Computation
Volume 7, Number 1, February 2017, 207–223 Website:http://jaac-online.com/
DOI:10.11948/2017014

PULLBACK ATTRACTORS FOR MODIFIED SWIFT-HOHENBERG EQUATION ON UNBOUNDED DOMAINS WITH NON-AUTONOMOUS DETERMINISTIC AND STOCHASTIC FORCING TERMS*

Zhi Wang[1,†] and Xianyun Du[1]

Abstract In this paper, the existence and uniqueness of pullback attractors for the modified Swift-Hohenberg equation defined on R^n driven by both deterministic non-autonomous forcing and additive white noise are established. We first define a continuous cocycle for the equation in $L^2(R^n)$, and we prove the existence of pullback absorbing sets and the pullback asymptotic compactness of solutions when the equation with exponential growth of the external force. The long time behaviors are discussed to explain the corresponding physical phenomenon.

Keywords Swift-Hohenberg equation, random dynamical systems, continuous cocycle, pullback attractors, asymptotic compactness.

MSC(2010) 35R60, 37L55, 60H10.

1. Introduction

The purpose of writing this article is to survey the behavior of asymptotically compact of solutions for the modified Swift-Hohenberg equation when time is large enough. J.B.Swift and P.C.Hohenberg in [14] were introduced the Swift-Hohenberg equation when studied the convective hydrodynamics and viscous film flow. Many authors treated the Swift-Hohenberg equation [9, 10, 12]. The Swift-Hohenberg equation is a partial differential equation for a scalar field which has been widely used as a model for the study of various issues in pattern formation. These include the effects of noise on bifurcations, pattern selection, spatiotemporal chaos and the dynamics of defects. It has been used to model patterns in simple fluids and in a variety of complex fluids and biological materials, such as neural tissues. We know that the important problem in infinite dimensional dynamical systems is to prove the existence of attractors and study the structure of attractors in the framework of a process. Random or Nonautonomous dynamical systems have been extensively studied by many researchers [1, 6, 7, 11, 15–17, 19]. Recently, the theory of pullback attractor has been successfully developed and applied [2, 5, 8, 18] in many ways. Because the Sobolev embedding are not compact in the domain R^n, so to obtain the $\mathcal{D}$-pullback asymptotic compactness of the equation, we will appeal to the idea

†the corresponding author.
Email address: wang111zhi@126.com(Z. Wang), duxy2011@foxmail.com(X. Du)
[1]College of Applied Mathematics, Chengdu university of information technology, Chengdu, Sichuan 610225, China
*The authors were supported by National Natural Science Foundation of China (11301043).

of uniform estimates on the tails of solutions. This idea can be found in the non-autonomous deterministic equations or the random equations with only stochastic forcing terms [3, 4, 20, 21]. Motivated by the ideas in [10, 12, 13], we accomplished this paper. It is shown that a pullback attractor exists when the equation with exponential growth of the external force. The master devote in this essay is to develop the theory of pullback attractor for the modified Swift-Hohenberg equation with both non-autonomous deterministic and random additive white noise terms. It is worth reminding that the structure of cocycle attractors is still a very interesting open problem which deserves much fuller treatment.

The paper is made as follows. In the section 2, we recall some main definitions and results concerning the pullback attractors theory. In section 3, we transform the stochastic equation into a deterministic one with random paramater and come into being a continuous cocycle. In section 4, we devote to obtaining uniform estimates of solution when $t \to \infty$. These estimates are necessary for proving the existence of bounded absorbing set and the asymptotic compactness of the solution operator by giving uniform estimates on the tails of solution. In the last section, the existence of a pullback random attractor is proved.

2. Preliminaries

In this section, we discuss the theory of random dynamical systems which is applicable to differential equations with both non-autonomous deterministic and random additive white noise terms [22]. The pullback attractors theory is an extension either for random systems with only stochastic terms or for only non-autonomous terms.

Let Ω_1 be a nonempty set, $(\Omega_2, \mathcal{F}_2, P)$ be a probability space, and (X, d) be a complete separable metric space with Borel σ-algebra $\mathcal{B}(X)$. Suppose that there are two groups $\{\theta_1(t)\}_{t\in R}$ and $\{\theta_2(t)\}_{t\in R}$ acting on Ω_1 and Ω_2. For convenience, we often write $\theta_1(t)$ and $\theta_2(t)$ as $\theta_{1,t}$ and $\theta_{2,t}$, respectively. In the sequel, we will call both $(\Omega_1, \{\theta_1(t)\}_{t\in R})$ and $(\Omega_2, \mathcal{F}_2, P, \{\theta_2(t)\}_{t\in R})$ a parametric dynamical system.

Definition 2.1. Let $(\Omega_1, \{\theta_1(t)\}_{t\in R})$ and $(\Omega_2, \mathcal{F}_2, P, \{\theta_2(t)\}_{t\in R})$ be parametric dynamical system. A mapping $\Phi : R^+ \times \Omega_1 \times \Omega_2 \times X \to X$ is called a continuous cocycle on X over $(\Omega_1, \{\theta_1(t)\}_{t\in R})$ and $(\Omega_2, \mathcal{F}_2, P, \{\theta_2(t)\}_{t\in R})$ if for all $\omega_1 \in \Omega_1, \omega_2 \in \Omega_2$ and $t, \tau \in R^+$, the following conditions are satisfied:
(i) $\Phi(\cdot, \omega_1, \cdot, \cdot) : R^+ \times \Omega_2 \times X \to X$ is $(\mathcal{B}(R^+) \times \mathcal{F}_2 \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;
(ii) $\Phi(0, \omega_1, \omega_2, \cdot)$ is the identity on X;
(iii) $\Phi(t + \tau, \omega_1, \omega_2, \cdot) = \Phi(t, \theta_{1,\tau}\omega_1, \theta_{2,\tau}\omega_2, \cdot) \circ \Phi(\tau, \omega_1, \omega_2, \cdot)$;
(iv) $\Phi(t, \omega_1, \omega_2, \cdot) : X \to X$ is continuous.

Definition 2.2. Let B and D be two families of subsets of X which are parametrized by $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$. Then B and D are said to be equal if $B(\omega_1, \omega_2) = D(\omega_1, \omega_2)$ for all $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$.

In the sequel, we use $\mathcal{D}$ to denote a collection of some families of nonempty subsets of X:

$$\mathcal{D} = \{D = \{\emptyset \neq D(\omega_1, \omega_2) \subseteq X : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}\}. \tag{2.1}$$

Definition 2.3. Let $B = \{B(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$ be a family of nonempty subsets of X. For every $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, let

$$\Omega(B, \omega_1, \omega_2) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \Phi(t, \theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2, B(\theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2))}. \quad (2.2)$$

Then the family $\{\Omega(B, \omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$ is called the Ω-limit set of B and is denoted by $\Omega(B)$.

Definition 2.4. Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X and $S = \{S(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\} \in \mathcal{D}$. Then S is called a $\mathcal{D}$-pullback absorbing set for Φ if for all $\omega_1 \in \Omega_1, \omega_2 \in \Omega_2$ and for every $B \in \mathcal{D}$, there exists $T = T(B, \omega_1, \omega_2) > 0$ such that

$$\Phi(t, \theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2, B(\theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2)) \subseteq S(\omega_1, \omega_2) \text{ for all } t \geq T. \quad (2.3)$$

Definition 2.5. Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X. Then Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for all $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, the sequence

$$\{\Phi(t_n, \theta_{1,-t_n}\omega_1, \theta_{2,-t_n}\omega_2, x_n)\}_{n=1}^{\infty} \text{ has a convergent subsequence in } X, \quad (2.4)$$

where $t_n \to \infty$, and $x_n \in B(\theta_{1,-t_n}\omega_1, \theta_{2,-t_n}\omega_2)$ with $\{B(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\} \in \mathcal{D}$.

Definition 2.6. Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X and $\mathscr{A} = \{\mathscr{A}(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\} \in \mathcal{D}$. Then $\mathscr{A}$ is called a $\mathcal{D}$-pullback attractor for Φ if the following conditions (i) $-$ (iii) are fulfilled:
(i) $\mathscr{A}$ is measurable with respect to the P-completion of $\mathcal{F}_2$ in Ω_2 and $\mathscr{A}(\omega_1, \omega_2)$ is compact for all $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$.
(ii) $\mathscr{A}$ is invariant, that is, for every $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$,

$$\Phi(t, \omega_1, \omega_2, \mathscr{A}(\omega_1, \omega_2)) = \mathscr{A}(\theta_{1,t}\omega_1, \theta_{2,t}\omega_2), \forall t \geq 0.$$

(iii) $\mathscr{A}$ attracts every member of $\mathcal{D}$, that is, for every$B = \{B(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\} \in \mathcal{D}$ and for every $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$,

$$\lim_{t \to \infty} d(\Phi(t, \theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2, B(\theta_{1,-t}\omega_1, \theta_{2,-t}\omega_2)), \mathscr{A}(\omega_1, \omega_2)) = 0.$$

Suppose now $\Omega_1 = R$. Define a family $\{\theta_{1,t}\}_{t \in R}$ of shift operators by

$$\theta_{1,t}(h) = h + t, \text{ for all } t, h \in R. \quad (2.5)$$

Proposition 2.1. *Let $\mathcal{D}$ be a neighborhood closed collection of some families of nonempty subsets of X and Φ be a continuous cocycle on X over $(R, \{\theta_1(t)\}_{t \in R})$ and $(\Omega_2, \mathcal{F}_2, P, \{\theta_2(t)\}_{t \in R})$, where $\{\theta_1(t)\}_{t \in R}$ is defined by* (2.5)*. Then Φ has a $\mathcal{D}$-pullback attractor $\mathscr{A}$ in $\mathcal{D}$ if and only if Φ is $\mathcal{D}$-pullback asymptotically compact in X and Φ has a closed measurable $\mathcal{D}$-pullback absorbing set S in $\mathcal{D}$. The $\mathcal{D}$-pullback attractor $\mathscr{A}$ is unique and is given by, for each $\tau \in R$ and $\omega \in \Omega_2$,*

$$\mathscr{A}(\tau, \omega) = \Omega(S, \tau, \omega) = \bigcup_{B \in \mathcal{D}} \Omega(B, \tau, \omega). \quad (2.6)$$

3. Cocycles for the SwiftCHohenberg equation on R^n

Given $\tau \in R$ and $t > \tau$, consider the following non-autonomous Swift-Hohenberg equation defined for $x \in R^n$,

$$du + (\Delta^2 u + 2\Delta u + au + b|\nabla u|^2 + u^3)dt = g(x,t)dt + h(x)d\omega, \tag{3.1}$$

with the initial date

$$u(x,\tau) = u_\tau(x), \quad x \in R^n, \tag{3.2}$$

where a, b are positive constant, $g \in L^2_{loc}(R, L^2(R^n))$, $h \in H^2(R^n) \bigcap W^{2,p}(R^n)$ for some $p \geq 2$, ω is a two-sided real-valued Wiener process on a probability space.

In the sequel, we consider the probability space $(\Omega, \mathcal{F}, P)$, where we write

$$\Omega = \{\omega \in C(R,R) : \omega(0) = 0\}.$$

Let $\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and P the corresponding Wiener measure on $(\Omega, \mathcal{F})$. We definite a group $\{\theta_{2,t}\}_{t\in R}$ acting on $(\Omega, \mathcal{F}, P)$, and the time shift by

$$\theta_{2,t}\omega(\cdot) = \omega(\cdot + t) - \omega(t), \ \omega \in \Omega, t \in R. \tag{3.3}$$

Then $(\Omega, \mathcal{F}, P, \{\theta_{2,t}\}_{t\in R})$ is a parametric dynamical system. To this end, we need to convert the stochastic equation with a random additive term into a corresponding non-autonomous deterministic one.

Given $\omega \in \Omega$, we consider the one-dimensional Ornstein-Uhlenbeck equation:

$$dz + azdt = d\omega.$$

From which we can have

$$dz(\theta_{2,t}\omega) + az(\theta_{2,t}\omega)dt = d\omega. \tag{3.4}$$

Then we can easy to check that the random variable z have a stationary solution denote by

$$z(\omega) = -a\int_{-\infty}^{0} e^{a\tau}\omega(\tau)d\tau. \tag{3.5}$$

There exists a set $\widetilde{\Omega}$ which is a $\theta_{2,t}$ invariant set of full P measure, it can make sure the $z(\theta_{2,t}\omega)$ is continuous in t for every $\omega \in \widetilde{\Omega}$, and the random variable $|z(\omega)|$ is tempered. From now on, we will write $\widetilde{\Omega}$ as Ω, and not distinguish them.

Formally, if u is the solution of equation (3.1), we let the variable $v(t) = u(t) - hz(\theta_{2,t}\omega)$, which can satisfy

$$\begin{aligned} \frac{\partial v}{\partial t} + \Delta^2 v + 2\Delta v + av =& g(x,t) - z(\theta_{2,t}\omega)(\Delta^2 h + 2\Delta h) \\ &- b|\nabla(v + hz(\theta_{2,t}\omega))|^2 - (v + hz(\theta_{2,t}\omega))^3. \end{aligned} \tag{3.6}$$

For $t > \tau, \tau \in R$ and $x \in R^n$, the equation of (3.6) is a deterministic equation, we can obtain a unique solution that when $\omega \in \Omega, \tau \in R$ and $v(\tau,\tau,\omega,v_\tau) = v_\tau, v_\tau \in$

$L^2(R^n)$, for every $T > 0$. $v(\cdot,\tau,\omega,v_\tau) \in C([\tau,\infty), L^2(R^n)) \bigcap L^2((\tau,\tau+T); H^1(R^n))$. For each $t \geq \tau$, we let $u(t,\tau,\omega,u_\tau) = v(t,\tau,\omega,v_\tau) + hz(\theta_{2,t}\omega)$ with $u_\tau = v_\tau + hz(\theta_{2,t}\omega)$. Then we obtain that u is a continuous and which is $(\mathcal{F}, \mathcal{B}(L^2(R^n)))$-measurable in $\omega \in \Omega$.

So we can define a cocycle $\Phi : R^+ \times R \times \Omega \times L^2(R^n) \to L^2(R^n)$, and we let

$$\Phi(t,\tau,\omega,u_\tau) = u(t+\tau,\tau,\theta_{2,-\tau}\omega,u_\tau) = v(t+\tau,\tau,\theta_{2,-\tau}\omega,v_\tau) + hz(\theta_{2,t}\omega), \quad (3.7)$$

where $v_\tau = u_\tau - hz(\omega)$. By above analysis we can check that Φ is a continuous cocycle on $L^2(R^n)$ over $(R, \{\theta_1(t)\}_{t\in R})$ and $(\Omega, \mathcal{F}, P, \{\theta_2(t)\}_{t\in R})$. To this end, we need to specify a collection D of families of subsets of $L^2(R^n)$.

Let B be a bounded nonempty subset of $L^2(R^n)$, and denote by

$$\|B\| = \sup_{\varphi\in B} \|\varphi\|_{L^2(R^n)}.$$

Let $D = \{D(\tau,\omega) : \tau \in R, \omega \in \Omega\}$ is a family of subsets as B, which satisfy that

$$\lim_{s\to-\infty} e^{\lambda s}\|D(\tau+s,\theta_{2,s}\omega)\|^2 = 0, \quad (3.8)$$

where λ is a positive constant. We let $D_\lambda = \{D = \{D(\tau,\omega) : \tau \in R, \omega \in \Omega\}\}$. It is shown that D_λ is neighborhood closed. When deriving uniform estimate of solution, the following condition will be employed. For $\forall\tau \in R$,

$$\int_{-\infty}^{\tau} e^{\lambda s}\|g(\cdot,s)\|^2_{L^2(R^n)} ds < \infty. \quad (3.9)$$

From the formula (3.9), we can obtain that

$$\lim_{k\to\infty} \int_{-\infty}^{\tau} \int_{|x|\geq k} e^{\lambda s}|g(x,s)|^2 dxds = 0. \quad (3.10)$$

We can refer literature [13] to get the following inequality.

Lemma 3.1 (Gagliardo-Nirenberg Inequality). *Let Ω be an open, bounded domain of the lipschitz class in R^n. Assume that $1 \leq p, q \leq \infty, 1 \leq r, 0 < \theta \leq 1$ and let*

$$k - \frac{n}{p} \leq \theta(m - \frac{n}{q}) + (1-\theta)\frac{n}{r}.$$

Then the following inequality holds

$$\|u\|_{k,p} \leq c(\Omega)\|u\|_r^{1-\theta}\|u\|_{m,q}^{\theta}.$$

Here after, c is a arbitary positive constant, which may change it's value from line to line or even in the same line.

Throughout the paper, we denote $(\cdot,\cdot)$ and $\|\cdot\|$ as the inner product and norm of $L^2(R^n)$. We also respectively denote $\|\cdot\|_{W^{m,p}(R^n)}$ and $\|\cdot\|_{L^p(R^n)}$ by $\|\cdot\|_{m,p}$ and $\|\cdot\|_p$. For the external force $g \in L^2(R^n)$, we also suppose that $\|g(t)\|^2 \leq \beta e^{\alpha|t|}, \alpha, \beta > 0$.

4. Uniform estimates of solutions

By the below estimates of solutions in $L^2(R^n)$, we will obtain the existence of $\mathcal{D}_\lambda$-pullback absorbing sets.

Lemma 4.1. *According to the assumption of the front. Then for every $\tau \in R, \omega \in \Omega$ and $D = \{D(\tau,\omega) : \tau \in R, \omega \in R\} \in \mathcal{D}_\lambda$, there exists $T = T(\tau,\omega,D) > 0$, such that for all $t > T$, the solution v of equation satisfies*

$$\|v(t)\|^2 \le e^{-\delta(t-\tau)}\|v_\tau\|^2 + \frac{M}{\delta} + \frac{e^{-\delta t}}{2\delta}\int_{-\infty}^t e^{\delta s}\|g(s)\|^2 ds + ce^{-\delta t}Z_1(t), \tag{4.1}$$

$$\begin{aligned}\int_\tau^t e^{\delta s}\|\Delta v(s)\|^2 ds \le 2[1-(t-\tau)]e^{\delta\tau}\|v_\tau\|^2 + \frac{4M}{\delta}e^{\delta t} + \int_{-\infty}^t\int_{-\infty}^s e^{\delta r}\|g(r)\|^2 dr ds \\ + \delta c\int_\tau^t Z_1(s)ds + \frac{1}{\delta}\int_{-\infty}^t e^{\delta s}\|g(s)\|^2 ds + cZ_1(t).\end{aligned} \tag{4.2}$$

In the two inequalities, $M > 0$ and the $Z_1(t) = \int_\tau^t e^{\delta s}(\|z(\theta_{2,t}\omega)\|^2 + \|z(\theta_{2,t}\omega)\|_4^2)ds$.

Proof. Taking the inner product of (3.6) with v in $L^2(R^n)$, we get that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 + \|\Delta v\|^2 + a\|v\|^2 =& 2\|\nabla v\|^2 + (g(x,t),v) - z(\theta_{2,t}\omega)((\Delta^2 h + 2\Delta h),v) \\ &- b\int_{R^n}|\nabla(v + hz(\theta_{2,t}\omega))|^2 v dx \\ &- \int_{R^n}(v + hz(\theta_{2,t}\omega))^3 v dx.\end{aligned} \tag{4.3}$$

Hölder and Poincaré inequality give that

$$\lambda\|v(x,t)\|^2 \le \|\Delta u(x,t)\|^2,$$

$$(g(x,t),v(x,t)) \le \|g(x,t)\|\|v(x,t)\| \le \frac{\lambda}{2}\|v(x,t)\|^2 + \frac{1}{2\lambda}\|g(x,t)\|^2.$$

By the above two inequalities, so (4.3) can be translate into the following,

$$\begin{aligned}\frac{d}{dt}\|v\|^2 + \|\Delta v\|^2 \le& 4\|\nabla v\|^2 + 2|a|\|v\|^2 - 2z(\theta_{2,t}\omega)((\Delta^2 h + 2\Delta h),v) + \frac{1}{\lambda}\|g\|^2 \\ &+ 2|b|\int_{R^n}|\nabla(v + hz(\theta_{2,t}\omega))|^2 v dx - 2\int_{R^n}(v + hz(\theta_{2,t}\omega))^3 v dx.\end{aligned} \tag{4.4}$$

We start estimates the above inequality. Applying the lemma 3.1, we have

$$4\|\nabla v\|^2 \le c\|\Delta v(t)\|\|v(t)\| \le \frac{1}{4}\|\Delta v(t)\|^2 + c\|v(t)\|^2, \tag{4.5}$$

$$\|z(\theta_{2,t}\omega)(\Delta^2 h + 2\Delta h)\|\|v\| \le \frac{1}{2}\|z(\theta_{2,t}\omega)(\Delta^2 h + 2\Delta h)\|^2 + \frac{1}{2}\|v\|^2, \tag{4.6}$$

$$\begin{aligned}&\int_{R^n}(v + hz(\theta_{2,t}\omega))^3 v dx \\ =& \int_{R^n} v^4 + 3v^3(hz(\theta_{2,t}\omega)) + 3v^2(hz(\theta_{2,t}\omega))^2 + v(hz(\theta_{2,t}\omega))^3 dx \\ \le& \|v\|_4^4 + 3\|v\|_3^4\|hz(\theta_{2,t}\omega)\|_4 + 3\|v\|_2^4\|hz(\theta_{2,t}\omega)\|_4 + \|v\|_4\|hz(\theta_{2,t}\omega)\|_4^3 \\ \le& \|v\|_4^4 + \frac{3}{4}\|v\|_4^6 + 2\|hz(\theta_{2,t}\omega)\|_4^2 + \frac{3}{4}\|v\|_4^4 + \frac{1}{4}\|v\|_4^2 + \|hz(\theta_{2,t}\omega)\|_4^6.\end{aligned} \tag{4.7}$$

Similarly, we can obtain that

$$
\begin{aligned}
&2|b|\int_{R^n}|\nabla(v+hz(\theta_{2,t}\omega))|^2 v dx\\
\le&2|b|\|\nabla(v+hz(\theta_{2,t}\omega))\|_4^2\|v(t)\|^2\\
\le&c\|\Delta(v+hz(\theta_{2,t}\omega))\|^{2\theta}\|v+hz(\theta_{2,t}\omega)\|_4^{2(1-\theta)}\|v+hz(\theta_{2,t}\omega)-hz(\theta_{2,t}\omega)\|^2 \qquad (4.8)\\
\le&\frac{1}{8}\|\Delta(v+hz(\theta_{2,t}\omega))\|^2+M_0\\
\le&\frac{1}{4}\|\Delta v\|^2+\frac{1}{4}\|z(\theta_{2,t}\omega)\Delta h\|^2+M_1,
\end{aligned}
$$

$$
\begin{aligned}
&M_0:=c(\|v+hz(\theta_{2,t}\omega)\|_4^{3-2\theta}+\|v+hz(\theta_{2,t}\omega)\|_4^{2(1-\theta)}\|hz(\theta_{2,t}\omega)\|_4)^{\frac{1}{1-\theta}},\\
&M_1:=c(\|v+hz(\theta_{2,t}\omega)\|_4^{7-6\theta}+\frac{1}{4}\|hz(\theta_{2,t}\omega)\|_4^2)^{\frac{1}{1-\theta}}.
\end{aligned} \qquad (4.9)
$$

So, from the above we can get that there exists $M>0$ such that

$$
\frac{d}{dt}\|v\|^2+\frac{1}{2}\|\Delta v\|^2\le M+\frac{1}{\lambda}\|g(x,t)\|^2+c(\|z(\theta_{2,t}\omega)(\Delta^2h+2\Delta h)\|^2+\|hz(\theta_{2,t}\omega)\|_4^2) \qquad (4.10)
$$

and

$$
\frac{d}{dt}\|v\|^2+\frac{\lambda}{2}\|v\|^2\le M+\frac{1}{\lambda}\|g(x,t)\|^2+c(\|z(\theta_{2,t}\omega)(\Delta^2h+2\Delta h)\|^2+\|hz(\theta_{2,t}\omega)\|_4^2). \qquad (4.11)
$$

Letting $\delta=\frac{\lambda}{2}$, multiplying (4.11) by $e^{\delta(t-\tau)}$ and integrating it over (τ,t), we obtain

$$
\begin{aligned}
\|v(t)\|^2\le&e^{-\delta(t-\tau)}\|v_\tau\|^2+\frac{M}{\delta}+\frac{e^{-\delta t}}{2\delta}\int_{-\infty}^t e^{\delta s}\|g(s)\|^2ds\\
&+ce^{-\delta t}\int_\tau^t e^{\delta s}(\|z(\theta_{2,t}\omega)(\Delta^2h+2\Delta h)\|^2+\|hz(\theta_{2,t}\omega)\|_4^2)ds. \qquad (4.12)
\end{aligned}
$$

In (4.12), we estimate the last term, we can get that

$$
\begin{aligned}
&ce^{-\delta t}\int_\tau^t e^{\delta s}(\|z(\theta_{2,t}\omega)(\Delta^2h+2\Delta h)\|^2+\|hz(\theta_{2,t}\omega)\|_4^2)ds\\
=&\ ce^{-\delta t}\int_\tau^t e^{\delta s}(\|z(\theta_{2,t}\omega)\|^2\|\Delta^2h+2\Delta h\|^2+\|h\|_4^2\|z(\theta_{2,t}\omega)\|_4^2)ds\\
=&\ ce^{-\delta t}\int_\tau^t e^{\delta s}(\|z(\theta_{2,t}\omega)\|^2+\|z(\theta_{2,t}\omega)\|_4^2)ds\\
:=&\ ce^{-\delta t}Z_1(t). \qquad (4.13)
\end{aligned}
$$

From (4.12) and (4.13), we can get that

$$
\|v(t)\|^2\le e^{-\delta(t-\tau)}\|v_\tau\|^2+\frac{M}{\delta}+\frac{e^{-\delta t}}{2\delta}\int_{-\infty}^t e^{\delta s}\|g(s)\|^2ds+ce^{-\delta t}Z_1(t). \qquad (4.14)
$$

Thus, we get the desired result (4.1). Now, multiplying (4.12) by $e^{\delta t}$ and integrating it over (τ,t), we get that

$$\int_\tau^t e^{\delta s}\|v(s)\|^2 ds \le (t-\tau)e^{\delta\tau}\|v_\tau\|^2 + \frac{M}{\delta^2}e^{\delta t} + \frac{1}{2\delta}\int_{-\infty}^t\int_{-\infty}^s e^{\delta r}\|g(r)\|^2 drds$$
$$+ c\int_\tau^t\int_\tau^s e^{\delta r}(\|z(\theta_{2,t}\omega)(\Delta^2 h + 2\Delta h)\|^2 + \|hz(\theta_{2,t}\omega)\|_4^2)drds. \tag{4.15}$$

Similarly, multiplying (4.10) by $e^{\delta t}$ and integrating it over (τ,t), we get that

$$\frac{1}{2}\int_\tau^t e^{\delta s}\|\Delta v(s)\|^2 ds \le e^{\delta\tau}\|v_\tau\|^2 + \delta\int_\tau^t e^{\delta s}\|v(s)\|^2 ds + \frac{M}{\delta}e^{\delta t}$$
$$+ c\int_\tau^t e^{\delta s}(\|z(\theta_{2,t}\omega)(\Delta^2 h + 2\Delta h)\|^2 + \|hz(\theta_{2,t}\omega)\|_4^2)ds$$
$$+ \frac{1}{2\delta}\int_{-\infty}^t e^{\delta s}\|g(s)\|^2 ds. \tag{4.16}$$

From (4.15) and (4.16), we can obtain that

$$\int_\tau^t e^{\delta s}\|\Delta v(s)\|^2 ds$$
$$\le 2[1-(t-\tau)]e^{\delta\tau}\|v_\tau\|^2 + \frac{4M}{\delta}e^{\delta t} + \int_{-\infty}^t\int_{-\infty}^s e^{\delta r}\|g(r)\|^2 drds$$
$$+ 2c\delta\int_\tau^t\int_\tau^s e^{\delta r}(\|z(\theta_{2,t}\omega)(\Delta^2 h + 2\Delta h)\|^2 + \|hz(\theta_{2,t}\omega)\|_4^2)drds$$
$$+ \frac{1}{\delta}\int_{-\infty}^t e^{\delta s}\|g(s)\|^2 ds + 2c\int_\tau^t e^{\delta s}(\|z(\theta_{2,t}\omega)(\Delta^2 h + 2\Delta h)\|^2 + \|hz(\theta_{2,t}\omega)\|_4^2)ds$$
$$\le 2[1-(t-\tau)]e^{\delta\tau}\|v_\tau\|^2 + \frac{4M}{\delta}e^{\delta t} + \int_{-\infty}^t\int_{-\infty}^s e^{\delta r}\|g(r)\|^2 drds$$
$$+ \delta c\int_\tau^t Z_1(s)ds + \frac{1}{\delta}\int_{-\infty}^t e^{\delta s}\|g(s)\|^2 ds + cZ_1(t). \tag{4.17}$$

So, the desired result (4.2) of lemma is proved. □

Now we can derive the uniform estimates of solutions in $H^2(R^n)$.

Lemma 4.2. *According to the assumption of the front. Then for every $\tau \in R, \omega \in \Omega$ and $D = \{D(\tau,\omega) : \tau \in R, \omega \in R\} \in \mathcal{D}_\lambda$, there exist $T = T(\tau,\omega,D,\epsilon) \ge 1$ such that for all $t \ge T$, the solution v of equation (3.6) satisfies*

$$\|\Delta v(t)\|^2 \le ce^{-\delta(t-\tau)}(\|v_\tau\|^2 + \|v_\tau\|^6 + \|v_\tau\|^{10})$$
$$+ ce^{-\delta t}\int_\tau^t e^{\delta s}(\|z(\theta_{2,s}\omega)\|^2 + \|z(\theta_{2,s}\omega)\|^6 + \|z(\theta_{2,s}\omega)\|^{10})ds$$
$$+ \frac{c}{\delta}(2 + \frac{1}{t-\tau})e^{-\delta t}\int_{-\infty}^t e^{\delta s}\|g(s)\|^2 ds + c(1 + \frac{1}{t-\tau})e^{-\delta t}Z_1(t)$$
$$+ c(1 + \frac{1}{t-\tau})\frac{4M}{\delta} + \frac{c}{\delta}[1 - e^{-\delta(t-\tau)}] + G_1(t) + G_2(t). \tag{4.18}$$

Proof. Taking the inner product of (3.6) with $\Delta^2 v$ in L^2, we get that

$$L_1(t,\omega) = R_1(t,\omega), \tag{4.19}$$

where

$$L_1(t,\omega) = \frac{1}{2}\frac{d}{dt}\|\Delta v\|^2 + \|\Delta^2 v\|^2 + 2\int_{R^n} \Delta v \Delta^2 v dx + a\int_{R^n} v\Delta^2 v dx,$$
$$R_1(t,\omega) = \int_{R^n} g(x,t)\Delta^2 v dx - \int_{R^n} \Delta^2 v z(\theta_{2,t}\omega)(\Delta^2 h + 2\Delta h)dx$$
$$- b\int_{R^n} |\nabla(v + hz(\theta_{2,t}\omega))|^2 \Delta^2 v dx - \int_{R^n} (v + hz(\theta_{2,t}\omega))^3 \Delta^2 v dx.$$

We use the Young inequality estimate the above. We can obtain

$$\int_{R^n} \Delta v \Delta^2 v dx \le \|\Delta v\|\|\Delta^2 v\| \le \frac{1}{12}\|\Delta^2 v\|^2 + 3\|\Delta v\|^2, \tag{4.20}$$

$$\int_{R^n} g(x,t)\Delta^2 v dx \le \|g(x,t)\|\|\Delta^2 v\| \le \frac{1}{12}\|\Delta^2 v\|^2 + 3\|g(x,t)\|^2, \tag{4.21}$$

$$\begin{aligned}\int_{R^n} \Delta^2 v z(\theta_{2,t}\omega)(\Delta^2 h + 2\Delta h)dx &\le \|\Delta^2 v\|\|z(\theta_{2,t}\omega)(\Delta^2 h + 2\Delta h)\| \\ &\le \frac{1}{12}\|\Delta^2 v\|^2 + 3\|z(\theta_{2,t}\omega)(\Delta^2 h + 2\Delta h)\|^2,\end{aligned} \tag{4.22}$$

$$\begin{aligned}|b|\int_{R^n} |\nabla(v + hz(\theta_{2,t}\omega))|^2 \Delta^2 v dx &\le |b|\|\nabla(v + hz(\theta_{2,t}\omega))\|_4^2 \|\Delta^2 v\| \\ &\le \frac{1}{12}\|\Delta^2 v\|^2 + 3b^2\|\nabla(v + hz(\theta_{2,t}\omega))\|_4^4,\end{aligned} \tag{4.23}$$

$$\begin{aligned}\int_{R^n} (v + hz(\theta_{2,t}\omega))^3 \Delta^2 v dx &\le \|v + hz(\theta_{2,t}\omega)\|_6^3 \|\Delta^2 v\| \\ &\le \frac{1}{12}\|\Delta^2 v\|^2 + 3\|v + hz(\theta_{2,t}\omega)\|_6^6.\end{aligned} \tag{4.24}$$

From above we can get that

$$\begin{aligned}\frac{d}{dt}\|\Delta v\|^2 + \|\Delta^2 v\|^2 \le& 12\|\Delta v\|^2 - 2a\|\Delta v\|^2 + 6\|z(\theta_{2,t}\omega)(\Delta^2 h + 2\Delta h)\|^2 + 6\|g(x,t)\|^2 \\ &+ 6b^2\|\nabla(v + hz(\theta_{2,t}\omega))\|_4^4 + 6\|v + hz(\theta_{2,t}\omega)\|_6^6.\end{aligned} \tag{4.25}$$

In Lemma 3.1, letting $k = 1, p = 4, n = m = q = r = 2, \theta = \frac{1}{4}$, we can obtain that

$$\begin{aligned}6b^2\|\nabla(v + hz(\theta_{2,t}\omega))\|_4^4 &\le c\|v + hz(\theta_{2,t}\omega)\|^3 \|\Delta(v + hz(\theta_{2,t}\omega))\| \\ &\le \frac{\lambda}{8}\|\Delta(v + hz(\theta_{2,t}\omega))\|^2 + c\|v + hz(\theta_{2,t}\omega)\|^6.\end{aligned} \tag{4.26}$$

Similarly, letting $k = 0, p = 6, n = m = q = r = 2, \theta = \frac{1}{6}$, we can obtain that

$$\begin{aligned}6\|v + hz(\theta_{2,t}\omega)\|_6^6 &\le c\|v + hz(\theta_{2,t}\omega)\|^5 \|\Delta(v + hz(\theta_{2,t}\omega))\| \\ &\le \frac{\lambda}{8}\|\Delta(v + hz(\theta_{2,t}\omega))\|^2 + c\|v + hz(\theta_{2,t}\omega)\|^{10}.\end{aligned} \tag{4.27}$$

Applying these estimates in (4.25), and letting $\delta = \frac{\lambda}{2}$, so we can obtain that

$$\begin{aligned}\frac{d}{dt}\|\Delta v(t)\|^2 + \delta\|\Delta v\|^2 \le& c\{\|\Delta v\|^2 + \|g(x,t)\|^2 + \|z(\theta_{2,t}\omega(\Delta^2 h + 2\Delta h))\|^2 \\ &+ \|\Delta hz(\theta_{2,t}\omega)\|^2 + \|v + hz(\theta_{2,t}\omega)\|^6 + \|v + hz(\theta_{2,t}\omega)\|^{10}\}.\end{aligned} \tag{4.28}$$

Multiplying (4.28) by $(t-\tau)e^{\delta t}$ and integrating it over (τ, t), we can obtain that

$$
\begin{aligned}
(t-\tau)e^{\delta t}\|\Delta v(t)\|^2 \le & c\{\int_\tau^t [1+(s-\tau)]e^{\delta s}\|\Delta v\|^2 ds + \int_\tau^t (s-\tau)e^{\delta s}\|g(s)\|^2 ds \\
& + \int_\tau^t (s-\tau)e^{\delta s}\|z(\theta_{2,t}\omega)(\Delta^2 h + 2\Delta h)\|^2 ds \\
& + \int_\tau^t (s-\tau)e^{\delta s}\|\Delta h z(\theta_{2,t}\omega)\|^2 ds \\
& + \int_\tau^t (s-\tau)e^{\delta s}(\|v+hz(\theta_{2,t}\omega)\|^6 + \|v+hz(\theta_{2,t}\omega)\|^{10}) ds\},
\end{aligned} \tag{4.29}
$$

and hence

$$
\begin{aligned}
\|\Delta v(t)\|^2 \le & c(1+\frac{1}{t-\tau})e^{-\delta t}\int_\tau^t e^{\delta s}\|\Delta v(s)\|^2 ds + ce^{-\delta t}\int_\tau^t e^{\delta s}\|z(\theta_{2,s}\omega)\Delta h\|^2 ds \\
& + ce^{-\delta t}\int_\tau^t e^{\delta s}\|z(\theta_{2,s}\omega)(\Delta^2 h + 2\Delta h)\|^2 ds + ce^{-\delta t}\int_{-\infty}^t e^{\delta s}\|g(s)\|^2 ds \\
& + ce^{-\delta t}\int_\tau^t e^{\delta s}\|v+hz(\theta_{2,s}\omega)\|^6 ds + ce^{-\delta t}\int_\tau^t e^{\delta s}\|v+hz(\theta_{2,s}\omega)\|^{10} ds \\
:= & I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned} \tag{4.30}
$$

Now, we estimate the terms on the right hand side of (4.30), and we can obtain that

$$
\begin{aligned}
I_1 = & c(1+\frac{1}{t-\tau})e^{-\delta t}\int_\tau^t e^{\delta s}\|\Delta v(s)\|^2 ds \\
\le & c[\frac{1}{t-\tau} - (t-\tau)]e^{-\delta(t-\tau)}\|v_\tau\|^2 + c(1+\frac{1}{t-\tau})\frac{4M}{\delta} \\
& + c(1+\frac{1}{t-\tau})e^{-\delta t}\int_{-\infty}^t\int_{-\infty}^s e^{\delta r}\|g(r)\|^2 dr ds + c(1+\frac{1}{t-\tau})e^{-\delta t}Z_1(t) \\
& + c\delta(1+\frac{1}{t-\tau})e^{-\delta t}\int_\tau^t Z_1(s) ds + \frac{c}{\delta}(1+\frac{1}{t-\tau})e^{-\delta t}\int_{-\infty}^t e^{\delta s}\|g(s)\|^2 ds.
\end{aligned} \tag{4.31}
$$

By (4.1), through calculation and reduction, we can get that

$$
\begin{aligned}
I_5 = & ce^{-\delta t}\int_\tau^t e^{\delta s}\|v+hz(\theta_{2,s}\omega)\|^6 ds \\
\le & ce^{-\delta t}\int_\tau^t e^{\delta s}(\|v\|^2)^3 ds + ce^{-\delta t}\int_\tau^t e^{\delta s}\|z(\theta_{2,s}\omega)\|^6 ds \\
\le & \frac{c}{-2\delta}[e^{-3\delta(t-\tau)} - e^{-\delta(t-\tau)}]\|v_\tau\|^6 + \frac{c}{\delta}[1-e^{-\delta(t-\tau)}] + ce^{-\delta t}\int_\tau^t e^{-2\delta s}[Z_1(s)]^3 ds \\
& + ce^{-\delta t}\int_\tau^t e^{-2\delta s}(\int_{-\infty}^s e^{\delta r}\|g(r)\|^2 dr)^3 ds + ce^{-\delta t}\int_\tau^t e^{\delta s}\|z(\theta_{2,s}\omega)\|^6 ds.
\end{aligned} \tag{4.32}
$$

Similarly, we get that

$$\begin{aligned}
I_6 =&ce^{-\delta t}\int_\tau^t e^{\delta s}\|v+hz(\theta_{2,s}\omega)\|^{10}ds\\
\le& ce^{-\delta t}\int_\tau^t e^{\delta s}(\|v\|^2)^5 ds + ce^{-\delta t}\int_\tau^t e^{\delta s}\|z(\theta_{2,s}\omega)\|^{10}ds\\
\le& \frac{c}{-4\delta}[e^{-5\delta(t-\tau)}-e^{\delta\tau}]\|v_\tau\|^{10}+\frac{c}{\delta}[1-e^{-\delta(t-\tau)}]+ce^{-\delta t}\int_\tau^t e^{-4\delta s}[Z_1(s)]^5 ds\\
&+ce^{-\delta t}\int_\tau^t e^{-4\delta s}(\int_{-\infty}^s e^{\delta r}\|g(r)\|^2dr)^5 ds + ce^{-\delta t}\int_\tau^t e^{\delta s}\|z(\theta_{2,s}\omega)\|^{10}ds. \quad (4.33)
\end{aligned}$$

We let

$$\begin{aligned}
G_1(t) =&c\delta(1+\frac{1}{t-\tau})e^{-\delta t}\int_\tau^t Z_1(s)ds + ce^{-\delta t}\int_\tau^t e^{-2\delta s}[Z_1(s)]^3 ds\\
&+ce^{-\delta t}\int_\tau^t e^{-4\delta s}[Z_1(s)]^5 ds, \quad (4.34)\\
G_2(t) =&c(1+\frac{1}{t-\tau})e^{-\delta t}\int_{-\infty}^t\int_{-\infty}^s e^{\delta r}\|g(r)\|^2 drds\\
&+ce^{-\delta t}\int_\tau^t e^{-2\delta s}(\int_{-\infty}^s e^{\delta r}\|g(r)\|^2dr)^3 ds \quad (4.35)\\
&+ce^{-\delta t}\int_\tau^t e^{-4\delta s}(\int_{-\infty}^s e^{\delta r}\|g(r)\|^2dr)^5 ds.
\end{aligned}$$

From (4.30)–(4.35), we can get that

$$\begin{aligned}
\|\Delta v(t)\|^2 \le& ce^{-\delta(t-\tau)}(\|v_\tau\|^2+\|v_\tau\|^6+\|v_\tau\|^{10})\\
&+ce^{-\delta t}\int_\tau^t e^{\delta s}(\|z(\theta_{2,s}\omega)\|^2+\|z(\theta_{2,s}\omega)\|^6+\|z(\theta_{2,s}\omega)\|^{10})ds\\
&+\frac{c}{\delta}(2+\frac{1}{t-\tau})e^{-\delta t}\int_{-\infty}^t e^{\delta s}\|g(s)\|^2ds + c(1+\frac{1}{t-\tau})e^{-\delta t}Z_1(t)\\
&+c(1+\frac{1}{t-\tau})\frac{4M}{\delta}+\frac{c}{\delta}[1-e^{-\delta(t-\tau)}]+G_1(t)+G_2(t). \quad (4.36)
\end{aligned}$$

So the proof is completed. □

The following uniform estimates are necessary condition for getting the asymptotic compactness of equation defined on unbounded domains. We derive uniform estimates on the tails of solution when time and space variables are large enough.

Lemma 4.3. *According to the assumption of the front. Then for every $\tau\in R$, $\omega\in\Omega$ and $D=\{D(\tau,\omega):\tau\in R,\omega\in R\}\in\mathcal{D}_\lambda$. Then for every $\epsilon>0$, there exist $T=T(\tau,\omega,D,\epsilon)\ge 1$ and $K=K(\tau,\omega,\epsilon)\ge 1$ such that for all $t\ge T$, the solution v of equation (3.6) with ω replaced by $\theta_{2,-\tau}\omega$ satisfies*

$$\int_{|x|\ge K}|v(\tau,\tau-t,\theta_{2,-\tau}\omega,v_{\tau-t})(x)|^2dx\le\epsilon,$$

where $v_{\tau-t}\in D(\tau-t,\theta_{2,-\tau}\omega)$.

Proof. Let ρ be a smooth function defined on R^+, such that $0 \le \rho(s) \le 1$ for all $s \in R^+$, and

$$\rho(s) = \begin{cases} 0 & \text{for } 0 \le s \le 1, \\ 1 & \text{for } s \ge 2. \end{cases} \tag{4.37}$$

Then there exists a positive constant c such that $|\rho'(s)| \le c$ for all $s \in R^+$. Taking the inner product of(3.6) with $\rho(\frac{|x|^2}{k^2})v$ in $L^2(R^n)$, we get that

$$L_2(t,\omega) = R_2(t,\omega), \tag{4.38}$$

where

$$\begin{aligned} L_2(t,\omega) =& \frac{1}{2}\frac{d}{dt}\int_{R^n} \rho(\frac{|x|^2}{k^2})|v|^2 dx + \int_{R^n} \Delta^2 v \rho(\frac{|x|^2}{k^2}) v dx + \int_{R^n} 2\Delta v \rho(\frac{|x|^2}{k^2}) v dx \\ &+ a\int_{R^n} \rho(\frac{|x|^2}{k^2}) v^2 dx, \\ R_2(t,\omega) =& -b\int_{R^n} \rho(\frac{|x|^2}{k^2})|\nabla(v + hz(\theta_{2,t}\omega))|^2 v dx - \int_{R^n} \rho(\frac{|x|^2}{k^2}) v(v + hz(\theta_{2,t}\omega))^3 dx \\ &- \int_{R^n} \rho(\frac{|x|^2}{k^2}) v z(\theta_{2,t}\omega)(\Delta^2 h + 2\Delta h) dx + \int_{R^n} \rho(\frac{|x|^2}{k^2}) v g(x,t) dx. \end{aligned}$$

We estimate $L_2(t,\omega)$, we can obtain

$$|\int_{R^n} \Delta^2 v \rho(\frac{|x|^2}{k^2}) v dx| \le \frac{\lambda}{2}\int_{R^n} \rho(\frac{|x|^2}{k^2}) v^2 dx + \frac{1}{2\lambda}\int_{R^n} (\Delta^2 v)^2 \rho(\frac{|x|^2}{k^2}) dx, \tag{4.39}$$

$$|\int_{R^n} 2\Delta v \rho(\frac{|x|^2}{k^2}) v dx| \le \lambda\int_{R^n} \rho(\frac{|x|^2}{k^2}) v^2 dx + \frac{1}{\lambda}\int_{R^n} \rho(\frac{|x|^2}{k^2})(\Delta v)^2 dx. \tag{4.40}$$

We estimate $R_2(t,\omega)$, we can obtain

$$\begin{aligned} &|\int_{R^n} \rho(\frac{|x|^2}{k^2}) v[g(x,t) - z(\theta_{2,t}\omega)(\Delta^2 h + 2\Delta h)] dx| \\ \le& \frac{\lambda}{2}\int_{R^n} \rho(\frac{|x|^2}{k^2}) v^2 dx + \frac{1}{\lambda}\int_{R^n} \rho(\frac{|x|^2}{k^2})[(g(x,t))^2 + (z(\theta_{2,t}\omega)(\Delta^2 h + 2\Delta h))^2] dx, \end{aligned} \tag{4.41}$$

$$\begin{aligned} &|b\int_{R^n} \rho(\frac{|x|^2}{k^2})|\nabla(v + hz(\theta_{2,t}\omega))|^2 v dx| \\ \le& \frac{\lambda}{2}|b|\int_{R^n} \rho(\frac{|x|^2}{k^2}) v^2 dx + \frac{1}{2\lambda}|b|\int_{R^n} \rho(\frac{|x|^2}{k^2})|\nabla(v + hz(\theta_{2,t}\omega))|^4 dx, \end{aligned} \tag{4.42}$$

$$\begin{aligned} &|\int_{R^n} \rho(\frac{|x|^2}{k^2}) v(v + hz(\theta_{2,t}\omega))^3 dx| \\ \le& \frac{\lambda}{2}|b|\int_{R^n} \rho(\frac{|x|^2}{k^2}) v^2 dx + \frac{1}{2\lambda}\int_{R^n} \rho(\frac{|x|^2}{k^2})(v + hz(\theta_{2,t}\omega))^6 dx. \end{aligned} \tag{4.43}$$

By above estimate we let

$$
\begin{aligned}
M(t,\omega) =&\frac{1}{\lambda}\int_{R^n}\rho(\frac{|x|^2}{k^2})[(g(x,t))^2+(z(\theta_{2,t}\omega)(\Delta^2 h+2\Delta h))^2]dx\\
&+\frac{1}{2\lambda}|b|\int_{R^n}\rho(\frac{|x|^2}{k^2})|\nabla(v+hz(\theta_{2,t}\omega))|^4dx\\
&+\frac{1}{2\lambda}\int_{R^n}\rho(\frac{|x|^2}{k^2})(v+hz(\theta_{2,t}\omega))^6dx. \qquad (4.44)
\end{aligned}
$$

Let $\lambda_1 = 2a - 5\lambda - \lambda|b|$, the final reduction to obtain that

$$
\begin{aligned}
&\frac{d}{dt}\int_{R^n}\rho(\frac{|x|^2}{k^2})|v|^2dx+\lambda_1\int_{R^n}\rho(\frac{|x|^2}{k^2})v^2dx\\
\leq&\frac{1}{\lambda}\|\Delta^2 v\|^2+\frac{2}{\lambda}\|\Delta v\|^2+\frac{2}{\lambda}\|g(x,t)\|^2+\frac{2}{\lambda}\|z(\theta_{2,t}\omega)(\Delta^2 h+2\Delta h)\|^2\\
&+\frac{|b|}{\lambda}\|\nabla(v+hz(\theta_{2,t}\omega))\|^4_{L^4(R^n)}+\frac{1}{\lambda}\|v+hz(\theta_{2,t}\omega)\|^6_{L^6(R^n)}. \qquad (4.45)
\end{aligned}
$$

By the inequalities (4.26) and (4.27), we can ulteriorly obtain that

$$
\begin{aligned}
&\frac{d}{dt}\int_{R^n}\rho(\frac{|x|^2}{k^2})|v|^2dx+\lambda_1\int_{R^n}\rho(\frac{|x|^2}{k^2})v^2dx\\
\leq&\frac{1}{\lambda}\|\Delta^2 v\|^2+(\frac{2}{\lambda}+\frac{\lambda}{2})\|\Delta v\|^2+\frac{2}{\lambda}\|g(x,t)\|^2+\frac{2}{\lambda}\|z(\theta_{2,t}\omega)(\Delta^2 h+2\Delta h)\|^2\\
&+\frac{\lambda}{2}\|\Delta hz(\theta_{2,t}\omega)\|^2+c\|v+hz(\theta_{2,t}\omega)\|^6+c\|v+hz(\theta_{2,t}\omega)\|^{10}. \qquad (4.46)
\end{aligned}
$$

We multiply (4.46) by $e^{\lambda t}$ and then integrate the inequality on $(\tau-t,\tau)$ with $t\geq 0$, we get that for each $\omega\in\Omega$,

$$
\begin{aligned}
&\int_{R^n}\rho(\frac{|x|^2}{k^2})|v(\tau,\tau-t,\omega,v_{\tau-t})|^2dx-e^{-\lambda t}\int_{R^n}\rho(\frac{|x|^2}{k^2})|v_{\tau-t}(x)|^2dx\\
\leq&\frac{1}{\lambda}e^{-\lambda t}\int_{\tau-t}^{\tau}e^{\lambda s}\|\Delta^2 v\|^2ds+H_1(t,\omega)+H_2(t,\omega). \qquad (4.47)
\end{aligned}
$$

In the above the

$$
\begin{aligned}
H_1(t,\omega) =&(\frac{2}{\lambda}+\frac{\lambda}{2})e^{-\lambda t}\int_{\tau-t}^{\tau}e^{\lambda s}\|\Delta v\|^2ds+\frac{2}{\lambda}\int_{\tau-t}^{\tau}e^{\lambda(s-\tau)}\|g(x,s)\|^2ds\\
&+\frac{2}{\lambda}\int_{\tau-t}^{\tau}e^{\lambda(s-\tau)}\|z(\theta_{2,s}\omega)(\Delta^2 h+2\Delta h)\|^2ds\\
&+\frac{\lambda}{2}\int_{\tau-t}^{\tau}e^{\lambda(s-\tau)}\|z(\theta_{2,s}\omega)\Delta h\|^2ds,\\
H_2(t,\omega) =&ce^{-\lambda t}\int_{\tau-t}^{\tau}e^{\lambda s}\|v+hz(\theta_{2,s}\omega)\|^6ds+ce^{-\lambda t}\int_{\tau-t}^{\tau}e^{\lambda s}\|v+hz(\theta_{2,s}\omega)\|^{10}ds.
\end{aligned}
\qquad (4.48)
$$

In (4.47), we let $H_0(t,\omega)=e^{-\lambda t}\int_{\tau-t}^{\tau}e^{\lambda s}\|\Delta^2 v\|^2ds$.

From the front inequality of (4.25), we multiply it by $e^{\lambda s}$, and integrate it over $(\tau-t,\tau)$, we can obtain that

$$\begin{aligned}&\int_{\tau-t}^{\tau} e^{\lambda s}\|\Delta^2 v\|^2 ds\\ \leq&\int_{\tau-t}^{\tau}(12-2a)e^{\lambda s}\|\Delta v\|^2 ds+6\int_{\tau-t}^{\tau} e^{\lambda s}\|z(\theta_{2,s}\omega)(\Delta^2 h+2\Delta h)\|^2 ds\\ &+\int_{\tau-t}^{\tau}6e^{\lambda s}\|g(x,s)\|^2 ds+6b^2\int_{\tau-t}^{\tau} e^{\lambda s}\|\nabla(v+hz(\theta_{2,s}\omega))\|_4^4 ds\\ &+6\int_{\tau-t}^{\tau} e^{\lambda s}\|v+hz(\theta_{2,s}\omega)\|_6^6 ds. \end{aligned}\tag{4.49}$$

Replacing ω by $\theta_{2,-\tau}\omega$ in (4.47) and (4.48). By the front estimate of (4.26), (4.27), and (4.31)–(4.33), we can get that $H_0(t,\theta_{2,-\tau}\omega)$, $H_1(t,\theta_{2,-\tau}\omega)$ and $H_2(t,\theta_{2,-\tau}\omega)$ are arbitrarily samll when the time t is large enough. So we can get that the right hand of the below inequality is also arbitrarily samll. There exists a large time T, when $t\geq T$, there have a arbitrarily small variable $\epsilon>0$, we can let the $H_0(t,\theta_{2,-\tau}\omega)\leq\frac{\lambda\epsilon}{3}$, $H_1(t,\theta_{2,-\tau}\omega)\leq\frac{\epsilon}{3}$ and $H_2(t,\theta_{2,-\tau}\omega)\leq\frac{\epsilon}{3}$.

$$\begin{aligned}&\int_{R^n}\rho(\frac{|x|^2}{k^2})|v(\tau,\tau-t,\theta_{2,-\tau}\omega,v_{\tau-t})|^2dx-e^{-\lambda t}\int_{R^n}\rho(\frac{|x|^2}{k^2})|v_{\tau-t}(x)|^2dx\\ \leq&\frac{1}{\lambda}H_0(t,\theta_{2,-\tau}\omega)+H_1(t,\theta_{2,-\tau}\omega)+H_2(t,\theta_{2,-\tau}\omega).\end{aligned}\tag{4.50}$$

Since $v_{\tau-t}\in D(\tau-t,\theta_{2,-\tau}\omega)$ and $D\in\mathcal{D}_\lambda$, we can obtain that

$$\limsup_{t\to\infty}e^{-\lambda t}\int_{R^n}\rho(\frac{|x|^2}{k^2})|v_{\tau-t}(x)|^2dx\leq\limsup_{t\to\infty}e^{-\lambda t}\|D(\tau-t,\theta_{2,-\tau}\omega)\|^2=0.\tag{4.51}$$

Which along with (4.50), we can obtain that there exist $T(\tau,\omega,D)$ such that for all $t\geq T$

$$\begin{aligned}&\int_{|x|\geq\sqrt{2}k}|v(\tau,\tau-t,\theta_{2,-\tau}\omega,v_{\tau-t})|^2dx\\ \leq&\int_{R^n}\rho(\frac{|x|^2}{k^2})|v(\tau,\tau-t,\theta_{2,-\tau}\omega,v_{\tau-t})|^2dx\\ \leq&\frac{1}{\lambda}H_0(t,\theta_{2,-\tau}\omega)+H_1(t,\theta_{2,-\tau}\omega)+H_2(t,\theta_{2,-\tau}\omega)\\ \leq&\frac{1}{\lambda}\cdot\frac{\lambda\epsilon}{3}+\frac{2\epsilon}{3}=\epsilon.\end{aligned}\tag{4.52}$$

So the lemma is proved. □

Now, we start to derive uniform estimates on the solutions u of the equation (3.1), we will base on the estimate of the solutions v of the equation (3.6). By (3.7), we now define another family $\widetilde{D}$ of subsets of $L^2(R^n)$ from D. Given $\tau\in R$ and $\omega\in\Omega$, set

$$\widetilde{D}(\tau,\omega)=\{\varphi\in L^2(R^n):\|\varphi\|^2\leq 2\|D(\tau,\omega)\|^2+2|z(\omega)|^2\|h\|^2\}.\tag{4.53}$$

Let $\widetilde{D}$ be the family consisting of those sets given by above, that is to say,

$$\widetilde{D}=\{\widetilde{D}(\tau,\omega):\widetilde{D}(\tau,\omega)\ is\ defined\ by\ (4.53),\tau\in R,\omega\in\Omega\}.\tag{4.54}$$

Since $|z(\omega)|$ is tempered and $D \in \mathcal{D}_\lambda$, it is easy to check that $\widetilde{D}$ given by (4.54) also belongs to $\mathcal{D}_\lambda$. Furthermore, if $u_{\tau-t} \in D(\tau - t, \theta_{2,-t}\omega)$, then $v_{\tau-t} = u_{\tau-t} - hz(\theta_{2,-t}\omega)$ belongs to $\widetilde{D}(\tau - t, \theta_{2,-t}\omega)$. So we can obtain that the uniform estimates of the solutions of equation (3.1) in $H^2(R^n)$ from (3.7), Lemma 4.1 and Lemma 4.2.

5. Existence of attractors for the Swift-Hohenberg equation

We next show that Φ is $\mathcal{D}_\lambda$-pullback asymptotically compact in $L^2(R^n)$.

Lemma 5.1. *According to the assumption of the front. Then Φ is $\mathcal{D}_\lambda$-pullback asymptotically compact in $L^2(R^n)$, that is to say, for every $\tau \in R, \omega \in \Omega, D = \{D(\tau,\omega) : \tau \in R, \omega \in \Omega\} \in \mathcal{D}_\lambda$, and $t_n \to \infty, u_{0,n} \in D(\tau - t_n, \theta_{2,-t_n}\omega, u_{0,n})$, the sequence $\Phi(t_n, \tau - t_n, \theta_{2,-t_n}\omega, u_{0,n})$ has a convergent subsequence in $L^2(R^n)$.*

Proof. In order to prove the Φ is pullback asymptotically compact, we need to show that for every $\epsilon > 0$, there has a finite covering of balls of radius less than ϵ for the sequence $\Phi(t_n, \tau - t_n, \theta_{2,-t_n}\omega, u_{0,n})$. Let $K > 0$, denote by $Q_K = \{x \in R^n : |x| \le K\}$ and $Q_K^c = R^n \backslash Q_K$. It follows form Lemma 4.3 that there exist $K = K(\tau,\omega,\epsilon) \ge 1$ and $N_1 = N_1(\tau,\omega,D,\epsilon) \ge 1$ such that for all $n \ge N_1$,

$$\|\Phi(t_n, \tau - t_n, \theta_{2,-t_n}\omega, u_{0,n})\|_{L^2(Q_K^c)} \le \frac{\epsilon}{2}. \tag{5.1}$$

By Lemma 4.2 and the final analysis of the fourth part, we can obtain that there exists $N_2 = N_2(\tau,\omega,D,\epsilon) \ge N_1$ such that for all $n \ge N_2$,

$$\|\Phi(t_n, \tau - t_n, \theta_{2,-t_n}\omega, u_{0,n})\|_{H^2(Q_K)} \le C(\tau,\omega), \tag{5.2}$$

where $C(\tau,\omega)$ is the constant which can control by the right-hand side of (4.36). By the compactness of embedding $H^2(Q_K) \hookrightarrow L^2(Q_K)$, the sequence $\Phi(t_n, \tau - t_n, \theta_{2,-t_n}\omega, u_{0,n})$ is precompact in $L^2(Q_K)$, and hence it has a finite covering in $L^2(Q_K)$ of balls of radius less than $\frac{\epsilon}{2}$. So together with (5.1), we can obtain the conclution that $\Phi(t_n, \tau - t_n, \theta_{2,-t_n}\omega, u_{0,n})$ has a finite covering in $L^2(R^n)$ of balls of radius less than ϵ, we finish the proof. □

In the next section, we prove the opinion that the existence of attractors for the cocycle Φ of the modified Swift-Hohenberg equation.

Theorem 5.1. *According to the assumption of the front. Then the cocycle Φ associated with problem* (3.1)–(3.2) *has a unique $\mathcal{D}_\lambda$-pullback attractor $\mathscr{A} \in \mathcal{D}_\lambda$ in $L^2(R^n)$.*

Proof. Attention that Φ has a closed measurable $\mathcal{D}_\lambda$-pullback absorbing set by lemma 4.1, and the Φ is $\mathcal{D}_\lambda$-pullback asymptotically compact in $L^2(R^n)$ by lemma 5.1, hence allow from proposition 2.7 we can immediately obtain the existence of a unique $\mathcal{D}_\lambda$-pullback attractor $\mathscr{A} \in \mathcal{D}_\lambda$ in $L^2(R^n)$. □

Acknowledgements

We would like to express our thanks to the referee for helpful comments and suggestions. This work is supported by The National Natural Science Foundation of China(No. 11301043).

References

[1] A. Adili and B. Wang, *Random attractors for stochastic fitzhugh-nagumo systems driven by deterministic non-autonomous forcing*, Discrete Continuous Dynamical Systems-Series B, 3(2013), 643–666.

[2] M. Anguiano, P. Marín-Rubio and J. Real, *Pullback attractors for non-autonomous reaction-diffusion equations with dynamical boundary conditions*, J. Math. Anal., 383(2011), 608–618.

[3] P. W. Bates, K. Lu and B. Wang, *Random attractors for stochastic reaction-diffusion equations on unbounded domains,* J. Differential Eq., 246(2009), 845–869.

[4] Z. Brzezniak and Y. Li, *Asymptotic compactness and absorbing sets for 2d stochastic Navier-Stokes equations on some unbounded domains*, Trans. Amer. Math. Soc., 358(2006), 5587–5629.

[5] T. Caraballo, G.Łukaszewicz and J. Real, *Pullback attractors for asymptotically compact non-autonomous dynamical systems*, Nonlinear Analysis, 64(2006), 484–498.

[6] H. Crauel, A. Debussche and F. Flandoli, *Random attractors*, J. Dynam. Differential Equations, 9(1997), 307–341.

[7] Y. Li, Z. Brzeźniak and J. Zhou, *Conceptual analysis and random attractor for dissipative random dynamical systems*, Acta Mathematica scientia, 2(2008), 253–268.

[8] Y. Li and C. Zhong, *Pullback attractors for the norm-to-weak continuous process and application to the nonautonomous reaction-diffusion equations*, Applied Mathematics and Computation, 190(2007), 1020–1029.

[9] L. A. Peletier and V. Rottschäfer, *Large time behaviour of solutions of the Swift-Hohenberg equations*, C.R. Acad. Sci., 336(2003), 225–230.

[10] M. Polat, *Global attractor for a modified Swift-Hohenberg equation*, Comput. Math. Appl., 57(2009), 62–66.

[11] M. Scheutzow, *Comparison of various concepts of a random attractor: A case study*, Arch. Math.(Basel), 78(2002), 233–240.

[12] L. Song, Y. Zhang and T. Ma, *Global attractor for a modified Swift-Hohenberg equation in H^k spaces*, Nonlinear Anal., 72(2010), 183–191.

[13] H. P. Sun and Y. P. Jong, *Pullback attractor for a non-autonomous modified Swift-Hohenberg equation*, Computers and Mathematics with Applications, 67(2014), 542–548.

[14] J. B. Swift and P. C. Hohenberg, *Hydrodynamic fluctuations at the convective instability*. Phy. Rev. A, 15(1977), 319–328.

[15] B. Wang, *Random attractors for non-autonomous stochastic wave equations with multiplicative noise*, Discrete and continuous dynamical systems, 34(2013), 269–300.

[16] B. X. Wang, *Random attractors for the stochastic Benjamin-Bona-Mahony equation on unbounded domains*, J. Differential Eq., 246(2009), 2506–2537.

[17] Y. Wang, Y. Liu and Z. Wang, *Random attractors for the partly dissipative stochastic lattice dynamical system*, J. Diff. Equ. Appl., 8(2008), 799–817.

[18] J. Wang and Y. Wang, *Pullback attractors for reaction-diffusion delay equations on unbounded domains with non-autonomous deterministic and stochastic forcing terms*, Journal of Mathematical Physics, 8(2013), 1–26.

[19] Z. Wang and S. Zhou, *Random attractor for stochastic reactionCdiffusion equation with multiplicative noise on unbounded domains*, Journal of Mathematical Analysis and Applications, 1(2011), 160–172.

[20] B. Wang, *Existence and upper semicontinuity of attractors for stochastic equations with deterministic non-autonomous terms*, Stochastics and Dynamics, 4(2014), 1–31.

[21] B. Wang, *Pullback attractors for non-autonomous reaction-diffusion equations on R^n* , Frontiers of Mathematics in China, 4(2009), 563–583.

[22] B. Wang, *Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems*, J. Differential Equations, 253(2012), 1544–1583.