Journal of Applied Analysis and Computation
Volume 7, Number 1, February 2017, 291–308
Website:http://jaac-online.com/
DOI:10.11948/2017020

EXISTENCE RESULTS OF SECOND ORDER IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY AND FRACTIONAL DAMPING*

Shengli Xie

Abstract In this paper we prove the existence, uniqueness, regularity and continuous dependence of mild solutions for second order impulsive functional differential equations with infinite delay and fractional damping in Banach spaces. We generalize the existence theorem of integer order differential equations to the fractional order case. The results obtained here improve and generalize some known results.

Keywords Mild solutions, classical solutions, strong solution, continuous dependence, fixed point.

MSC(2010) 34G25, 34K37, 35L30, 35L15, 35L70.

1. Introduction

Consider the following second order impulsive functional differential equations Cauchy problem with infinite delay and fractional damping:

$$x''(t) - BD_t^{\alpha}x(t) = Ax(t) + f\big(t, x_t, D_t^{\gamma}x_t\big),\ t \in [0,b], t \neq t_i, \tag{1.1}$$

$$\begin{cases} \Delta x(t_i) = I_i(x_{t_i}, x'_{t_i}), \Delta x'(t_i) = J_i(x_{t_i}, x'_{t_i}),\ i = 1, 2, \dots, n, \\ x_0 = \varphi \in \mathcal{B},\ x'_0 = \psi \in \mathcal{B}, \end{cases} \tag{1.2}$$

where $D_t^{\alpha}, D_t^{\gamma}$ are the Caputo's fractional derivative operator of order $\alpha, \gamma \in (0,1)$, A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ on a Banach space $X, B : X \to X$ is a bounded linear operator. The history $x_t, x'_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$ and $x'_t(\theta) = x'(t+\theta)$ belong to some abstract phase space $\mathcal{B}$ defined axiomatically; $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = b$ are fixed numbers and the symbol $\triangle x(t_i)$ represent the jump of the function x at t_i, which is defined by $\triangle x(t_i) = x(t_i^+) - x(t_i^-)$ for $i = 1, 2, \dots, n$. $\triangle x'(t_i)$ has the same meaning.

Impulsive differential equations with delay and fractional derivatives have played an important role in describing dynamics of populations subject to abrupt changes as well as other phenomena such as harvesting, diseases, and so forth, which has been used for constructing many mathematical models in science and engineering. The theory of fractional differential equations has been extensively studied by many authors [3–5, 7, 15–17, 20–24, 27, 29, 30]. To obtain existence of mild solutions

Email address: slxie@ahjzu.edu.cn
School of Mathematics & Physics, Anhui University of Architecture, Ziyun Road, 230601 Hefei, China
*The author was supported by Natural Science Foundation of Education Office of Anhui Province (KJ2014A043, KJ2012A055), China.

in these papers, usually,the compactness condition on associated family of operators and pulse items, restrictive conditions on a priori estimation are used. For example, authors in [10] studied the existence of mild solutions for a damped impulsive system:

$$\begin{cases} x''(t) = Ax(t) + Bx'(t) + f(t, x(t), x(a(t)), x'(t), x'(b(t))),\ t \in [0, b], \\ \Delta x(t_i) = I_i(x(t_i), x'(t_i)),\ \Delta x'(t_i) = J_i(x(t_i), x'(t_i)),\ i = 1, 2, \dots, n, \\ x(0) = x_0,\ x'(0) = z, \end{cases} \tag{1.3}$$

the compactness conditions on associated family of operators and and pulse items, the restrictive conditions on a priori estimation

$$\mu = \sum_{i=1}^{n} \left[(N_1 + M)c_i^I + (M + N)c_i^J\right] < 1, \tag{1.4}$$

$$(M + N)\Big[\|B\|b + \liminf_{r\to\infty} \frac{W(4r)}{r} \int_0^b m(s)ds\Big] + \sum_{i=1}^{n} \left[(M + N_1)L_i + (M + N)K_i\right] < 1, \tag{1.5}$$

are used. Recently, authors [1] used the measure of noncompactness without the compactness assumption on associated family of operators, to obtain the existence of mild solutions for the following fractional order integro-differential system:

$$\begin{cases} D_t^q x(t) = Ax(t) + \displaystyle\int_0^t a(t, s)f(s, x_s, x(s))ds,\ t \in [0, b], \\ x(t) = \varphi(t),\ t \in (-\infty, 0], \end{cases}$$

the restrictive conditions on a priori estimation and measure of noncompactness estimation

$$\begin{aligned} &b^q aC_{q,M}\|\mu_2\|_{L^1(J,R^+)} < q, \\ &16a\eta^* < 1, \end{aligned} \tag{1.6}$$

are used in [1]. However, to the best of the author's knowledge, fractional derivatives are introduced here for such problems for the first time.

In this paper, using the Kuratowski measure of noncompactness and progressive estimation method, we prove the existence, uniqueness, regularity and continuous dependence of mild solutions for the problem (1.1)–(1.2). The compactness condition of pulse items, some restrictive conditions on a priori estimation and noncompactness measure estimation have been removed to obtain the existence and uniqueness of mild solutions. Our results improve and generalize the corresponding results in [2, 10]. Finally, an example of non-compact semigroups is given.

The paper is organized as follows. In section 2 we give some basic concepts and Lemmas. In section 3 we discuss the existence, uniqueness and regularity of mild solutions, in section 4 we discuss the continuous dependence of mild solutions. Our results are based on the properties of equicontinuous semigroups and the ideas and techniques in Xie [28].

2. Preliminaries

In this paper, X will be a Banach space with norm $\|\cdot\|$ and $A : D(A) \subset X \to X$ is the infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$ of bounded linear operators on X and $(S(t))_{t\in\mathbb{R}}$ is the sine function associated with $(C(t))_{t\in\mathbb{R}}$, which is defined by $S(t)x = \int_0^t C(s)xds, x \in X, t \in \mathbb{R}$. We designate by $N, \overline{N}$ certain constants such that $\|C(t)\| \le N$ and $\|S(t)\| \le \overline{N}$ for every $t \in J = [0, b]$. We refer the reader to [6] for the necessary concepts about cosine functions. As usual we denote by $[D(A)]$ the domain of A endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|, x \in D(A)$. Moreover, the notation E stands for the space formed by the vector $x \in X$ for which the function $C(\cdot)x$ is of class C^1. It was proved by Kisyński [18] that the space E endowed with the norm $\|x\|_E = \|x\| + \sup_{0\le t\le b} \|AS(t)x\|,\ x \in E$, is a Banach space. The operator valued function $G(t) = \left[\begin{smallmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{smallmatrix}\right]$ is a strongly continuous group of linear operators on the space $E \times X$ generated by the operator $\mathcal{A} = \left[\begin{smallmatrix} 0 & I \\ A & 0 \end{smallmatrix}\right]$ defined on $D(A) \times E$. It follows from this that $AS(t) : E \to X$ is a bounded linear operator and that $AS(t)x \to 0$ $(t \to 0)$ for each $x \in E$. Furthermore, if $x : [0,\infty) \to X$ is a locally integrable function, then $z(t) = \int_0^t S(t-s)x(s)ds$ defines an E-valued continuous function. This is a consequence of the fact that

$$\int_0^t G(t-s)[0, x(s)]^T ds = \Big[\int_0^t S(t-s)x(s)ds,\ \int_0^t C(t-s)x(s)ds\Big]^T,$$

defines an $(E \times X)$-valued continuous function. Next $N_1 =: \sup_{t\in J} \|AS(t)\|_{L(E,X)}$ in which $L(E, X)$ stands for the Banach space of bounded linear operators from E into X and we abbreviate this notation to $L(X)$ when $E = X$.

We say that a function $u : [\sigma, b] \to X$ is a normalized piecewise continuous function on $[\sigma, b]$ if u is piecewise continuous and left continuous on $(\sigma, b]$. We denote by $PC([\sigma, b], X)$ the space formed by the normalized piecewise continuous functions from $[\sigma, b]$ into X. In particular, we introduce the space PC formed by all functions $u : [0, b] \to X$ such that u is continuous at $t \ne t_i, u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists for all $i = 1, 2, \cdots, n$. It is clear that PC endowed with the norm $\|x\|_{pc} = \sup_{t\in J} \|x(t)\|$ is a Banach space. Similarly, We say that $x \in PC$ is piecewise smooth if x is continuously differentiable at $t \ne t_i, i = 1, 2, \cdots, n$, and for $t = t_i, i = 1, 2, \cdots, n$, there are the right derivative $x'(t_i^+) = \lim\limits_{s\to 0+} \dfrac{x(t_i + s) - x(t_i)}{s}$ and the left derivative $x'(t_i^-) = \lim\limits_{s\to 0-} \dfrac{x(t_i + s) - x(t_i)}{s}$. Furthermore, we denote the space by $PC^1 = \{x \in PC : x'(t)$ is continuous at $t \ne t_i, x'(t_i^-)$ and $x'(t_i^+)$ exist, $i = 1, 2, \cdots n\}$. Then PC^1 endowed with the norm $\|u\|_1 = \|u\|_{pc} + \|u'\|_{pc}$ is a Banach space. Next, for $u \in PC^1$ we represent by $u'(t)$ the left derivative at $t > 0$ and by $u'(0)$ the right derivative at zero.

Let $J_0 = [0, t_1], J_1 = (t_1, t_2], J_2 = (t_2, t_3], \cdots, J_n = (t_n, b],\ J_0 = \overline{J}_0, \overline{J}_1 = [t_1, t_2], \overline{J}_2 = [t_2, t_3], \cdots, \overline{J}_n = [t_n, b]$. For $x \in PC$, we denote by $\widetilde{x}_i, i = 0, 1, \ldots, n$, the function $\widetilde{x}_i \in C(\overline{J}_i, X)$ given by $\widetilde{x}_i(t) = x(t), t \in (t_i, t_{i+1}]$ and $\widetilde{x}_i(t_i) = x(t_i^+)$. Moreover, for $V \subset PC$ and $i = 0, 1, \ldots, n$, we use the notation $\widetilde{V}_i$ for $\widetilde{V}_i = \{\widetilde{x}_i : x \in V\}$. By Lemma 1.1 in [10], we know that a set $V \subseteq PC$ is relatively compact if and only if each set $\widetilde{V}_i = \{\widetilde{x}_i : x \in V\}$ is relatively compact in $C(\overline{J}_i, X)$ for every $i = 0, 1, \ldots, n$.

In this work we will employ an axiomatic definition of the phase space ß introduced by Hale and Kato [13] which appropriated to treat retarded impulsive differential equations.

Definition 2.1 ([13]). The phase space $\mathcal{B}$ is a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ and $\mathcal{B}$ satisfies the following axioms:

(A) If $x : (-\infty, \sigma+b] \to X$ $(b > 0)$ is such that $x_\sigma \in \mathcal{B}$ and $x|_{[\sigma,\sigma+b]} \in PC([\sigma, \sigma+b], X)$, then for every $t \in [\sigma, \sigma+b)$ the following conditions hold:

  (i) x_t is in $\mathcal{B}$,

  (ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$,

  (iii) $\|x_t\|_{\mathcal{B}} \leq K(t-\sigma)\sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$, where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, $K(\cdot)$ is continuous, $M(\cdot)$ is locally bounded and H, K, M are independent of $x(\cdot)$.

(B) The space $\mathcal{B}$ is complete.

Remark 2.1. If the function $x(\cdot)$ of (A) is continuous on $[\sigma, \sigma+b)$, x_t is a $\mathcal{B}$-valued continuous function on $[\sigma, \sigma+b)$. see [13].

Example 2.1. The phase space $PC_r(X) \times L^2(h, X)$. Let $h : (-\infty, -r] \to (0, \infty)$ be a positive function verifying the conditions (g_6) and (g_7) of [14]. This means that $h(\cdot)$ is Lebesgue integrable on $(-\infty, -r)$ and that there exists a non-negative and locally bounded function $\gamma : (-\infty, 0] \to [0, \infty)$ such that $h(\xi+\theta) \leq \gamma(\xi)h(\theta)$, for all $\xi \leq 0, \theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. Let the space $\mathcal{B} = PC_r \times L^2(h, X), (r \geq 0)$ be formed of all classes of functions $\psi : (-\infty, 0] \to X$ such that $\psi|_{[-r,0]} \in PC([-r, 0], X), \psi(\cdot)$ is Lebesgue measurable on $(-\infty, -r]$ and $h(\cdot)\|\psi(\cdot)\|^p$ is Lebesgue integrable on $(-\infty, -r]$. The seminorm $\|\cdot\|_{\mathcal{B}}$ is defined by

$$\|\psi\|_{\mathcal{B}} = \sup_{\theta\in[-r,0]} \|\psi(\theta)\| + \Big(\int_{-\infty}^{-r} h(s)\|\psi(s)\|^2 ds\Big)^{\frac{1}{2}}.$$

It follows from the proof of [14] (Th.1.3.8) that $\mathcal{B}$ is a phase space which verifies the axioms (A) and (B) of our work. Moreover, when $r = 0$ this space coincides with $C_0 \times L^2(h, X)$ and the parameters $H = 1, C(t) = \gamma(-t)^{\frac{1}{2}}$ and $K(t) = 1 + \big(\int_{-t}^{0} h(\theta)d\theta\big)^{\frac{1}{2}}, t \geq 0$.

In this paper, we denote by $\alpha(\cdot)$ and $\alpha_{pc}(\cdot)$ the Kuratowski measure of noncompactness of X and $PC(J, X)$. $K_b := \sup_{t\in J} K(t), M_b := \sup_{t\in J} M(t)$.

Lemma 2.1 ([9]). *(1) If $W \subset PC(J, X)$ is bounded, then $\alpha(W(t)) \leq \alpha_{pc}(W)$, $\alpha(W(t)) \in L^1(J, \mathbb{R}^+)$ and $\alpha\big(\int_0^t W(s)ds\big) \leq 2\int_0^t \alpha\big(W(s)\big)ds,\ t \in J$.*

(2) If $W \subset PC(J, X)$ is piecewise equicontinuous bounded set, then $\alpha(W(t))$ is piecewise continuous on J, $\alpha_{pc}(W) = \sup_{t\in J} \alpha(W(t))$ and $\alpha\big(\int_0^t W(s)ds\big) \leq \int_0^t \alpha(W(s))ds, t \in J$.

(3) If $W \subset PC^1(J, X)$ is bounded and the elements of W' are equicontinuous on each J_i $(i = 0, 1, \cdots, n)$, then $\alpha_{pc^1}(W) = \max\big\{\sup_{t\in J} \alpha(W(t)), \sup_{t\in J} \alpha(W'(t))\big\}$, where $\alpha_{pc^1}(\cdot)$ denotes the Kuratowski measure of noncompactness in space $PC^1(J, X)$.

Lemma 2.2 ([25, 26]). *Suppose that A is the infinitesimal generator of a strongly continuous cosine family $(C(t))_{t\in\mathbb{R}}$, $g : \mathbb{R} \to X$ is a continuously differentiable* *function and* $p(t) = \int_0^t S(t-s)g(s)ds$. *Then* $p(t) \in D(A), p'(t) = \int_0^t C(t-s)g(s)ds$ *and*

$$p''(t) = \int_0^t C(t-s)g'(s)ds + C(t)g(0) = Ap(t) + g(t).$$

Lemma 2.3 (Mónch, [19]). *Let* Ω *be a bounded open subset in a Banach space* X *and* $0 \in \Omega$. *Assume that the operator* $F : \overline{\Omega} \to X$ *is continuous and satisfies the following conditions:*

(1) $x \neq \lambda F x, \ \forall\ \lambda \in (0,1), x \in \partial\Omega$;

(2) V *is relatively compact if* $V \subset \overline{co}(\{0\} \cup F(V))$ *for any countable set* $V \subset \overline{\Omega}$. *Then* F *has a fixed point in* $\overline{\Omega}$.

3. Existence and uniqueness of mild solutions

Definition 3.1. A function $x : (-\infty, b] \to X$ is called a mild solution of the system (1.1)–(1.2) if $x(\cdot)|_J \in PC^1$, the condition (1.2) is satisfied and

$$x(t) = \begin{cases} \varphi(t), \ t \leq 0, \\ C(t)\varphi(0) + S(t)\psi(0) + \displaystyle\int_0^t S(t-s)\big[BI_0^{1-\alpha}x'(s) + f\big(s, x_s, I_0^{1-\gamma}x'_s\big)\big]ds \\ + \displaystyle\sum_{t_i<t} C(t-t_i)I_i(x_{t_i}, x'_{t_i}) + \sum_{t_i<t} S(t-t_i)J_i(x_{t_i}, x'_{t_i}), \ t \in J, \end{cases} \tag{3.1}$$

where $I_0^{1-\alpha}x'(s) = \int_0^s \frac{(s-r)^{-\alpha}}{\Gamma(1-\alpha)}x'(r)dr$ is the Riemann-Liouville fractional integration.

When $\varphi(0), I_i(x_{t_i}, x'_{t_i}) \in E$, $x(t)$ is continuously differentiable on J_i and

$$x'(t) = \begin{cases} \psi(t), \ t \leq 0, \\ AS(t)\varphi(0) + C(t)\psi(0) + \displaystyle\int_0^t C(t-s)\big[BI_0^{1-\alpha}x'(s) + f\big(s, x_s, I_0^{1-\gamma}x'_s\big)\big]ds \\ + \displaystyle\sum_{t_i<t} AS(t-t_i)I_i(x_{t_i}, x'_{t_i}) + \sum_{t_i<t} C(t-t_i)J_i(x_{t_i}, x'_{t_i}), \ t \in J. \end{cases} \tag{3.2}$$

We make the following hypotheses.

(H_1) $f : J \times \mathcal{B} \times X \to X$ satisfies the following conditions:

(1) For every $(u, v) \in \mathcal{B} \times X$, the function $f(t, u, v) : J \to X$ is strongly measurable and $f(t, \cdot, \cdot) : \mathcal{B} \times X \to X$ is continuous for a.e. $t \in J$;

(2) There is an integrable function $q(\cdot) : J \to \mathbb{R}^+$ such that

$$\|f(t, u, v)\| \leq q(t)(\|u\|_{\mathcal{B}} + \|v\|), \ t \in J, u \in \mathcal{B}, v \in X;$$

(3) For any bounded set $U, V \subset PC$, there is a constant $l > 0$ such that

$$\alpha\big(f(t, V_t, U(t))\big) \leq l[\alpha(V_t) + \alpha(U(t))], \ t \in J;$$

where $V_t = \{x_t : x \in V\} \subset \mathcal{B}$ $(t \in J)$.

(H_1') The function $f(\cdot)$ is continuous, $f(t,0,0)=0$ $(t\in J)$ and there is a positive constant L_f such that

$$\begin{aligned}&\|f(t,u_1,v_1)-f(t,u_2,v_2)\|\\ &\le L_f(\|u_1-u_2\|_{\mathcal{B}}+\|v_1-v_2\|),\ t\in J,(u_k,v_k)\in\mathcal{B}\times X\ (k=1,2).\end{aligned}$$

(H_2) $I_i\in C(\mathcal{B}\times\mathcal{B},E), J_i\in C(\mathcal{B}\times\mathcal{B},X)$, there are constants $c_i>0$ and $d_i>0$ such that, for each $i=1,2,\cdots,n$,

$$\|I_i(u,v)\|_E\le c_i(\|u\|_{\mathcal{B}}+\|v\|_{\mathcal{B}}),\ \|J_i(u,v)\|\le d_i(\|u\|_{\mathcal{B}}+\|v\|_{\mathcal{B}}).$$

(H_2') $I_i(0,0)=J_i(0,0)=0$ $(i=1,\cdots,n)$, there are constants $c_i>0$ and $d_i>0$ such that, for each $i=1,2,\cdots,n,(u_1,v_1),(u_2,v_2)\in\mathcal{B}\times\mathcal{B}$,

$$\begin{aligned}&\|I_i(u_1,v_1)-I_i(u_2,v_2)\|_E\le c_i(\|u_1-u_2\|_{\mathcal{B}}+\|v_1-v_2\|_{\mathcal{B}}),\\ &\|J_i(u_1,v_1)-J_i(u_2,v_2)\|\le d_i(\|u_1-u_2\|_{\mathcal{B}}+\|v_1-v_2\|_{\mathcal{B}}).\end{aligned}$$

Let $S(b)=\{x:(-\infty,b]\to X: x_0=0, x(\cdot)|_J\in PC\}$. Then $S(b)$ is a Banach space endowed with the norm $\|x\|_b=\|x_0\|_{\mathcal{B}}+\sup_{t\in J}\|x(t)\|=\sup_{t\in J}\|x(t)\|$. The space $S^1(b)=:\{x:(-\infty,b]\to X: x_0=0, x_0'=0, x(\cdot)|_J\in PC^1\}$ endowed with the norm $\|x\|_{PC^1}$.

Theorem 3.1. *Assume that* $\varphi(0), I_i(x_{t_i},x'_{t_i})\in E$ *and conditions* $(H_1),(H_2)$ *are satisfied. Then the system* (1.1)–(1.2) *has at least one mild solution.*

Proof. If $x(\cdot)$ satisfies the equations (3.1) and (3.2), we can decompose $x(\cdot)$ as $x(t)=y(t)+\overline{u}(t), x'(t)=y'(t)+\overline{v}(t)$, where

$$y(t)=\begin{cases}\varphi(t), & t\le 0,\\ C(t)\varphi(0)+S(t)\psi(0), & t\in J,\end{cases}\quad y'(t)=\begin{cases}\psi(t), & t\le 0,\\ AS(t)\varphi(0)+C(t)\psi(0), & t\in J,\end{cases}$$
$$\overline{u}(t)=\begin{cases}0, & t\le 0,\\ u(t), & t\in J,\end{cases}\qquad \overline{v}(t)=\begin{cases}0, & t\le 0,\\ v(t), & t\in J,\end{cases}\tag{3.3}$$

which implies that $y_0=\varphi, y_0'=\psi$ and $x_t=y_t+u_t, x_t'=y_t'+v_t, t\in J$. Clearly, $\|y(t)\|\le N\|\varphi(0)\|_E+\overline{N}\|\psi(0)\|=:L_1, \|y'(t)\|\le N_1\|\varphi(0)\|_E+N\|\psi(0)\|=:L'$, $\|y_t\|_{\mathcal{B}}\le K_b\|y\|_b+M_b\|\varphi\|_{\mathcal{B}}=:M_1, \|y_t'\|_{\mathcal{B}}\le K_b\|y'\|_b+M_b\|\psi\|_{\mathcal{B}}=:M'$ for $t\in J$, where $\|y\|_b=\sup_{t\in J}\|y(t)\|, \|y'\|_b=\sup_{t\in J}\|y'(t)\|$.

Let the map $T=(T_1,T_2):S(b)\times S(b)\to S(b)$ be defined by

$$T_1(u,v)(t)=\begin{cases}\displaystyle\int_0^t S(t-s)\big[BI_0^{1-\alpha}(v(s)+y'(s))+f\big(s,u_s+y_s,I_0^{1-\gamma}(v_s+y_s')\big)\big]ds\\ \displaystyle+\sum_{t_i<t}C(t-t_i)I_i(u_{t_i}+y_{t_i},v_{t_i}+y'_{t_i})\\ \displaystyle+\sum_{t_i<t}S(t-t_i)J_i(u_{t_i}+y_{t_i},v_{t_i}+y'_{t_i}),\ t\in J,\end{cases}\tag{3.4}$$

$$T_2(u,v)(t)=\begin{cases}\displaystyle\int_0^t C(t-s)\big[BI_0^{1-\alpha}(v(s)+y'(s))+f\big(s,u_s+y_s,I_0^{1-\gamma}(v_s+y_s')\big)\big]ds\\ \displaystyle+\sum_{t_i<t}AS(t-t_i)I_i(u_{t_i}+y_{t_i},v_{t_i}+y'_{t_i})\\ \displaystyle+\sum_{t_i<t}C(t-t_i)J_i(u_{t_i}+y_{t_i},v_{t_i}+y'_{t_i}),\ t\in J.\end{cases}\tag{3.5}$$

The product space $S(b)\times S(b)$ endows the norm $\|(u,v)\|_b=\|u\|_b+\|v\|_b$. Then T_1,T_2 are well defined with values in $S(b)$. In addition, from the axioms of phase space, the Lebesgue dominated convergence theorem,the condition (H_1) and (H_2), it is easy to show that $T=(T_1,T_2)$ is continuous. It is easy to see that if (u,v) is a fixed point of T, then $u+y$ is a mild solution of the system (1.1)–(1.2). First, we show that the set

$$\Omega_0=\{(u,v)\in S(b)\times S(b):(u,v)=\lambda T(u,v)\text{ for some }\lambda\in(0,1)\}$$

is bounded. If $(u,v)\in\Omega_0$, there exists a $\lambda\in(0,1)$ such that $(u,v)=\lambda(T_1(u,v),T_2(u,v))$.

When $t\in J_0=[0,t_1]$, it follows from (3.4), (3.5) and (H_1)(2) that

$$\begin{aligned}\|u(t)\|\le\|T_1(u,v)(t)\|&\le\overline{N}\|B\|\int_0^t I_0^{1-\alpha}\|v(s)+y'(s)\|ds\\&\quad+\overline{N}\int_0^t q(s)\big(\|u_s+y_s\|_{\mathcal{B}}+I_0^{1-\gamma}\|v_s+y_s'\|_{\mathcal{B}}\big)ds\\&\le\overline{N}\big[L'\Gamma\|B\|+(M_1+M'\Gamma)qb\big]\\&\quad+\overline{N}[H\Gamma\|B\|+K_b(\Gamma+q)]\int_0^t(\|u\|_s+\|v\|_s)ds,\end{aligned}\tag{3.6}$$

$$\begin{aligned}\|v(t)\|&\le N\big[L'\Gamma\|B\|+(M_1+M'\Gamma)qb\big]\\&\quad+N[H\Gamma\|B\|+K_b(\Gamma+q)]\int_0^t(\|u\|_s+\|v\|_s)ds,\end{aligned}\tag{3.7}$$

where $\Gamma=\max\{\frac{b^{1-\alpha}}{\Gamma(2-\alpha)},\frac{b^{1-\gamma}}{\Gamma(2-\gamma)}\},\Gamma(2-\alpha)$ is the Gamma function, $q=\sup_{t\in J}\|q(t)\|$, $\|u\|_s=\sup_{0\le r\le s}\|u(r)\|$, $\|v\|_s=\sup_{0\le r\le s}\|v(r)\|, s\in J_0$. Since $\|u\|_s$ and $\|v\|_s$ are continuous nondecreasing on J_0, (3.6) and (3.7) imply that

$$\begin{aligned}\|u\|_t+\|v\|_t&\le(N+\overline{N})[L'\Gamma\|B\|+(M_1+M'\Gamma)qb\big]\\&\quad+(N+\overline{N})[H\Gamma\|B\|+K_b(\Gamma+q)]\int_0^t(\|u\|_s+\|v\|_s)ds.\end{aligned}\tag{3.8}$$

By the Gronwall Lemma and (3.8), there is a constant $G_0>0$ independent of u,v and $\lambda\in(0,1)$, such that $\|u\|_t+\|v\|_t\le G_0,t\in J_0$. Hence $\|u_t\|_{\mathcal{B}}\le K_bG_0,\|v_t\|_{\mathcal{B}}\le K_bG_0,t\in J_0$. It follows from this and the condition (H_2) that

$$\begin{aligned}&\|I_1(u_{t_1}+y_{t_1},v_{t_1}+y'_{t_1})\|_E\le 2c_1(K_bG_0+M_1+M')=:\delta_1,\\&\|J_1(u_{t_1}+y_{t_1},v_{t_1}+y'_{t_1})\|\le 2d_1(K_bG_0+M_1+M')=:\eta_1,\\&\|u(t_1^+)\|=\|u(t_1)+I_1(u_{t_1}+y_{t_1},v_{t_1}+y'_{t_1})\|\le G_0+\delta_1,\\&\|v(t_1^+)\|=\|v(t_1)+J_1(u_{t_1}+y_{t_1},v_{t_1}+y'_{t_1})\|\le G_0+\eta_1.\end{aligned}$$

When $t \in J_1$, $\widetilde{u}_1, \widetilde{v}_1 \in C([\overline{J}_1, , X)$. It is similar to (3.6) and (3.7), we get

$$\begin{aligned}\|\widetilde{u}_1(t)\| &\le N\delta_1 + \overline{N}\eta_1 + \overline{N}\|B\| \int_0^t I_0^{1-\alpha}\|v(s) + y(s)\|ds \\ &\quad + \overline{N}q \int_0^t \big(\|u_s + y_s\|_{\mathcal{B}} + I_0^{1-\gamma}\|v_s + y'_s\|_{\mathcal{B}}\big)ds \\ &\le N\delta_1 + \overline{N}\eta_1 + G_0 + Q_1 \\ &\quad + \overline{N}[H\Gamma\|B\| + qK_b(\Gamma+1)] \int_{t_1}^t (\|\widetilde{u}_1\|_s + \|\widetilde{v}_1\|_s)ds,\end{aligned} \tag{3.9}$$

$$\begin{aligned}\|\widetilde{v}_1(t)\| &\le N_1\delta_1 + N\eta_1 + G_0 + Q'_1 \\ &\quad + N[H\Gamma\|B\| + qK_b(\Gamma+1)] \int_{t_1}^t (\|\widetilde{u}_1\|_s + \|\widetilde{v}_1\|_s)ds,\end{aligned} \tag{3.10}$$

where $\|\widetilde{u}_1\|_s = \sup_{t_1 \le r \le s}\|\widetilde{u}_1(r)\|, \|\widetilde{v}_1\|_s = \sup_{t_1 \le r \le s}\|\widetilde{v}_1(r)\|$, $Q_1 > 0$ and $Q'_1 > 0$ are fixed constants. We have by (3.9) and (3.10),

$$\begin{aligned}\|\widetilde{u}_1\|_t + \|\widetilde{v}_1\|_t &\le (N + \overline{N})[H\Gamma\|B\| + qK_b(\Gamma+1)] \int_{t_1}^t (\|\widetilde{u}_1\|_s + \|\widetilde{v}_1\|_s)ds \\ &\quad + (N + N_1)\delta_1 + (N + \overline{N})\eta_1 + 2G_0 + Q_1 + Q'_1,\ t \in \overline{J}_1.\end{aligned} \tag{3.11}$$

Using the Gronwall Lemma once again and (3.11), there exists a constant $G_1 > 0$ such that $\|\widetilde{u}_1\|_t + \|\widetilde{v}_1\|_t \le G_1, t \in [t_1, t_2]$, so $\|u\|_t + \|v\|_t \le G_1$ and $\|u_t\|_{\mathcal{B}} \le K_b(G_0 + G_1), \|v_t\|_{\mathcal{B}} \le K_b(G_0 + G_1), t \in J_1$.

Similarly, there is a constant $G_i > 0$ such that $\sup\limits_{t_i \le s \le t}\|\widetilde{u}_i(s)\| + \sup\limits_{t_i \le s \le t}\|\widetilde{v}_i(s)\| \le G_i, t \in \overline{J}_i$ $(i = 2, 3, \cdots, n)$. Let $G \ge G_0 + G_1 + \cdots + G_n$, then $\|(u, v)\|_b \le G$ and Ω_0 is bounded.

Second, we verify that the conditions of Lemma 2.3 are satisfied. Let $R > G$ and

$$\Omega_R = \{(u, v) \in S(b) \times S(b) : \|(u, v)\|_b < R\}.$$

Then Ω_R is a bounded open set and $(0, 0) \in \Omega_R$. Since $R > G$, we know that $(u, v) \ne \lambda T(u, v)$ for any $(u, v) \in \partial\Omega_R$ and $\lambda \in (0, 1)$.

Suppose that $V \subset \overline{\Omega}_R$ is a countable set and $V \subset \overline{co}(\{(0, 0)\} \cup T(V))$. Let

$$V_1 = \{x \in S(b) : \exists\ z \in S(b), (x, z) \in V\},\ V_2 = \{z \in S(b) : \exists\ x \in S(b), (x, z) \in V\}.$$

Then we have

$$\begin{aligned}V &\subset V_1 \times V_2 \subset \overline{co}(\{0\} \cup T_1(V)) \times \overline{co}(\{0\} \cup T_2(V)) \\ &\subset \overline{co}(\{0\} \cup T_1(V_1 \times V_2)) \times \overline{co}(\{0\} \cup T_2(V_1 \times V_2)).\end{aligned} \tag{3.12}$$

Since $C(t)$ and $S(t)$ are strongly continuous, it follows from (3.4) and (3.5) that $T_k\big((\widetilde{V}_1)_i \times (\widetilde{V}_2)_i\big)$ is equicontinuous on every $\overline{J}_i = [t_i, t_{i+1}]$, which together with (3.12) implies that $(\widetilde{V}_k)_i$ is equicontinuous on every $\overline{J}_i$ $(k = 1, 2, i = 0, 1, \cdots, n)$.

In the following, we verify that the sets V_1, V_2 are relatively compact in PC. Without loss of generality, we do not distinguish $V_k|_{J_i}$ and $(\widetilde{V}_k)_i$, where $V_k|_{J_i}$ is the restriction of V_k on $J_i = (t_i, t_{i+1}]$ $(k = 1, 2, i = 1, 2, \cdots, n)$.

When $t \in J_0$, by properties of noncompactness measure, Lemma 2.1 and $(H_1)(3)$,

$$\begin{aligned}\alpha(V_1(t)) &\le \alpha(T_1(V_1\times V_2)(t))\\ &\le 2\overline{N}\textstyle\int_0^t \Gamma\|B\|\alpha(V_2(s))ds + 2l\overline{N}\int_0^t\big[\alpha(V_{1s}+\Gamma\alpha(V_{2s})\big]ds\\ &\le 2\overline{N}K_b\big[H\Gamma\|B\|+l(1+\Gamma)\big]\int_0^t\Big(\sup_{0\le\tau\le s}\alpha(V_1(\tau))+\sup_{0\le\tau\le s}\alpha(V_2(\tau))\Big)ds,\end{aligned}\tag{3.13}$$

$$\begin{aligned}\alpha(V_2(t)) &\le \alpha(\Gamma_2(V_1\times V_2)(t))\\ &\le 2NK_b\big[H\Gamma\|B\|+l(1+\Gamma)\big]\int_0^t\Big(\sup_{0\le\tau\le s}\alpha(V_1(\tau))+\sup_{0\le\tau\le s}\alpha(V_2(\tau))\Big)ds.\end{aligned}\tag{3.14}$$

Since $\alpha(V_k(t))$ $(k=1,2)$ is continuous nondecreasing on J_0, (3.13) and (3.14) imply that

$$\begin{aligned}&\sup_{0\le s\le t}\alpha(V_1(s))+\sup_{0\le s\le t}\alpha(V_2(s))\\ &\le 2(N+\overline{N})\big[H\Gamma\|B\|+l(1+\Gamma)\big]\int_0^t\Big(\sup_{0\le\tau\le s}\alpha(V_1(\tau))+\sup_{0\le\tau\le s}\alpha(V_2(\tau))\Big)ds.\end{aligned}\tag{3.15}$$

From (3.15) we know that $\alpha(V_k(t))=0$ $(k=1,2), t\in J_0$. Thence V_1 and V_2 are relatively compact in $C(J_0,X)$. Since

$$\begin{aligned}\alpha(V_{1t_1}+y_{t_1}) &\le \alpha(V_{1t_1}) \le K_b\sup\nolimits_{s\in J_0}\alpha(V_1(s))=0,\\ \alpha(V_{2t_1}+y'_{t_1}) &\le \alpha(V_{2t_1}) \le K_b\sup\nolimits_{s\in J_0}\alpha(V_2(s))=0,\end{aligned}$$

$I_1(\cdot,\cdot)$ and $J_1(\cdot,\cdot)$ are continuous,

$$\alpha\big(I_1(V_{1t_1}+y_{t_1},V_{2t_1}+y'_{t_1})\big)=\alpha\big(J_1(V_{1t_1}+y_{t_1},V_{2t_1}+y'_{t_1})\big)=0.\tag{3.16}$$

When $t\in\overline{J}_1$, we have by (3.4), (3.5) and (3.16),

$$\begin{aligned}&\alpha(V_1(t))\\ &\le 2\overline{N}K_b\big[H\Gamma\|B\|+l(1+\Gamma)\big]\int_{t_1}^t\Big(\sup_{t_1\le\tau\le s}\alpha(V_1(\tau))+\sup_{t_1\le\tau\le s}\alpha(V_2(\tau))\Big)ds,\\ &\alpha(V_2(t))\\ &\le 2NK_b\big[H\Gamma\|B\|+l(1+\Gamma)\big]\int_{t_1}^t\Big(\sup_{t_1\le\tau\le s}\alpha(V_1(\tau))+\sup_{t_1\le\tau\le s}\alpha(V_2(\tau))\Big)ds.\end{aligned}\tag{3.17}$$

(3.17) implies that

$$\begin{aligned}&\sup_{t_1\le s\le t}\alpha(V_1(s))+\sup_{t_1\le s\le t}\alpha(V_2(s))\\ &\le 2(N+\overline{N})K_b\big[H\Gamma\|B\|+l(1+\Gamma)\big]\int_{t_1}^t\Big(\sup_{t_1\le\tau\le s}\alpha(V_1(\tau))+\sup_{t_1\le\tau\le s}\alpha(V_2(\tau))\Big)ds.\end{aligned}$$

Consequently, $\alpha(V_k(t))=0, t\in\overline{J}_1$ $(k=1,2)$ and V_k is relatively compact in $C(\overline{J}_1,X)$.

Similarly, we can show that V_k is relatively compact in $C(\overline{J}_i,X)$ $(k=1,2,i=2,3,\cdots,n)$. So V_k is relatively compact in $S(b)$. Lemma 2.3 concludes that T has a fixed point (u,v) in $\overline{\Omega}_R$. Then the system (1.1)–(1.2) has a mild solution. □

Theorem 3.2. *Assume that* $\varphi(0), I_i(x_{t_i}, x'_{t_i}) \in E$ *and conditions* $(H_1'), (H_2)$ *are satisfied. Then the system* (1.1)–(1.2) *has a unique mild solution.*

Proof. We have by (H_1'),

$$\|f(t,u,v)\| \le L_f(\|u\|_{\mathcal{B}} + \|v\|),\ \ t \in J, u \in \mathcal{B}, v \in X,$$
$$\alpha(f(t,V_t,U(t))) \le L_f[\alpha(V_t) + \alpha(U(t))],\ \ t \in J,\ \text{bounded } U, V \subset PC(J,X).$$

Then the conditions (H_1) and (H_2) of Theorem 3.1 are satisfied. According to Theorem 3.1, the system (1.1)–(1.2) has at least one mild solution.

We now prove the uniqueness. Let $(u,v),(w,z) \in \overline{\Omega}_R$ be two fixed points of the operator T defined by (3.3) and (3.4). When $t \in J_0$, we have

$$\begin{aligned}
&\|u(t)-w(t)\| = \|T_1(u,v)(t) - T_1(w,z)(t)\| \\
&\le \overline{N}\|B\|\Gamma \int_0^t \|v(s)-z(s)\|ds + \overline{N}L_f\int_0^t (\|u_s - w_s\|_{\mathcal{B}} + \Gamma\|v_s - z_s\|_{\mathcal{B}})ds \\
&\le \overline{N}K_b(1+\Gamma)(H\|B\| + L_f)\int_0^t (\|u-w\|_s + \|v-z\|_s)ds,
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
&\|v(t)-z(t)\| = \|T_2(u,v)(t) - T_2(w,z)(t)\| \\
&\le NK_b(1+\Gamma)(H\|B\| + L_f)\int_0^t (\|u-w\|_s + \|v-z\|_s)ds.
\end{aligned} \tag{3.19}$$

(3.18) and (3.19) imply that

$$\|u-w\|_t + \|v-z\|_t \le (N+\overline{N})K_b(1+\Gamma)(H\|B\| + L_f)\int_0^t (\|u-w\|_s + \|v-z\|_s)ds.$$

Consequently, $u(t) = w(t), v(t) = z(t), t \in J_0$, and so $u_{t_1} = w_{t_1}, v_{t_1} = z_{t_1}$.

When $t \in \overline{J}_1 = [t_1, t_2]$, it is easy to get

$$\begin{aligned}
&\|\widetilde{u}_1(t) - \widetilde{w}_1(t)\| = \|T_1(\widetilde{u}_1, \widetilde{v}_1)(t) - T_1(\widetilde{w}_1, \widetilde{z}_1)(t)\| \\
&\le N\|I_1(u_{t_1} + y_{t_1}, v_{t_1} + y'_{t_1)}) - I_1(w_{t_1} + y_{t_1}, z_{t_1} + y'_{t_1)})\|_E \\
&\quad + \overline{N}\big[\|J_1(u_{t_1} + y_{t_1}, v_{t_1} + y'_{t_1)}) - J_1(w_{t_1} + y_{t_1}, z_{t_1} + y'_{t_1)})\| \\
&\quad + \overline{N}K_b(1+\Gamma)(H\|B\| + L_f)\int_0^t (\|u-w\|_s + \|v-z\|_s)ds \\
&= \overline{N}K_b(1+\Gamma)(H\|B\| + L_f)\int_{t_1}^t (\|\widetilde{u}_1 - \widetilde{w}_1\|_s + \|\widetilde{v}_1 - \widetilde{z}_1\|_s)ds, \\
&\|\widetilde{v}_1(t) - \widetilde{z}_1(t)\| = \|T_2(\widetilde{u}_1, \widetilde{v}_1)(t) - T_2(\widetilde{w}_1, \widetilde{z}_1)(t)\| \\
&\le NK_b(1+\Gamma)(H\|B\| + L_f)\int_{t_1}^t (\|\widetilde{u}_1 - \widetilde{w}_1\|_s + \|\widetilde{v}_1 - \widetilde{z}_1\|_s)ds, \\
&\|\widetilde{u}_1 - \widetilde{w}_1\|_t + \|\widetilde{v}_1 - \widetilde{z}_1\|_t \\
&\le (N+\overline{N})K_b(1+\Gamma)(H\|B\| + L_f)\int_{t_1}^t (\|\widetilde{u}_1 - \widetilde{w}_1\|_s + \|\widetilde{v}_1 - \widetilde{z}_1\|_s)ds.
\end{aligned}$$

Consequently, $u(t) = w(t), v(t) = z(t), t \in J_1$ and $u_{t_2} = w_{t_2}, v_{t_2} = z_{t_2}$.

Similarly, we can prove that $u(t) = w(t), v(t) = z(t), t \in J_i$ $(i = 2, 3, \cdots, m)$. Hence $u(t) = w(t), v(t) = z(t), t \in J$. □

It is similar to the proof of Theorem 3.2, we can obtain the following theorem.

Theorem 3.3. *Assume that $\varphi(0), I_i(x_{t_i}, x'_{t_i}) \in E$ and conditions $(H'_1), (H'_2)$ are satisfied. Then the system* (1.1)–(1.2) *has a unique mild solution.*

In the next result, for $x \in X, \mathcal{X}_x : (-\infty, 0] \to X$ represents the function defined by $\mathcal{X}_x(\theta) = 0$ for $\theta < 0$ and $\mathcal{X}_x(0) = x$.

Theorem 3.4. *Assume that the conditions of Theorem 3.2 are satisfied and $u(\cdot)$ is a mild solution of the system* (1.1)–(1.2)*. Then, for $i = 1, 2, \cdots, n$, $\widetilde{u}_i(\cdot)$ is a mild solution of the following abstract Cauchy problem*

$$\begin{cases} x''(t) = Ax(t) + BD_t^{\alpha}\widetilde{u}_i(t) + f\big(t, (\widetilde{u}_i)_t, D_t^{\gamma}(\widetilde{u}_i)_t\big), \ t \in \overline{J}_i, \\ x_{t_i} = u_{t_i} + \mathcal{X}_{I_i(u_{t_i}, u'_{t_i})} \in \mathcal{B}, \ x'_{t_i} = u'_{t_i} + \mathcal{X}_{J_i(u_{t_i}, u'_{t_i})} \in \mathcal{B}. \end{cases} \tag{3.20}$$

Proof. *For $i = 1, 2, \cdots, n$, the mild solution of the problem (3.20) can be expressed as*

$$\begin{aligned} \widetilde{u}_i(t) &= C(t - t_i)[u(t_i) + I_i(u_{t_i}, u'_{t_i})] + S(t - t_i)[u'_{t_i} + J_i(u_{t_i}, u'_{t_i})] \\ &\quad + \int_{t_i}^{t} S(t - s)\big[BI_0^{1-\alpha}(\widetilde{u}_i)'(s) + f\big(s, x_s, I_0^{1-\gamma}(\widetilde{u}_i)'_s\big)\big]ds, \ t \in \overline{J}_i. \end{aligned} \tag{3.21}$$

From (3.2) *and* (3.21) *it is easily to verify that $\widetilde{u}_i(\cdot) \in C^1(\overline{J}_i, X)$ is the mild solution of the problem* (3.20) *for $i = 1, 2, \cdots, n$. (refer Corollary 2.2 [10]).* □

In the following, we study the regularity of mild solutions for the system (1.1)–(1.2).

Definition 3.2. A function $u : (-\infty, b] \to X$ is called a classical solution of the system (1.1)–(1.2) if $u \in PC^1, u(\cdot)$ is a function of class C^2 on $J \setminus \{t_1, t_2, \cdots, t_n\}$ and that verifies the equation (1.1) on $J \setminus \{t_1, t_2, \cdots, t_n\}$ and the condition (1.2).

Definition 3.3. A function $u : (-\infty, b] \to X$ is called a strong solution of the system (1.1)–(1.2) if $u \in PC^1, \widetilde{u}_i \in W^{2,1}([t_i, t_{i+1}], X)$ for every $i = 0, 1, \cdots, n$, the equation (1.1) is satisfied a.e. on J and the condition (1.2) is satisfied.

A Banach space X has the Radon-Nikodym property if for each λ-continuous vector measure $\mu : \Sigma \to X$ of bounded variation there exists $h \in L^1(\mu, X)$ such that $\mu(B) = \int_B hd\lambda$ for all $B \in \Sigma$.

Theorem 3.5. *Assume that the conditions of Theorem 3.2 are satisfied, the space X has Radon-Nikodym property and $Bx \in L^1(J, E)$. If $u(\cdot)$ is a mild solution of the system* (1.1)–(1.2) *and the following conditions are satisfied:*

(a) $\varphi(0), u_{t_i} + I_i(u_{t_i}, u'_{t_i}) \in D(A)$ and $\psi(0), u'_{t_i} + J_i(u_{t_i}, u'_{t_i}) \in E$ for every $i = 1, 2, \cdots, n$;

(b) The function $f : J \times \Omega \to X$ (Ω is a open set in $\mathcal{B} \times X$) is continuous and satisfies the Carathédodory conditions on E:

(i) $f(\cdot, u, v)$ is measurable on J for each $(u, v) \in \Omega$,

(ii) $f(t, \cdot, \cdot)$ is continuous on Ω for $t \in J$ a.e.,

(iii) For each $R > 0$, there is a integrable function $\beta_R : J \to \mathbb{R}^+$ such that $\|f(t, u, v)\|_E \le \beta_R(t)$ a.e. for $t \in J$ and for all $(u, v) \in \Omega$ such that $\|u\|_{\mathcal{B}} + \|v\| \le R$. Then $u(\cdot)$ is a classical solution of the system (1.1)–(1.2).

Proof. Theorem 3.4 implies that $\widetilde{u}_i(\cdot)$ is the mild solution of the system (3.20) for each $i = 1, 2, \cdots, n$. Then $\widetilde{u}_i, \widetilde{u}'_i \in C(\overline{J}_i, X)$, and so $(\widetilde{u}_i)_t, (\widetilde{u}_i)'_t$ are continuous on $\overline{J}_i$.

Since $\widetilde{u}_i(\cdot)$ is the mild solution of the following Cauchy problem

$$\begin{cases} x''(t) = Ax(t) + BD_t^\alpha \widetilde{u}_i(t) + f\big(t, (\widetilde{u}_i)_t, D_t^\gamma(\widetilde{u}_i)_t\big), \ t \in \overline{J}_i, \\ x(t_i) = u_{t_i} + I_i(u_{t_i}, u'_{t_i}), x'(t_i) = u'_{t_i} + J_i(u_{t_i}, u'_{t_i}), \ i = 1, 2, \ldots, n, \end{cases}$$

the function $h_i(t) =: BD_t^\alpha \widetilde{u}_i(t) + f\big(t, (\widetilde{u}_i)_t, D_t^\gamma(\widetilde{u}_i)_t\big) \in C(\overline{J}_i, X) \cap L^1(\overline{J}_i, E)$. Similarly, we have $h_0(t) =: BD_t^\alpha u(t) + f(t, u_t, D_t^\gamma u_t) \in C([0, t_1], X) \cap L^1([0, t_1], E)$ for the following Cauchy problem

$$\begin{cases} x''(t) = Ax(t) + BD_t^\alpha u(t) + f\big(t, u_t, D_t^\gamma u_t\big), \ t \in J_0, \\ x(0) = \varphi(0), x'(0) = \psi(0). \end{cases} \tag{3.22}$$

It follows from Theorem 3.1 [11] that $u(\cdot)$ is a classical solution of the system (1.1)–(1.2). □

Theorem 3.6. *Assume that the conditions of Theorem 3.2 are satisfied and $u(\cdot)$ is a mild solution of the system (1.1)–(1.2). If the space X has Radon-Nikodym property and the following conditions hold:*

(a) $\varphi(0), u(t_i) + I_i(u_{t_i}, u'_{t_i}) \in D(A)$ and $\psi(0), u'(t_i) + J_i(u_{t_i}, u'_{t_i}) \in E$ for every $i = 1, 2, \cdots, n$. (b) For every bounded set $D_1 \subseteq \mathcal{B}, D_2 \subseteq X, C(\cdot)f(t, x, y), t \in J, (x, y) \in D_1 \times D_2$, is uniformly Lipschitz continuous.

Then $u(t)$ is a strong solution of the system (1.1)–(1.2).

Proof. For any $t \in (t_i, t_{i+1})$ and $\varepsilon > 0$ such that $t + \varepsilon \in (t_i, t_{i+1})$, we get from (3.21) that

$$\begin{aligned} &\|\widetilde{u}_i{}'(t+\varepsilon) - \widetilde{u}_i{}'(t)\| \\ \le &\|A[S(t+\varepsilon-t_i) - S(t-t_i)][u(t_i) + I_i(u_{t_i}, u'_{t_i})]\| \\ &+ \|[C(t+\varepsilon-t_i) - C(t-t_i)][u'(t_i) + J_i(u_{t_i}, u'_{t_i})]\| \\ &+ \int_{t_i}^{t} \|C(t-s)\| \|BI_0^{1-\alpha}[\widetilde{u}_i{}'(s+\varepsilon) - \widetilde{u}_i{}'(s)]\| ds \\ &+ \int_{t_i}^{\varepsilon+t_i} \|C(t+\varepsilon-s)\| \|BI_0^{1-\alpha}\widetilde{u}_i{}'(s))\| ds \\ &+ \int_{t_i}^{t} \|[C(t+\varepsilon-s) - C(t-s)]f(s, \widetilde{u}_i(s), I_0^{1-\gamma}(\widetilde{u}_i)'_s)\| ds \\ &+ \int_{t}^{\varepsilon+t} \|C(t+\varepsilon-s)f(s, \widetilde{u}_i(s), I_0^{1-\gamma}(\widetilde{u}_i)'_s)\| ds \\ \le &C_1\varepsilon + N\|B\|\Gamma \int_{t_i}^{t} \|\widetilde{u}_i{}'(s+\varepsilon) - \widetilde{u}_i{}'(s))\| ds, \end{aligned}$$

where $C_1 > 0$ is a constant independent of t and ε, and the fact that $AS(t - t_i)[u(t_i) + I_i(u_{t_i}, u'_{t_i})], C(t-t_i)[u'(t_i) + J_i(u_{t_i}, u'_{t_i})]$ are Lipschitz continuous on $\overline{J}_i$ has been used (see [12]). Therefore, from the Gronwall Lemma we know that $\widetilde{u}_i{}'(t)$ is Lipschitz continuous on $\overline{J}_i$ for each $i = 1, 2, \cdots, n$. Since X has Radon-Nikodym property, it follows from Proposition 3.3 [11] that $\widetilde{u}_i(t)$ is a strong solution of the system (1.1)–(1.2) for each $i = 1, 2, \cdots, n$. A similar argument permit us to prove that $\widetilde{u}_0(t)$ is a strong solution of the system (1.1)–(1.2). The proof is completed. □

4. Continuous dependence of mild solutions

Theorem 4.1. *Let the conditions* $(H_1'), (H_2')$ *be satisfied and* $I_i(x_{t_i}, x'_{t_i}) \in E$. *Then for each* $(\varphi, \psi), (\varphi^*, \psi^*) \in \mathcal{B} \times \mathcal{B}, \varphi(0), \varphi^*(0) \in E$, $x(t) = y(t) + \overline{u}(t), x^*(t) = y^*(t) + \overline{w}(t)$ *are the corresponding mild solution of the system* (1.1)–(1.2), *the following inequalities hold:*

$$\|(u, v) - (w, z)\|_{J_i} \leq (\rho_i + \eta_i) e^{(N+\overline{N})\sigma b} (\|\varphi - \varphi^*\|_{\mathcal{B}} + \|\psi - \psi^*\|_{\mathcal{B}}), \tag{4.1}$$

for $i = 0, 1, \cdots, n$, *where* $\sigma = K_b\big[(1 + H\|B\|)\Gamma + L_f\big] > 0$, η_i, ρ_i $(i = 0, 1, \cdots, n)$ *stand different positive constants.*

Proof. Let $x'(t) = y'(t) + v(t), (x^*)'(t) = (y^*)'(t) + z(t), t \in J$. When $t \in J_0$, by (3.3), Definition 2.1 and simple calculation, there are constants $\rho_0 > 0$ and $\eta_0 > 0$ such that

$$\begin{aligned}
\|u(t) - w(t)\| \leq \overline{N}\|B\| &\int_0^t I_0^{1-\alpha}\big(\|v(s) - z(s)\| + \|y'(s) - (y^*)'(s)\|\big) ds \\
&+ \overline{N} L_f \int_0^t \big[\|u_s - w_s\|_{\mathcal{B}} + \|y_s - y_s^*\|_{\mathcal{B}} \\
&+ I_0^{1-\gamma}(\|v_s - z_s\|_{\mathcal{B}} + \|y_s' - (y^*)_s'\|_{\mathcal{B}})\big] ds \\
\leq{}& \rho_0(\|\varphi - \varphi^*\|_{\mathcal{B}} + \|\psi - \psi^*\|_{\mathcal{B}}) \\
&+ \overline{N}\sigma \int_0^t (\|u - w\|_s + \|v - z\|_s) ds,
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
\|v(t) - z(t)\| \leq{}& \eta_0(\|\varphi - \varphi^*\|_{\mathcal{B}} + \|\psi - \psi^*\|_{\mathcal{B}}) \\
&+ N\sigma \int_0^t (\|u - w\|_s + \|v - z\|_s) ds.
\end{aligned} \tag{4.3}$$

(4.2) and (4.3) imply that

$$\begin{aligned}
\|u - w\|_t + \|v - z\|_t \leq{}& (\rho_0 + \eta_0)(\|\varphi - \varphi^*\|_{\mathcal{B}} + \|\psi - \psi^*\|_{\mathcal{B}}) \\
&+ (N + \overline{N})\sigma \int_0^t (\|u - w\|_s + \|v - z\|_s) ds.
\end{aligned}$$

Consequently,

$$\|(u, v) - (w, z)\|_{J_0} \leq (\rho_0 + \eta_0) e^{(N+\overline{N})\sigma b} (\|\varphi - \varphi^*\|_{\mathcal{B}} + \|\psi - \psi^*\|_{\mathcal{B}}).$$

When $t \in J_1 = (t_1, t_2]$, we have by (H_2'),

$$\begin{aligned}
&\|I_1(u_{t_1} + y_{t_1}, v_{t_1} + (y_{t_1})') - I_1(w_{t_1} + y_{t_1}^*, z_{t_1} + (y_{t_1}^*)')\|_E \\
&\leq c_1\big[K_b(\|u - w\|_{t_1} + \|v - z\|_{t_1}) + \|y_{t_1} - y_{t_1}^*\|_{\mathcal{B}} + \|(y_{t_1})' - (y_{t_1}^*)'\|_{\mathcal{B}}\big] \\
&\leq e_1(\|\varphi - \varphi^*\|_{\mathcal{B}} + \|\psi - \psi^*\|_{\mathcal{B}}), \\
&\|J_1(u_{t_1} + y_{t_1}, v_{t_1} + (y_{t_1})') - J_1(w_{t_1} + y_{t_1}^*, z_{t_1} + (y_{t_1}^*)')\| \\
&\leq \gamma_1(\|\varphi - \varphi^*\|_{\mathcal{B}} + \|\psi - \psi^*\|_{\mathcal{B}}),
\end{aligned}$$

where e_1, γ_1 are positive constants. It is similar to (4.2) and (4.3), there are constants $\rho_1 > 0$ and $\eta_1 > 0$ such that

$$\begin{aligned}
&\|\widetilde{u}_1 - \widetilde{w}_1\|_t + \|\widetilde{v}_1 - \widetilde{z}_1\|_t \\
&\leq (\rho_1 + \eta_1)(\|\varphi - \varphi^*\|_{\mathcal{B}} + \|\psi - \psi^*\|_{\mathcal{B}}) \\
&\quad + (N + \overline{N})\sigma \int_{t_1}^{t} (\|\widetilde{u}_1 - \widetilde{w}_1\|_s + \|\widetilde{v}_1 - \widetilde{z}_1\|_s) ds.
\end{aligned} \tag{4.4}$$

Consequently,

$$\|(u,v) - (w,z)\|_{J_1} \leq (\rho_1 + \eta_1) e^{(N+\overline{N})\sigma b} (\|\varphi - \varphi^*\|_{\mathcal{B}} + \|\psi - \psi^*\|_{\mathcal{B}}).$$

Similarly, we can prove that there are constants $\rho_k > 0, \eta_k > 0$ $(k = 2, 3, \cdots, n)$ such that

$$\|(u,v) - (w,z)\|_{J_k} \leq (\rho_k + \eta_k) e^{(N+\overline{N})\sigma b} (\|\varphi - \varphi^*\|_{\mathcal{B}} + \|\psi - \psi^*\|_{\mathcal{B}}).$$

$\square$

Remark 4.1. Some restrictive conditions are not used in Theorem 3.2 and Theorem 3.3, which is different from the corresponding results in [4,8].

5. An example

Let $X = L^2([0, \pi])$ and A be the operator given by $Af = f''$ with domain

$$D(A) = \{f \in X : f, f' \text{are absolutely continuous}, f'' \in X, f(0) = f(\pi) = 0\}.$$

It is well known that A is the infinitesimal generator of a strongly continuous cosine family $C(t)), t \in \mathbb{R}$ on X. Moreover, A has discrete spectrum, the eigenvalues are $-n^2, n \in \mathbb{N}$, with corresponding normalized eigenvectors $z_n(\xi) := \sqrt{\frac{2}{\pi}} \sin(n\xi)$ and the following properties hold:

(i) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of X.

(ii) For $f \in X, C(t)f = \sum_{n=1}^{\infty} \cos(nt)(f; z_n) z_n$. Moreover, it follow from this expression that $S(t)f = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n}(f; z_n) z_n$, that $S(t)$ is compact for $t > 0$ and that $\|C(t)\| = 1$ and $\|S(t)\| = 1$ for every $t \in \mathbb{R}$.

(iii) If Φ denotes the group of translations on X defined by $\Phi(t)x(\xi) = \widetilde{x}(\xi + t)$, where $\widetilde{x}$ is the extension of x with period 2π, then $C(t) = \frac{1}{2}(\Phi(t) + \Phi(-t)); A = B^2$ where B is the infinitesimal generator of the group Φ and $E = \{x \in H^1(0, \pi) : x(0) = x(\pi) = 0\}$, see [6] for details.

(iv) The functions φ, ψ defined by $\varphi(\theta, \xi) = \varphi(\theta)(\xi), \psi(\theta, \xi) = \psi(\theta)(\xi)$ belong to $\mathcal{B}$.

In the next application, $\mathcal{B}$ should be the phase space $\mathcal{B} = PC_0 \times L^2(h, X)$ in 2.1, where $h : (-\infty, 0] \to \mathbb{R}$ is a positive Lebesgue integrable function. We can take $H = 1, M(t) = \gamma(-t)^{\frac{1}{2}}$ and $K(t) = 1 + \left(\int_{-t}^{0} h(\theta) d\theta\right)^{\frac{1}{2}}$ for $t \geq 0$.

Example 5.1. Consider second order impulsive differential system with fractional derivative damping:

$$
\begin{cases}
\dfrac{\partial^2}{\partial t^2}u(t,\xi) = \dfrac{\partial^2}{\partial \xi^2}u(t,\xi) + cD_t^{\alpha}u(t,\xi) + \displaystyle\int_0^{\xi} a(s)D_t^{\alpha}u(t,s)ds \\
\qquad\qquad + \displaystyle\int_{-\infty}^{t} \mu(s-t)u(s,\xi)ds + D_t^{\gamma}u_t(\xi),\ t \in [0,1], \xi \in [0,\pi], \\
u(t,0) = u(t,\pi) = 0,\ t \in [0,1], \\
u(\theta,\xi) = \varphi(\theta,\xi),\ \theta \in (-\infty,0], \xi \in [0,\pi], \\
\dfrac{\partial}{\partial t}u(\theta,\xi) = \psi(\theta,\xi),\ \theta \in (-\infty,0], \xi \in [0,\pi], \\
\Delta u(t_i,\xi) = \displaystyle\int_{-\infty}^{0} q_i(\theta)\dfrac{\partial}{\partial t}u(t_i+\theta,\xi)d\theta, i = 1,2,\cdots,n, \\
\Delta u'(t_i,\xi) = \displaystyle\int_{-\infty}^{0} \overline{q}_i(\theta))\dfrac{u(s,x)}{1+|u(s,x)|}d\theta, i = 1,2,\cdots,n,
\end{cases}
\tag{5.1}
$$

where $D_t^{\alpha}, D_t^{\gamma}$ are the Caputo's fractional derivative operator of order $\alpha, \gamma \in (0,1]$, $c \in \mathbb{R}, a \in L^2([0,\pi]), \varphi, \psi \in C_0 \times L^2(h,X), 0 < t_1 < \cdots < t_n < 1$. Assume that the following conditions are satisfied:

(a) The function $\mu : \mathbb{R} \to \mathbb{R}^+$ is continuous and $d = \big(\int_{-\infty}^0 h^{-1}(\theta)\mu^2(\theta)d\theta\big)^{\frac{1}{2}} < \infty$;

(b) $q_i \in C(\mathbb{R},\mathbb{R})$ and $c_i = \big(\int_{-\infty}^0 q_i^2(\theta)h^{-1}(\theta)d\theta\big)^{\frac{1}{2}} < \infty$ for each $i = 1,2,\cdots,n$;

(c) $\overline{q}_i \in C(\mathbb{R},\mathbb{R})$ and $d_i = \big(\int_{-\infty}^0 \overline{q}_i^2(\theta)h^{-1}(\theta)d\theta\big)^{\frac{1}{2}} < \infty$ for each $i = 1,2,\cdots,n$.

Then the system (5.1) has a unique PC^1–mild solution.

Proof. The system (5.1) can be modeled as the abstract Cauchy problem (1.1) by defining

$$
\begin{aligned}
Bx(\xi) &= cx(\xi) + \int_0^{\xi} a(s)x(s)ds, \\
f(t,\varphi,y)(\xi) &= \int_{-\infty}^{0} \mu(\theta)\varphi(\theta,\xi)d\theta + y(\xi), \\
I_i(\varphi,\psi)(\xi) &= \int_{-\infty}^{0} q_i(\theta)\psi(\theta,\xi)d\theta, \\
J_i(\varphi,\psi)(\xi) &= \int_{-\infty}^{0} \overline{q}_i(\theta)\frac{\varphi(\theta,\xi)}{1+|\varphi(\theta,\xi)|}d\theta,
\end{aligned}
$$

where $f(t,\cdot,\cdot), I_i(\cdot,\cdot)$ $(i = 1,2,\cdots,n)$ and B are bounded linear operators, $\|I_i\| \leq c_i, \|J_i\| \leq d_i$ $(i = 1,2,\cdots,n)$ and $\|B\| \leq |c| + \|a\|_{L^2}$.

Moreover, for $(t,\phi,u),(t,\psi,v)\in[0,1]\times\mathcal{B}\times X\to X$, we have

$$
\begin{aligned}
&\|f(t,\phi,u(t))-f(t,\psi,v(t))\|_{L^2}\\
\le&\Big(\int_0^\pi\Big(\int_{-\infty}^0\mu(t,\theta)[\phi(\theta,x)-\psi(\theta,x)]d\theta+[u(t,x)-v(t,x)]\Big)^2dx\Big)^{\frac12}\\
\le&\Big(\int_0^\pi\Big(\int_{-\infty}^0\mu(t,\theta)[\phi(\theta,x)-\psi(\theta,x)]d\theta\Big)^2dx\Big)^{\frac12}+\Big(\int_0^\pi[u(t,x)-v(t,x)]^2dx\Big)^{\frac12}\\
\le&\Big(\int_{-\infty}^0\frac{\mu^2(t,\theta)}{h(\theta)}d\theta\int_{-\infty}^0h(\theta)\|\phi(\theta,\cdot)-\psi(\theta,\cdot)\|_{L^2}^2d\theta\Big)^{\frac12}+\|u(t)-v(t)\|_{L^2}\\
\le&d(t)\|\phi-\psi\|_{\mathcal{B}}+\|u(t)-v(t)\|_{L^2}\\
\le&L_f\big(\|\phi-\psi\|_{\mathcal{B}}+\|u(t)-v(t)\|_{L^2}\big),
\end{aligned}
$$

where $L_f=\max\{\sup_{0\le t\le1}d(t),1\}$. Hence the condition (H_1') is satisfied and all conditions of Theorem 3.2 are satisfied, therefore the system (5.1) has a unique PC^1–mild solution. However,

$$(N+\overline{N})K_b(1+\Gamma)(H\|B\|+L_f)\ge 2(1+\Gamma)(\|B\|+d)>1,$$

the restrictive conditions(1.4), (1.5) and (1.6) are not hold. □

Remark 5.1. If $\alpha=\gamma=1$ is in the system (1.1)–(1.2), all of our conclusions still holds, it improves and generalize the results in [10].

Acknowledgements

The author would like to thank the referee(s) for their valuable suggestions to improve presentation of the paper.

References

[1] K. Aissani and M. Benchohra, *Semlinear fractional order integro-differential equations with infinite delay in Banach spaces*, Arch. Math. (Brno), 49(2013)(2), 105–117.

[2] M. Benchohra, J. Henderson, S. K. Ntouyas and A. Ouahab, *Existence results for impulsive semilinear damped differential inclusions*, Electron. J. Qual. Theory Differ. Equ., 11(2003), 1–19.

[3] J. Dabas, A. Chauhan and M. Kumar, *Existence of the mild solutions for impulsive fractional equations with infinite delay*, Int. J. Differ. Equ., 2011(2011).

[4] J. Dabas and A. Chauhan, *Existence and uniqueness of mild solution for an impulsive neutral fractional integro-differential equation with infinite delay*, Math. Comput. Model., 57(2013)(3-4), 754–763.

[5] M. Fečkan, Y. Zhou and J. R. Wang, *On the concept and existence of solution for impulsive fractional differential equations*, Commun. Nonlinear Sci. Numer. Simul., 17(2012)(7), 3050–3060.

[6] H. O. Fattorini, *Second Order Linear Differential Equations in Banach Spaces*, North-Holland Math. Stud., vol. 108, North-Holland, Amsterdam, 1985.

[7] Z. Y. Guo and M. Liu, *An integrod-differential equation with fractional derivatives in the nonlinearities*, Acta Math. Univ., Comenianae, LXXXII, 1(2013), 105–111.

[8] T. L. Guo and W. Jiang, *Impulsive fractional functional differential equations*, Comput. Math. Appl., 64(2012)(10), 3414–3424.

[9] D. J. Guo, V. Lakshmikantham and X. Z. Liu, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic Publishers, 1996.

[10] E. Hernández, K. Balachandran and N. Annapoorani, *Existence results for a damped second order abstract functional differential equation with impulses*, Math. Comput. Model., 50(2009)(11-12), 1583–1594.

[11] H. R. Henriquez and C. H. Vásquez, *Differentiability of solutions of the second order abstract Cauchy problem*, Semigroup Forum, 64(2002)(3), 472–488.

[12] H. R. Henriquez and C. H. Vásquez, *Differentiability of solutions of second-order functional differential equations with unbounded delay*, J. Math. Anal. Appl., 280(2003)(2), 284–312.

[13] J. K. Hale and J. Kato, *Phase space for retarded equations with infinite delay*, Funkcial Ekvac, 21(1978), 11–41.

[14] Y. Hino, S. Murakami and T. Naito, *Functional-Differential Equations with Infinite Delay*, in: Lecture Notes in Mathematics, 1473, Springer-Verlag, Berlin, 1991.

[15] Y. Jalilian and R. Jalilian, *Existence of solution for delay fractional differential equations*, Mediterr. J. Math., 10(2013)(4), 1731–1747.

[16] H. P. Jiang, *Existence results for fractional order functional differential equations with impulse*, Comput. Math. Appl., 64(2012)(10), 3477–3483.

[17] V. Kavitha, P. Z. Wang and R. Murugesu, *Existence results for neutral functional fractional differential equations with state dependent-delay*, Malaya Journal of Matematik, 1(2012)(1), 50–61.

[18] J. Kisy'nski, *On cosine operator functions and one parameter group of operators*, Studia Math., 49(1972), 93–105.

[19] H. Mönch, *Boundary value problems for nonlinear ordinary equations of second order in Banach spaces*, Nonlinear Analysis, 4(1980)(5), 985–999.

[20] C. Ravichandran and M. Mallika Arjunan, *Existence results for impulsive fractional semilinear functional integro-differential equations in Banach spaces*,J. Fract. Calc. Appl., 3(2012)(8), 1–11.

[21] X. B. Shu, Y. Z. Lai and Y. M. Chen, *The existence of mild solutions for impulsive fractional partial differential equations*, Nonlinear Analysis, 74(2011)(5), 2003–2011.

[22] X. B. Shu and Q. Q. Wang, *The existence and uniqueness of mild solutions for fractional differential equations with nonlocal conditions of order* $1 < \alpha < 2$, Comput. Math. Appl., 64(2012)(6), 2100–2110.

[23] X. B. Shu and F. Xu, *The existence of solutions for impulsive fractional partial neutral differential equations*, Journal of Mathematics, 2013(2013), Article ID 147193.

[24] N. E. Tatar, *Mild solutions for a problem involving fractional derivatives in the nonlinearity and in the nonlocal conditions*, Adv. Difference Equ., 2011(2011), 1–18.

[25] C. C. Travis and G. F. Webb, *An abstract second order semilinear Volterra integrodifferential equation*, SIAM J. Math. Anal., 10(1979)(2), 412–424.

[26] C. C. Travis and G. F. Webb, *Compactness, regularity and uniform continuity properties of strongly continuous cosine families*, Houston J. Math., 3(1977)(4), 555–567.

[27] J. R. Wang, M. Fečkan and Y. Zhou, *On the new concept of solutions and existence results for impulsive fractional evolution equations*, Dynamics of PDE, 8(2011)(4), 345–361.

[28] S. L. Xie, *Solvability of impulsive partial neutral second-order functional integro-differential equations with infinite delay*, Bound. Value Probl., 203(2013), 1–19.

[29] S. L. Xie, *Existence results of mild solutions for impulsive fractional integro-differential evolution equations with infinite delay*, Fract. Calc. Appl. Anal., 17(2014)(4), 1158–1174.

[30] X. M. Zhang, X. Y. Huang and Z. H. Liu, *The existence and uniqueness of mild solutions for impulsive fractional equations with nonlocal conditions and infinite delay*, Nonlinear Analysis: Hybrid Systems, 4(2010), 775–781.